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Certain Classes Of Analytic Functions with Varying Arguments Defined by Convolution

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Abstract

In this paper, we introduce the new class $VCR(g,\lambda,A,B)$ ***of analytic functions with varying arguments in*** $\mathbb{U}=\{z\in\mathbb{C}:z<|z|\}$ ***defined by convolution and detrmine various properties for functions in it.***

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1 Introduction

Let S denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1}$$

which are analytic in $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For functions $f(z)$ given by (1) and $g(z)\in S$ given by

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n, \tag{2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n. \tag{3}$$

For $\lambda\geq 0$, $-1\leq A<B\leq 1$, $0<B\leq 1$, $f,g\in S$ and for all $z\in\mathbb{U}$, Mostafa et al. [8], defined the class $S(g,\lambda,A,B)$ such that

$$(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)\prec\frac{1+Az}{1+Bz}, \tag{4}$$

where $\prec$ denotes subordination. For the same values of the above parameters, let $CR(g,\lambda,A,B)$ denotes the subclass of S satisfying

$$(f*g)'(z)+\lambda z(f*g)''(z)\prec\frac{1+Az}{1+Bz}. \tag{5}$$

From (4) and (5) we obtain

$$f(z)\in CR(g,\lambda,A,B)\Longleftrightarrow zf'(z)\in S(g,\lambda,A,B). \tag{6}$$

For suitable choices of $g(z),\lambda,A$ and B, we obtain the following subclasses:

(1) Putting $g(z)=\frac{z}{1-z}$ or $b_n=1$, in (5), the class $CR(\frac{z}{1-z},\lambda,A,B)$ reduces to the class $R(\lambda,A,B)(-1\leq A<B\leq 1,0<B\leq 1,\lambda\geq 0)$ $(see Aouf et al. [3]\, and\, [4, with \gamma=-1]);$

(2) Putting $g(z)=\frac{z}{1-z}$, $A=2\beta-1$ and $B=1$, the class $CR\left(\frac{z}{1-z},\lambda,2\beta-1,1\right)$ reduces to the class $R(\lambda,\beta)\,(0\leq\beta<1,\lambda\geq 0)$ (see Altintas [2]).

Also,

(1) Putting $g(z)=z+\sum\limits_{n=2}^{\infty}\left(\frac{1+l+\gamma(n-1)}{l+1}\right)^m z^n\,(\gamma,l\geq 0,m\in\mathbb{N}_0=\{0,1,2,...\})$, the class $CR\left(z+\sum\limits_{n=2}^{\infty}\left(\frac{1+l+\gamma(n-1)}{l+1}\right)^m z^n,\lambda,A,B\right)$ reduces to

$$CR^*(\gamma,l,m,\lambda,A,B)=\{f\in S:(I^m(\gamma,l)f(z))'+\lambda z(I^m(\gamma,l)f(z))''\prec\frac{1+Az}{1+Bz}\}, \tag{7}$$

where $I^m(\gamma,l)$ is the extended multiplier transformation (see [5]). We note that $I^m(\gamma,0)=D_\gamma^m$ (see [1]) and $I^m(1,0)=D^n$ (see [9]).

(2) For $g(z)=z+\sum\limits_{n=2}^{\infty}\Gamma_n(\alpha_1)z^n$

$$\Gamma_n(\alpha_1)=\frac{(\alpha_1)_{n-1},...(\alpha_q)_{n-1}}{(\beta_1)_{n-1},...(\beta_s)_{n-1}} \tag{8}$$

$$\left(\alpha_i>0,i=1,...,q;\beta_j>0,j=1,...,s;q\leq s+1;q,s\in\mathbb{N}_0\right), \tag{9}$$

$(\theta)_k$ is the Pochhamer symbol defined by

$$(\theta)_k = \frac{\Gamma(\theta+k)}{\Gamma(\theta)} = \begin{cases} 1 & (k=0) \\ \theta(\theta+1)\ldots(\theta+k-1) & (k\in\mathbb{N}) \end{cases}. \tag{10}$$

The class $CR(g,\lambda,A,B)$ reduces to the class $CR^{**}([\alpha_1],\lambda,A,B)$

$$(H_{q,s}(\alpha_1)f(z))' + \lambda z (H_{q,s}(\alpha_1)f(z))'' \prec \frac{1+Az}{1+Bz}, \tag{11}$$

where $H_{q,s}(\alpha_1)$ is the Dziok-Srivastava operator (see [6], and [7]).

Definition 1[10]. If $f(z)\in S$, then $f(z)\in V(\theta_n)$ if $\arg(a_n)=\theta_n$, $n\geq 2$. If furthermore there exists a real number β such that

$$\theta_n + (n-1)\beta \equiv \pi mod 2\pi, \tag{12}$$

then $f(z)\in V(\theta_n,\beta)$. The union of $V(\theta_n,\beta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers β is denoted by V.

Note that

(1) $V(\theta_n+2n\pi)=V(\theta_n)$, n is an integer;

(2) $V(\pi,0)=\{f(z)\in S : f(z)=z-\sum_{n=2}^{\infty}a_nz^n\}=T$.

Let $VCR(g,\lambda,A,B)=CR(g,\lambda,A,B)\cap V$,

$V^*CR(\gamma,l,m,\lambda,A,B)=CR^*(\gamma,l,m,\lambda,A,B)\cap V$,

$V_{q,s}([\alpha_1],\lambda,A,B)=CR^{**}([\alpha_1],\lambda,A,B)\cap V$

and $VR(\lambda,\beta)=R(\lambda,\beta)\cap V$.

We note that:

(1) $VCR\left(\frac{z}{1-z},0,2\alpha-1,1\right)=C(\alpha)\ (0\leq\alpha<1)$ see [11];

(2) $VCR\left(\frac{z}{1-z},\lambda,A,B\right)=VR(\lambda,A,B)$(see [3]).

2 Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that, $b_n\geq 0 (n\geq 2)$, $\lambda\geq 0, -1\leq A<B\leq 1, 0<B\leq 1$, $g(z)$ is given by (2) and $z\in\mathbb{U}$.

Theorem 1 *Suppose that $f(z)$ of the form (1). If*

$$\sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right](1+B)b_n\left|a_n\right| \leq B-A, \tag{13}$$

then $f(z)\in CR(g,\lambda,A,B)$.

Proof. A function $f(z) \in CR(g, \lambda, A, B)$ if and only if there exists a function w, $|w(z)| \leq |z|$, such that

$$(f * g)'(z) + \lambda z(f * g)''(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{14}$$

or, equivalently,

$$\left| \frac{(f * g)'(z) + \lambda z(f * g)''(z) - 1}{B\left[(f * g)'(z) + \lambda z(f * g)''(z)\right] - A} \right| < 1. \tag{15}$$

Thus, it is sufficient to prove that

$$|(f * g)'(z) + \lambda z(f * g)''(z) - 1| - |B\left[(f * g)'(z) + \lambda z(f * g)''(z)\right] - A| < 0. \tag{16}$$

Indeed, letting $|z| = r (0 \leq r < 1)$ we have

$$|(f * g)'(z) + \lambda z(f * g)''(z) - 1| - |B\left[(f * g)'(z) + \lambda z(f * g)''(z)\right] - A| \tag{17}$$

$$= \left| \sum_{n=2}^{\infty} n\left[1 + \lambda(n-1)\right] a_n b_n z^{n-1} \right| - \left| (B - A) + B \sum_{n=2}^{\infty} n(1 + \lambda(n-1)) a_n b_n z^{n-1} \right| \tag{18}$$

$$\leq \left(\sum_{n=2}^{\infty} n\left[1 + \lambda(n-1)\right] b_n |a_n| r^{n-1} - (B - A) + B \sum_{n=2}^{\infty} n(1 + \lambda(n-1)) b_n |a_n| r^{n-1} \right) \tag{19}$$

$$< \sum_{n=2}^{\infty} n\left[1 + \lambda(n-1)\right](1 + B) b_n |a_n| - (B - A) \leq 0, \tag{20}$$

by (13). This completes the proof of Theorem 1.

Theorem 2 *The function* $f(z) \in VCR(g, \lambda, A, B)$ *if and only if*

$$\sum_{n=2}^{\infty} n\left[1 + \lambda(n-1)\right](1 + B) b_n |a_n| \leq (B - A). \tag{21}$$

Proof. In view of Theorem 1 we need only to show that each function $f(z)$ from the class $VCR(g, \lambda, A, B)$ satisfies the coefficient inequality (21). Let $f(z) \in VCR(g, \lambda, A, B)$. Then, by (15) and (1), we have

$$\left| \frac{\sum_{n=2}^{\infty} n\left[1 + \lambda(n-1)\right] b_n a_n z^{n-1}}{(B - A) + \sum_{n=2}^{\infty} B\left[1 + \lambda(n-1)\right] b_n a_n z^{n-1}} \right| < 1. \tag{22}$$

Since $f(z) \in VCR(g, \lambda, A, B)$, $f(z)$ lies in the class $V(\theta_n, \beta)$ for some sequence $\{\theta_n\}$ and a real number β such that $\theta_n + (n-1)\beta \equiv \pi mod 2\pi$, $(n \geq 2)$. Setting $z = re^{i\beta}$ in the above inequality, we get

$$\left| \frac{-\sum\limits_{n=2}^{\infty} n\left[1+\lambda(n-1)\right] b_n a_n r^{n-1} e^{i[\theta_n+(n-1)\beta]}}{(B-A) - \sum\limits_{n=2}^{\infty} B\left[1+\lambda(n-1)\right] b_n \left|a_n\right| r^{n-1} e^{i[\theta_n+(n-1)\beta]}} \right| < 1. \tag{23}$$

Since $Re\{w(z)\} < |w(z)| < 1$, then

$$Re\left\{ \frac{\sum\limits_{n=2}^{\infty} n\left[1+\lambda(n-1)\right] b_n \left|a_n\right| r^{n-1}}{(B-A) - \sum\limits_{n=2}^{\infty} B\left[1+\lambda(n-1)\right] b_n \left|a_n\right| r^{n-1}} \right\} < 1. \tag{24}$$

Hence

$$\sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right](1+B)\, b_n \left|a_n\right| r^{n-1} \leq (B-A), \tag{25}$$

which, upon letting $r \longrightarrow 1^-$, readily yieds the assertion (15).

Remark 1 *We can obtain another proof of Theorem 2, by using (6) and* $[10, Theorem 2]$.

Corollary 1 *Suppose that* $f(z) \in VCR(g, \lambda, A, B)$, *then*

$$|a_n| \leq \frac{(B-A)}{n\left[1+\lambda(n-1)\right](1+B)\, b_n} (n \geq 2). \tag{26}$$

The result is sharp for

$$f(z) = z + \frac{(B-A)}{n\left[1+\lambda(n-1)\right](1+B)\, b_n} e^{i\theta_n} z^n (n \geq 2). \tag{27}$$

3 Distortion theorems

Theorem 3 *Let* $f(z) \in VCR(g, \lambda, A, B)$. *Then*

$$|z| - \frac{(B-A)}{2(1+\lambda)(1+B)\, b_2} |z|^2 \leq |f(z)| \leq |z| + \frac{(B-A)}{2(1+\lambda)(1+B)\, b_2} |z|^2, \tag{28}$$

provided $b_n \geq b_2 (n \geq 2)$. *The result is sharp.*

Proof. Since
$$\Phi(n) = n\left[1+\lambda(n-1)\right](1+B)\,b_n, \tag{29}$$
is an increasing function of $n \quad (n \geq 2)$, from Theorem 2, we have
$$2(1+\lambda)(1+B)\,b_2\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty} n\left[1+\lambda(n-1)\right](1+B)\,b_n\,|a_n| \leq (B-A)\,, \tag{30}$$
that is
$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(B-A)}{2(1+\lambda)(1+B)\,b_2}. \tag{31}$$
Thus
$$|f(z)| = \left|z+\sum_{n=2}^{\infty}a_n z^n\right| \leq |z| + |z|^2\sum_{n=2}^{\infty}a_n \tag{32}$$
$$\leq |z| + \frac{(B-A)}{2(1+\lambda)(1+B)\,b_2}\,|z|^2\,. \tag{33}$$
Similarly, we get
$$|f(z)| \geq |z| - \sum_{n=2}^{\infty}|a_n|\,|z|^n \geq |z| - |z|^2\sum_{n=2}^{\infty}|a_n| \tag{34}$$
$$\geq |z| - \frac{(B-A)}{2(1+\lambda)(1+B)\,b_2}\,|z|^2\,. \tag{35}$$
Finally the result is sharp for
$$f(z) = z + \frac{(B-A)}{2(1+\lambda)(1+B)\,b_2}e^{i\theta_2}z^2 \tag{36}$$
at $z = \pm|z|\,e^{-i\theta_2}$.

Corollary 2 *Under the hypotheses of Theorem* 3, $f(z)$ *is included in a disc with center at the origin and radius* r_1 *given by*
$$r_1 = 1 + \frac{(B-A)}{2(1+\lambda)(1+B)\,b_2}. \tag{37}$$

Theorem 4 . *Let* $f(z) \in VCR\,(g,\lambda,A,B)$. *Then*
$$1 - \frac{(B-A)}{(1+\lambda)(1+B)\,b_2}\,|z| \leq |f'(z)| \leq 1 + \frac{(B-A)}{(1+\lambda)(1+B)\,b_2}\,|z|\,. \tag{38}$$
The result is sharp.

Proof. Similarly $n\Phi(n)$ is an increasing function of $n(n \geq 2)$, where $\Phi(n)$ is defined by (29). In view of Theorem 2, we have

$$(1+\lambda)(1+B)\,b_2\sum_{n=2}^{\infty} n\,|a_n| \leq \sum_{n=2}^{\infty} n\,[1+\lambda(n-1)]\,(1+B)\,b_n\,|a_n| \leq (B-A)\,, \tag{39}$$

that is

$$\sum_{n=2}^{\infty} n\,|a_n| \leq \frac{(B-A)}{(1+B)\,(1+\lambda)b_2}. \tag{40}$$

Thus

$$|f'(z)| = \left|1+\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq 1+|z|\sum_{n=2}^{\infty} n\,|a_n| \tag{41}$$

$$\leq 1+\frac{(B-A)}{(1+B)\,(1+\lambda)b_2}\,|z|\,. \tag{42}$$

Similarly, we get

$$|f'(z)| \geq 1+|z|\sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1} \geq 1-|z|\sum_{n=2}^{\infty} n\,|a_n| \tag{43}$$

$$\geq 1-\frac{(B-A)}{(1+B)\,(1+\lambda)b_2}\,|z|\,. \tag{44}$$

Finally the result is sharp for the function $f(z)$ given by (36).

Corollary 3 *Let $f(z) \in VCR\,(g,\lambda,A,B)$. Then $f'(z)$ is included in a disc with center at the origin and radius r_2 given by*

$$r_2 = 1+\frac{(B-A)}{(1+B)\,(1+\lambda)b_2}. \tag{45}$$

4 Extreme points

Theorem 5 *Suppose that $f(z) \in VCR\,(g,\lambda,A,B)$, with $arg(a_n) = \theta_n$ where $\theta_n + (n-1)\beta \equiv \pi mod 2\pi$. Let*

$$f_1(z) = z \tag{46}$$

and

$$f_n\,(z) = z+\frac{(B-A)}{n[1+\lambda\,(n-1)]\,(1+B)\,b_n}e^{i\beta n}z^n (n \geq 2). \tag{47}$$

Then $f(z)$ is in the class $VCR\left(g,\lambda,A,B\right)$, if and only if it can be expressed in the form

$$f\left(z\right)=\sum_{n=1}^{\infty}\mu_{n}f_{n}\left(z\right), \tag{48}$$

where $\mu_{n}\geq 0$ $(n\geq 1)$ and $\sum\limits_{n=1}^{\infty}\mu_{n}=1.$

Proof. If $f\left(z\right)=\sum\limits_{n=1}^{\infty}\mu_{n}f_{n}\left(z\right)$ with $\sum\limits_{n=1}^{\infty}\mu_{n}=1$ and $\mu_{n}\geq 0$, then

$$\sum_{n=2}^{\infty}n[1+\lambda\left(n-1\right)]\left(1+B\right)b_{n}\frac{\left(B-A\right)}{n[1+\lambda\left(n-1\right)]\left(1+B\right)b_{n}}\mu_{n} \tag{49}$$

$$\sum_{n=2}^{\infty}\left(B-A\right)\mu_{n}=\left(B-A\right)\left(1-\mu_{n}\right)\leq\left(B-A\right). \tag{50}$$

Hence $f(z)\in VCR\left(g,\lambda,A,B\right).$

Conversely, let $f(z)\in VCR\left(g,\lambda,A,B\right)$. Then a_{n} are given by (26). Setting

$$\mu_{n}=\frac{n[1+\lambda\left(n-1\right)]\left(1+B\right)b_{n}}{\left(B-A\right)}\left|a_{n}\right|, \tag{51}$$

and

$$\mu_{1}=1-\sum_{n=2}^{\infty}\mu_{n}. \tag{52}$$

From Theorem 2, $\sum\limits_{n=2}^{\infty}\mu_{n}\leq 1$ and so $\mu_{n}\geq 0$. Since $\mu_{n}f_{n}(z)=\mu_{n}z+a_{n}z^{n}$, then

$$\sum_{n=1}^{\infty}\mu_{n}f_{n}(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}=f\left(z\right). \tag{53}$$

This completes the proof of Theorem 5.

Remark 1. Putting $g(z)=\frac{z}{1-z},\lambda=0,A=2\alpha-1(0\leq\alpha<1)$ and $B=1$ in all the above results, we obtain the results obtained by Srivastava and Owa [11] ;

Remark 2. Putting $g(z)=\frac{z}{1-z},A=2\alpha-1(0\leq\alpha<1)$ and $B=1$ in all the above results, we obtain results obtained by Aouf et al. $(see\,[3]\,and\,[4,with\gamma=-1])$.

Remark 3. Specializing the parameters λ,A,B and a function $g(z)$, in the above results, we obtain new results for the classes $V^{*}CR(\gamma,l,m;\lambda,A,B)$, $V_{q,s}([\alpha_{1}]\,;\lambda,A,B)$ and $VR(\lambda,\beta)$ defined in the introduction.

5 Open Problem

The authors suggest studying properties for the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, satisfying

$$\frac{(f*g)'(z)}{pz^{p-1}} + \frac{\lambda (f*g)''(z)}{p(p-1)z^{p-1}} \prec \frac{1+Az}{1+Bz} (-1 \leq A < B \leq 1, 0 < B \leq 1, p \in \mathbb{N}, z \in \mathbb{U}), \tag{54}$$

when their coefficients are of varying arguments.

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