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Calderón's reproducing formula for the generalized wavelet transform on $\mathbb{R}^d$ associated to the Heckman-Opdam theory

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Abstract

We consider the harmonic analysis associated with the Heckman-Opdam theory on $\mathbb{R}^d$. Through this theory, we have defined and studied in [4], generalized wavelet transform on $\mathbb{R}^d$. In this paper, we prove a Calderón type reproducing formula, which gives rise to new representation for $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$-functions.

Keywords: *Heckman-Opdam theory; Generalized wavelet transform; Calderón's reproducing formula*

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1 Introduction

Calderón's reproducing formula was originally used in the so-called Calderón-Zygmund theory of singular integral operators (see [1]). Besides other applications in decomposition of certain function spaces (see [3]), the Calderón's formula was proved to be a powerful tool for recovering any L^2- function f from its wavelet transform $\Phi_g(f)$ (see [2]), given for a scale $a > 0$ and position $b \in \mathbb{R}^d$, as follows: For $g \in L^2(\mathbb{R}^d)$ a classical wavelet, we have

$$\Phi_g(f)(a,b) = \int_{\mathbb{R}^d} f(x)\overline{g_{a,b}}(x)dx, \quad (a,b) \in]0,+\infty[\times\mathbb{R}^d, \qquad (1.1)$$

where $g_{a,b}$ is the wavelet defined by

$$g_{a,b}(x) = \mathcal{T}_b g_a(x), \quad x \in \mathbb{R}^d, \tag{1.2}$$

with g_a the function given by

$$g_a(x) = \frac{1}{a^d} g(\frac{x}{a}). \tag{1.3}$$

Which satisfies

$$\mathcal{F}(g_a)(\lambda) = \mathcal{F}(g)(a\lambda), \quad \lambda \in \mathbb{R}^d, \tag{1.4}$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathbb{R}^d$ and $\mathcal{T}_b$, $b \in \mathbb{R}^d$, the classical translation operator defined by

$$\mathcal{T}_b g(x) = g(b - x), \quad x \in \mathbb{R}^d. \tag{1.5}$$

From [6], Calderón's reproducing formula is expressed in this way:

$$f(x) = \frac{1}{C_g} \int_0^{+\infty} (\int_{\mathbb{R}^d} \Phi_g(f)(a,b) g_{a,b}(x) dx) \frac{da}{a}, \tag{1.6}$$

strongly in $L^2(\mathbb{R}^d)$, where C_g is the constant given for almost all $\lambda \in \mathbb{R}^d$, by

$$C_g = \int_0^{+\infty} |\mathcal{F}(g)(a\lambda)|^2 \frac{da}{a}, \tag{1.7}$$

and which satisfies

$$0 < C_g < +\infty. \tag{1.8}$$

In [5][7], Heckman and Opdam have developed a harmonic analysis associated to the Cherednik operators on $\mathbb{R}^d$, which generalizes the harmonic analysis on symmetric spaces called the Heckman-Opdam theory on $\mathbb{R}^d$.

We have studied in [4], generalized wavelets and the generalized wavelet transform on $\mathbb{R}^d$ associated to the Heckman-Opdam theory. In this paper, we prove a Calderón's reproducing formula for this generalized wavelet transform.

2 Harmonic analysis associated to the Heckman-Opdam theory on $\mathbb{R}^d$

In this section, we cite basic results of the harmonic analysis associated to the Heckman-Opdam theory on $\mathbb{R}^d$. More details can be found in [9][10].

We consider $\mathbb{R}^d$ with the standard basis $\{e_i, i = 1, 2, ..., d\}$ and the inner product $\langle ., . \rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^d$.

2.1 The root system, the multiplicity function and the Cherednik operators

Let $\alpha \in \mathbb{R}^d \backslash \{0\}$ and $\check{\alpha} = \frac{2}{\|\alpha\|^2}\alpha$. We denote by

$$r_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha, \quad x \in \mathbb{R}^d, \tag{2.1}$$

the reflection in the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $\mathcal{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $r_\alpha \mathcal{R} = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. For a given root system $\mathcal{R}$, the reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$. For a given $\beta \in \mathbb{R}^d$ which belongs to no hyperplane H_α,$\alpha \in \mathcal{R}$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle \alpha, \beta \rangle > 0\}$. Then for each $\alpha \in \mathcal{R}$, either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$. We denote by $\mathcal{R}_+^0$, the set of positive indivisible roots. Let

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^d, \forall\, \alpha \in \mathcal{R}, \langle \alpha, x \rangle > 0\} \tag{2.2}$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}^+}$ its closure.
Let also $\mathbb{R}^d_{reg} = \mathbb{R}^d \backslash \cup_{\alpha \in \mathcal{R}} H_\alpha$ be the set of regular elements in $\mathbb{R}^d$.

A function $k : \mathcal{R} \to [0, +\infty[$ on the root system $\mathcal{R}$ is called a multiplicity function, if it is invariant under the action of the reflection group W. We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha). \tag{2.3}$$

Moreover, let $\mathcal{A}_k$ be the weight function

$$\forall\, x \in \mathbb{R}^d, \;\; \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |2 \sinh \langle \frac{\alpha}{2}, x \rangle|^{2k(\alpha)}, \tag{2.4}$$

which is W-invariant.

The Cherednik operators $T_j, j = 1, 2, ..., d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k, are defined for f of class C^1 on $\mathbb{R}^d$ and $x \in \mathbb{R}^d_{reg}$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j f(x), \tag{2.5}$$

where

$$\rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \text{ and } \alpha^j = \langle \alpha, e_j \rangle. \tag{2.6}$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the operators $T_j, j = 1, 2, ...d$, reduce to the corresponding partial derivatives. We suppose in the following that $k \neq 0$.

The Cherednik operators form a commutative system of differential-difference operators.

For f of class C^1 on $\mathbb{R}^d$ with compact support and g of class C^1 on $\mathbb{R}^d$, we have for $j = 1, 2, ..., d$:

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \mathcal{A}_k(x) dx = -\int_{\mathbb{R}^d} f(x)(T_j + S_j) g(x) \mathcal{A}_k(x) dx, \tag{2.7}$$

with

$$\forall\, x \in \mathbb{R}^d, S_j g(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha^j g(r_\alpha x). \tag{2.8}$$

2.2 The Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function

We denote by $G_\lambda, \lambda \in \mathbb{C}^d$, the eigenfunction of the operators $T_j, j = 1, 2, ..., d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda(x) & = i\lambda_j G_\lambda(x), \quad j = 1, 2, ..., d, x \in \mathbb{R}^d, \\ G_\lambda(0) & = 1. \end{cases} \tag{2.9}$$

It is called the Opdam-Cherednik kernel.
We consider the function F_λ defined by

$$\forall\, x \in \mathbb{R}^d, \;\; F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx). \tag{2.10}$$

This function is the unique analytic function on $\mathbb{R}^d$, which satisfies the differential system

$$\begin{cases} p(T) F_\lambda(x) & = p(i\lambda) F_\lambda(x), \quad x \in \mathbb{R}^d, \\ F_\lambda(0) & = 1 \end{cases} \tag{2.11}$$

for all W-invariant polynomials p on $\mathbb{C}^d$ and $p(T) = p(T_1, T_2, ..., T_d)$.

The function $F_\lambda(x)$ called the Heckman-Opdam hypergeometric function, is W-invariant both in λ and x. (For more properties of F_λ see [8]).

2.3 The Hypergeometric Fourier transform

Notations. We denote by

- $\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant.

- $\mathcal{D}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, with compact support and W-invariant.

- $\mathcal{S}(\mathbb{R}^d)^W$ the space of W-invariant functions from the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

The spaces $\mathcal{E}(\mathbb{R}^d)^W$, $\mathcal{D}(\mathbb{R}^d)^W$ et $\mathcal{E}(\mathbb{R}^d)^W$ are equipped with their classical topologies.

- $\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant, and such that for all $\ell, n \in \mathbb{N}$,

$$p_{\ell,n}(f) = \sup_{\substack{|\mu|\leq n \\ x\in\mathbb{R}^d}} (1+\|x\|)^\ell (F_0(x))^{-1}|D^\mu f(x)| < +\infty, \tag{2.12}$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}...\partial x_d^{\mu_d}}, \quad \mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d, \quad |\mu| = \sum_{i=1}^d \mu_i.$$

Its topology is defined by the semi-norms $p_{\ell,n}, \ell, n \in \mathbb{N}$.

- $PW_a(\mathbb{C}^d)^W$, $a > 0$, the space of entire functions g on $\mathbb{C}^d$, which are W-invariant and satisfying

$$\forall\, m \in \mathbb{N}, q_m(g) = \sup_{\lambda\in\mathbb{C}^d}(1+\|\lambda\|)^m e^{-a\|Im\lambda\|}|g(\lambda)| < +\infty. \tag{2.13}$$

The topology of $PW_a(\mathbb{C}^d)$ is defined by the semi-norms $q_m, m \in \mathbb{N}$.
We set

$$PW(\mathbb{C}^d)^W = \cup_{a>0} PW_a(\mathbb{C}^d)^W. \tag{2.14}$$

This space is called the Paley-Wiener space. It is equipped with the inductive limit topology.

Definition 1 *The hypergeometric Fourier transform $\mathcal{H}^W$ is defined for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) by*

$$\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_{-\lambda}(x)\mathcal{A}_k(x)dx. \tag{2.15}$$

Remark 1 *We have also the relation*

$$\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_\lambda(-x)\mathcal{A}_k(x)dx. \tag{2.16}$$

Proposition 1 *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have the following relations*

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(\bar{f})(\lambda) = \overline{\mathcal{H}^W(\check{f})(\lambda)}, \tag{2.17}$$

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f)(\lambda) = \mathcal{H}^W(\check{f})(-\lambda), \tag{2.18}$$

where $\check{f}$ is the function defined by

$$\forall\, x \in \mathbb{R}^d, \quad \check{f}(x) = f(-x).$$

Theorem 1
i) The hypergeometric Fourier transform $\mathcal{H}^W$ *is a topological isomorphism from*

- $\mathcal{D}(\mathbb{R}^d)^W$ *onto* $PW(\mathbb{C}^d)^W$.
- $\mathcal{S}_2(\mathbb{R}^d)^W$ *onto* $\mathcal{S}(\mathbb{R}^d)^W$.

ii) A function f *belongs to* $\mathcal{D}(\mathbb{R}^d)^W$ *with supp* $f \subset B(0,a)$ *the closed ball of center 0 and radius* $a > 0$*, if and only if its hypergeometric Fourier transform* $\mathcal{H}^W(f)$ *belongs to* $PW_a(\mathbb{C}^d)^W$.
iii) The inverse transform $(\mathcal{H}^W)^{-1}$ *is given by*

$$\forall\, x \in \mathbb{R}^d, (\mathcal{H}^W)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda) F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda, \tag{2.19}$$

where

$$\mathcal{C}_k^W(\lambda) = c_o |c_k(\lambda)|^{-2}, \tag{2.20}$$

with c_o *a positive constant chosen in such a way that* $\mathcal{C}_k^W(-\rho) = 1$*, and*

$$c_k(\lambda) = \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma(\langle i\lambda, \check{\alpha}\rangle + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(\langle i\lambda, \check{\alpha}\rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))}\,, \tag{2.21}$$

with the convention that $k(\frac{\alpha}{2}) = 0$ *if* $\frac{\alpha}{2} \notin \mathcal{R}$.

Remark 2 *The function* $\mathcal{C}_k^W$ *is continuous on* $\mathbb{R}^d$ *and satisfies the estimate*

$$\forall\, \lambda \in \mathbb{R}^d, |\mathcal{C}_k^W(\lambda)| \leq const.(1 + \|\lambda\|)^s, \tag{2.22}$$

for some $s > 0$.

Notations. We denote by

- $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfying

$$\begin{aligned} \|f\|_{\mathcal{A}_k,p} &= \Big(\int_{\mathbb{R}^d} |f(x)|^p \mathcal{A}_k(x) dx \Big)^{1/p} < +\infty, \quad 1 \leq p < +\infty, \\ \|f\|_{\mathcal{A}_k,\infty} &= ess \sup_{x \in \mathbb{R}^d} |f(x)| < +\infty. \end{aligned}$$

- $L^p_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfying

$$\begin{aligned} \|f\|_{\mathcal{C}_k^W,p} &= \Big(\int_{\mathbb{R}^d} |f(\lambda)|^p \mathcal{C}_k^W(\lambda) d\lambda \Big)^{1/p} < +\infty, \quad 1 \leq p < +\infty, \\ \|f\|_{\mathcal{C}_k^W,\infty} &= ess \sup_{\lambda \in \mathbb{R}^d} |f(\lambda)| < +\infty. \end{aligned}$$

Theorem 2
i) (Plancherel formulas). For all f, g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)}\mathcal{C}_k^W(\lambda)d\lambda, \tag{2.23}$$

and

$$||f||_{\mathcal{A}_k,2} = ||\mathcal{H}^W(f)||_{\mathcal{C}_k^W,2}. \tag{2.24}$$

ii) (Plancherel theorem). The hypergeometric Fourier transform $\mathcal{H}^W$ extends uniquely to an isometric isomorphism from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ onto $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$.

Corollary 1 *For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, we have the inversion formula*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)F_\lambda(x)\mathcal{C}_k^W(\lambda)d\lambda, \ \text{a.e.} \quad x \in \mathbb{R}^d . \tag{2.25}$$

2.4 The hypergeometric translation operator and the hypergeometric convolution product

Definition 2 *The hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, is defined on $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by*

$$\mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x)\mathcal{H}^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d. \tag{2.26}$$

Proposition 2
i) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\|\mathcal{T}_x^W(f)\|_{\mathcal{A}_k,2} \le |W|^{1/2}\|f\|_{\mathcal{A}_k,2}. \tag{2.27}$$

ii) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\mathcal{T}_x^W(f)(y) = \lim_{n\to+\infty}\int_{B(0,n)} F_\lambda(x)F_\lambda(y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda,$$

where $B(0,n)$ is the closed ball of center 0 and radius n. The limit is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
iii) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$ and $x \in \mathbb{R}^d$, we have

$$\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_\lambda(x)F_\lambda(y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda, \ \text{a.e.} \quad y \in \mathbb{R}^d. \tag{2.28}$$

iv) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\overline{\mathcal{T}_x^W(f)(y)} = \mathcal{T}_x^W(\overline{f})(y), \quad x, y \in \mathbb{R}^d, \tag{2.29}$$

and

$$\mathcal{T}_x^W(f)(y) = \mathcal{T}_y^W(f)(x), \quad x, y \in \mathbb{R}^d. \tag{2.30}$$

Definition 3 *The hypergeometric convolution product $f *_{\mathcal{H}^W} g$ of the functions f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) is defined by*

$$\forall\, x \in \mathbb{R}^d,\ \ f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy. \tag{2.31}$$

Proposition 3 *Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then, the function $f *_{\mathcal{H}^W} g$ defined all most everywhere on $\mathbb{R}^d$ by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy, \tag{2.32}$$

belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,2} \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,1}, \tag{2.33}$$

and

$$\mathcal{H}^W(f *_{\mathcal{H}^W} g) = \mathcal{H}^W(f).\mathcal{H}^W(g). \tag{2.34}$$

Proposition 4 *Let f and g be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then, the function $f *_{\mathcal{H}^W} g$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ if and only if the function $\mathcal{H}^W(f).\mathcal{H}^W(g)$ is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have*

$$\mathcal{H}^W(f *_{\mathcal{H}^W} g) = \mathcal{H}^W(f).\mathcal{H}^W(g), \tag{2.35}$$

in the L^2-case.

3 Calderón's reproducing formula

3.1 Generalized wavelets and the generalized wavelet transform on $\mathbb{R}^d$

Definition 4 *We say that a function g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ is a generalized wavelet on $\mathbb{R}^d$, if there exists a constant C_g such that*
i) $0 < C_g < +\infty$.
ii) For almost all $\lambda \in \mathbb{R}^d$, we have

$$C_g = \int_0^{+\infty} |\mathcal{H}^W(g)(a\lambda)|^2 \frac{da}{a}. \tag{3.1}$$

Example 1 *Let $t > 0$. We consider the function g defined by*

$$\forall x \in \mathbb{R}^d,\quad g(x) = -\mathcal{L}_k^W E_t^W(x),$$

where $\mathcal{L}_k^W$ is the Heckman-Opdam Laplacian defined for a function f on $\mathbb{R}^d$ of class C^2 and W-invariant, by

$$\mathcal{L}_k^W f = \sum_{j=1}^{d} T_j^2 f. \tag{3.2}$$

It has the following form : For $x \in \mathbb{R}^d_{reg}$

$$\mathcal{L}_k^W f(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \coth(\frac{\langle \alpha, x \rangle}{2}) \langle \nabla f(x), \alpha \rangle + ||\rho||^2 f(x),$$

where Δ and ∇ are respectively the Laplacian and the gradient on $\mathbb{R}^d$, and E_t^W, $t > 0$, the heat kernel given by

$$\forall x \in \mathbb{R}^d, \quad E_t^W(x) = \int_{\mathbb{R}^d} e^{-t(||\lambda||^2 + \|\rho\|^2)} F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda. \tag{3.3}$$

By using (2.9),(2.10),(3.2),(3.3), we obtain

$$\forall x \in \mathbb{R}^d, \quad g(x) = \int_{\mathbb{R}^d} ||\lambda||^2 e^{-t(||\lambda||^2 + \|\rho\|^2)} F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda.$$

The function g belongs to $S_2(\mathbb{R}^d)^W$ and we have

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(g)(\lambda) = ||\lambda||^2 e^{-t(||\lambda||^2 + \|\rho\|^2)}.$$

For $\lambda \in \mathbb{R}^d \backslash \{0\}$, we have

$$\begin{aligned} C_g &= \int_0^{+\infty} |\mathcal{H}^W(g)(a\lambda)|^2 \frac{da}{a} \\ &= e^{-2t||\rho||^2} \int_0^{+\infty} ||\lambda||^4 e^{-2ta^2||\lambda||^2} a^3 da, \end{aligned}$$

By change of variables we obtain, for almost all $\lambda \in \mathbb{R}^d$:

$$C_g = \frac{e^{-2t||\rho||^2}}{8t^2}.$$

Definition 5 *We define the function l_k on $]0, +\infty[$ by*

$$l_k(a) = \sup_{\lambda \in \mathbb{R}^d \backslash \{0\}} \frac{|\mathcal{C}_k^W(\frac{\lambda}{a})|}{|\mathcal{C}_k^W(\lambda)|} = \sup_{\lambda \in \mathbb{R}^d \backslash \{0\}} \frac{|c_k(\lambda)|^2}{|c_k(\frac{\lambda}{a})|^2}, \tag{3.4}$$

where $\mathcal{C}_k^W$ and c_k the functions given by the relations (2.20),(2.21).

Remark 3 *When $k(\alpha) \in \mathbb{N}$, for all $\alpha \in \mathcal{R}$, the function l_k has the follouwing form*

$$l_k(a) = \sup_{\lambda \in \mathbb{R}^d \setminus \{0\}} \prod_{\alpha \in \mathcal{R}_+} \prod_{n=1}^{k(\alpha)} \frac{(\langle \lambda, \check{\alpha} \rangle)^2 + (\frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha) - n)^2}{(\frac{1}{a}\langle \lambda, \check{\alpha} \rangle)^2 + (\frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha) - n)^2}.$$

It satisfies the estimates
i) If $a \in [1, +\infty[$

$$0 < l_k(a) \leq a^{2\gamma},$$

with γ defined by the relation (2.3).
ii) If $a \in]0, 1[$

$$0 < l_k(a) \leq \prod_{\alpha \in \mathcal{R}_+} k(\alpha).$$

Theorem 3 *Let $a > 0$ and g a generalized wavelet on $\mathbb{R}^d$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then,*
i) The function $\lambda \longrightarrow \mathcal{H}^W(g)(a\lambda)$ belongs to $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, and we have

$$\int_{\mathbb{R}^d} |\mathcal{H}^W(g)(a\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda \leq \frac{l_k(a)}{a^d} ||g||^2_{\mathcal{A}_k,2}, \tag{3.5}$$

where l_k is the function given by the relation (3.4).
ii) There exists a function g_a in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that

$$\mathcal{H}^W(g_a)(\lambda) = \mathcal{H}^W(g)(a\lambda), \quad \lambda \in \mathbb{R}^d, \tag{3.6}$$

and we have

$$||g_a||^2_{\mathcal{A}_k,2} \leq \frac{l_k(a)}{a^d} ||g||^2_{\mathcal{A}_k,2}. \tag{3.7}$$

Proposition 5 *Let g be a generalized wavelet on $\mathbb{R}^d$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then, for $a > 0$ and $b \in \mathbb{R}^d$, the function*

$$g_{a,b}(x) = \mathcal{T}_b^W g_a(x), \quad x \in \mathbb{R}^d, \tag{3.8}$$

is a generalized wavelet on $\mathbb{R}^d$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$C_{g_{a,b}} \leq |W| C_g. \tag{3.9}$$

Definition 6 *The generalized wavelet transform Φ_g on $\mathbb{R}^d$ is defined, for f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, by*

$$\Phi_g(f)(a, b) = \int_{\mathbb{R}^d} f(x) \overline{g_{a,b}(x)} \mathcal{A}_k(x) dx, \quad (a, b) \in]0, +\infty[\times \mathbb{R}^d. \tag{3.10}$$

We can also write it in the form

$$\Phi_g(f)(a, b) = \check{f} *_{\mathcal{H}^W} \overline{g_a}(b), \tag{3.11}$$

where $\check{f}$ is the function defined by

$$\check{f}(x) = f(-x), \quad x \in \mathbb{R}^d.$$

3.2 Calderón's reproducing formula

Theorem 4 *(Calderón's formula). Let g be a generalized wavelet in* $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ *such that* $\mathcal{H}^W(g)$ *belongs to* $L^\infty_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$. *Then, for f in* $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ *and* $0 < \epsilon < \delta < +\infty$, *the function*

$$f^{\epsilon,\delta}(x) = \frac{1}{C_g}\int_\epsilon^\delta \int_{\mathbb{R}^d} \Phi_g(a,b) g_{a,b}(x)\mathcal{A}_k(b)db\frac{da}{a}, \quad x \in \mathbb{R}^d, \tag{3.12}$$

belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, *and satisfies*

$$\lim_{\epsilon\to 0,\delta\to+\infty} ||f^{\epsilon,\delta} - f||_{\mathcal{A}_k,2} = 0. \tag{3.13}$$

To prove this theorem we need the following Lemmas.

Lemma 1 *Let g be the generalized wavelet satisfying the conditions of Theorem 4 and f in* $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. *Then,*
i) The functions $(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{}$ *and* $(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a$ *are in* $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, *and we have*

$$\mathcal{H}^W((\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a)(\lambda) = \mathcal{H}^W(f)(\lambda)|\mathcal{H}^W(g_a)(\lambda)|^2, \quad \lambda \in \mathbb{R}^d. \tag{3.14}$$

ii) We have

$$||(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a||_{\mathcal{A}_k,2} \le ||\mathcal{H}^W(g)||^2_{\mathcal{C}_k^W,\infty}||f||_{\mathcal{A}_k,2}. \tag{3.15}$$

Proof
i) From the relations (2.17)(2.18) and Proposition 4 we have

$$\begin{aligned}\mathcal{H}^W((\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{})(\lambda) &= \mathcal{H}^W(\check{f} *_{\mathcal{H}^W} \overline{g_a})(-\lambda)\\ &= \mathcal{H}^W(\check{f})(-\lambda)\mathcal{H}^W(\overline{g_a})(-\lambda)\\ &= \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(\check{g_a})(-\lambda)}.\end{aligned}$$

Thus,

$$\mathcal{H}^W((\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{})(\lambda) = \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g_a)(\lambda)}. \tag{3.16}$$

On the other hand, we put

$$Z(x) = (\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{}(x), \quad x \in \mathbb{R}^d.$$

Thus,

$$\mathcal{H}^W((\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a)(\lambda) = \mathcal{H}^W(Z *_{\mathcal{H}^W} g_a)(\lambda), \quad \lambda \in \mathbb{R}^d.$$

By using Proposition 4, we deduce that the function Z belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$\mathcal{H}^W(Z *_{\mathcal{H}^W} g_a)(\lambda) = \mathcal{H}^W(Z)(\lambda)\mathcal{H}^W(g_a)(\lambda), \quad \lambda \in \mathbb{R}^d. \tag{3.17}$$

We deduce (3.14) from (3.16),(3.17).
ii) From the i) we have

$$\int_{\mathbb{R}^d} |\mathcal{H}^W((\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a)(\lambda)|^2 \mathcal{C}_k^W(\lambda)d\lambda = \int_{\mathbb{R}^d} |\mathcal{H}^W(f)(\lambda)|^2 |\mathcal{H}^W(g_a)(\lambda)|^4 \mathcal{C}_k^W(\lambda)d\lambda.$$

Then, from the Plancherel formula (2.24) and the fact that $\mathcal{H}^W(g_a)$ belongs to $L^\infty_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, we obtain

$$||(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a||_{\mathcal{A}_k,2} \leq ||\mathcal{H}^W(g_a)||^2_{\mathcal{C}_k^W,\infty}||f||_{\mathcal{A}_k,2}.$$

We deduce the result from the relation (3.6).

Lemma 2 *Let g be the generalized wavelet satisfying the conditions of Theorem 4. Then, the function $K_{\epsilon,\delta}$ defined by*

$$K_{\epsilon,\delta}(\lambda) = \frac{1}{C_g}\int_\epsilon^\delta |\mathcal{H}^W(g_a)(\lambda)|^2 \frac{da}{a}, \quad \lambda \in \mathbb{R}^d, \tag{3.18}$$

satisfies, for almost all $\lambda \in \mathbb{R}^d$:

$$0 < K_{\epsilon,\delta}(\lambda) \leq 1, \tag{3.19}$$

and

$$\lim_{\epsilon\to 0,\delta\to+\infty} K_{\epsilon,\delta}(\lambda) = 1. \tag{3.20}$$

Proof
From the relation (3.1), for almost all $\lambda \in \mathbb{R}^d$, we have

$$|K_{\epsilon,\delta}(\lambda)| \leq \frac{1}{C_g}\int_0^{+\infty} |\mathcal{H}^W(g_a)(\lambda)|^2 \frac{da}{a} = 1.$$

On the other hand, for almost all $\lambda \in \mathbb{R}^d$, we have

$$\lim_{\epsilon\to 0,\delta\to+\infty} K_{\epsilon,\delta}(\lambda) = 1.$$

This completes the proof.

Lemma 3 *We consider the functions f and g satisfying the conditions of Theorem 4. Then the function $f^{\epsilon,\delta}$ defined by the relation (3.12) belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and satisfies*

$$\mathcal{H}^W(f^{\epsilon,\delta})(\lambda) = \mathcal{H}^W(f)(\lambda)K_{\epsilon,\delta}(\lambda), \quad \lambda \in \mathbb{R}^d, \tag{3.21}$$

where $K_{\epsilon,\delta}$ is the function given by the relation (3.18).

Proof

- We prove first, that the function $f^{\epsilon,\delta}$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
From Definition 6, Proposition 5 and the relation (2.30) we have

$$f^{\epsilon,\delta}(x) = \frac{1}{C_g}\int_\epsilon^\delta \int_{\mathbb{R}^d} (\check{f} *_{\mathcal{H}^W} \overline{g_a})(b)\mathcal{T}^W_x(g_a)(b)\mathcal{A}_k(b)db\frac{da}{a}. \tag{3.22}$$

But, from the relation (2.32) we have

$$\begin{aligned}\int_{\mathbb{R}^d} (\check{f} *_{\mathcal{H}^W} \overline{g_a})(b)\mathcal{T}^W_x(g_a)(b)\mathcal{A}_k(b)db &= \int_{\mathbb{R}^d} (\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{}(b)\mathcal{T}^W_x(g_a)(-b)\mathcal{A}_k(b)db \\ &= (\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x).\end{aligned}$$

Then,

$$f^{\epsilon,\delta}(x) = \frac{1}{C_g}\int_\epsilon^\delta (\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)\frac{da}{a}. \tag{3.23}$$

By using Hölder's inequality for the measure $\frac{da}{a}$, we get

$$|f^{\epsilon,\delta}(x)|^2 \leq \frac{1}{C_g^2}(\int_\epsilon^\delta \frac{da}{a})\int_\epsilon^\delta |(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)|^2\frac{da}{a}.$$

So, by applying Fubini-Tonelli's theorem, we obtain

$$\int_{\mathbb{R}^d} |f^{\epsilon,\delta}(x)|^2\mathcal{A}_k(x)dx \leq \frac{1}{C_g^2}(\int_\epsilon^\delta \frac{da}{a})\int_\epsilon^\delta (\int_{\mathbb{R}^d} |(\check{f}*_{\mathcal{H}^W}\overline{g_a})\check{}*_{\mathcal{H}^W} g_a(x)|^2\mathcal{A}_k(x)dx)\frac{da}{a}.$$

From the Plancherel formula (2.24) and the relation (3.14), we deduce that

$$\int_{\mathbb{R}^d} |f^{\epsilon,\delta}(x)|^2\mathcal{A}_k(x)dx \leq \frac{1}{C_g^2}(\int_\epsilon^\delta \frac{da}{a})\int_{\mathbb{R}^d} |\mathcal{H}^W(f)(\lambda)|^2(\int_\epsilon^\delta |\mathcal{H}^W(g_a)(\lambda)|^4\frac{da}{a})\mathcal{C}^W_k(\lambda)d\lambda.$$

On the other hand, from the relations (3.1),(3.6), we have

$$\int_\epsilon^\delta |\mathcal{H}^W(g_a)(\lambda)|^4\frac{da}{a} \leq C_g||\mathcal{H}^W(g)||^2_{\mathcal{C}^W_k,\infty}.$$

Thus,

$$\int_{\mathbb{R}^d} |f^{\epsilon,\delta}(x)|^2 \mathcal{A}_k(x)dx \leq \frac{1}{C_g}(\int_\epsilon^\delta \frac{da}{a})||\mathcal{H}^W(g)||^2_{\mathcal{C}_k^W,\infty}||\mathcal{H}^W(f)||^2_{\mathcal{C}_k^W,2},$$

and the Plancherel formula (2.24) implies

$$\int_{\mathbb{R}^d} |f^{\epsilon,\delta}(x)|^2 \mathcal{A}_k(x)dx \leq \frac{1}{C_g}(\int_\epsilon^\delta \frac{da}{a})||\mathcal{H}^W(g)||^2_{\mathcal{C}_k^W,\infty}||f||^2_{\mathcal{A}_k,2} < +\infty.$$

Then, $f^{\epsilon,\delta}$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
- We prove now the relation (3.21). Let ψ in $S(\mathbb{R}^d)^W$. From Theorem 1.i), the function $(\mathcal{H}^W)^{-1}(\psi)$ is in $S_2(\mathbb{R}^d)^W$. From the relation (3.23), we have

$$\int_{\mathbb{R}^d} f^{\epsilon,\delta}(x)(\mathcal{H}^W)^{-1}(\psi)(x)\mathcal{A}_k(x)dx$$
$$= \int_{\mathbb{R}^d}(\frac{1}{C_g}\int_\epsilon^\delta (\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)\frac{da}{a})(\mathcal{H}^W)^{-1}(\psi)(x)\mathcal{A}_k(x)dx. \tag{3.24}$$

We consider

$$\frac{1}{C_g}\int_{\mathbb{R}^d}\int_\epsilon^\delta |(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)(\mathcal{H}^W)^{-1}(\psi)(x)|\mathcal{A}_k(x)dx\frac{da}{a}$$
$$= \frac{1}{C_g}\int_\epsilon^\delta [\int_{\mathbb{R}^d} |(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)||(\mathcal{H}^W)^{-1}(\psi)(x)|\mathcal{A}_k(x)dx]\frac{da}{a}. \tag{3.25}$$

By applying Hölder's inequality to the second member, we get

$$\frac{1}{C_g}\int_\epsilon^\delta [\int_{\mathbb{R}^d} |(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)||(\mathcal{H}^W)^{-1}(\psi)(x)|\mathcal{A}_k(x)dx]\frac{da}{a}$$
$$\leq \frac{1}{C_g}\int_\epsilon^\delta ||(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a||_{\mathcal{A}_k,2}||(\mathcal{H}^W)^{-1}(\psi)||_{\mathcal{A}_k,2}\frac{da}{a}.$$

From the relation (3.15) and the Plancherel formula (2.24), we obtain

$$\frac{1}{C_g}\int_\epsilon^\delta [\int_{\mathbb{R}^d} |(\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)||(\mathcal{H}^W)^{-1}(\psi)(x)|\mathcal{A}_k(x)dx]\frac{da}{a}$$
$$\leq \frac{1}{C_g}(\int_\epsilon^\delta \frac{da}{a})||\mathcal{H}^W(g)||_{\mathcal{C}_k^W,\infty}||\psi||_{\mathcal{C}_k^W,2}||f||_{\mathcal{A}_k,2} < +\infty.$$

Then, from Fubini theorem, the second member of the relation (3.24) can also be written in the form

$$\frac{1}{C_g}\int_\epsilon^\delta (\int_{\mathbb{R}^d} (\check{f} *_{\mathcal{H}^W} \overline{g_a})\check{} *_{\mathcal{H}^W} g_a(x)(\mathcal{H}^W)^{-1}(\psi)(x)\mathcal{A}_k(x)dx)\frac{da}{a}. \tag{3.26}$$

But, by using the Plancherel formula (2.24) and the relation (3.14), the relation (3.26) is equal to

$$\frac{1}{C_g}\int_{\epsilon}^{\delta}(\int_{\mathbb{R}^d}\mathcal{H}^W(f)(\lambda)|\mathcal{H}^W(g_a)|^2\psi(\lambda)\mathcal{C}_k^W(\lambda)d\lambda)\frac{da}{a}.$$

By applying Fubini-Tonelli's theorem and next Fubini's theorem to this integral, it takes the form

$$\int_{\mathbb{R}^d}\mathcal{H}^W(f)(\lambda)(\frac{1}{C_g}\int_{\epsilon}^{\delta}|\mathcal{H}^W(g_a)|^2\frac{da}{a})\psi(\lambda)\mathcal{C}_k^W(\lambda)d\lambda = \int_{\mathbb{R}^d}\mathcal{H}^W(f)(\lambda)K_{\epsilon,\delta}(\lambda)\psi(\lambda)\mathcal{C}_k^W(\lambda)d\lambda. \tag{3.27}$$

On the other hand, by applying the Plancherel formula (2.24) to the first member of the relation (3.24), we get

$$\int_{\mathbb{R}^d}\mathcal{H}^W(f^{\epsilon,\delta})(\lambda)\psi(\lambda)\mathcal{C}_k^W(\lambda)d\lambda. \tag{3.28}$$

From the relations (3.27),(3.28), we obtain for all ψ in $S(\mathbb{R}^d)^W$:

$$\int_{\mathbb{R}^d}(\mathcal{H}^W(f^{\epsilon,\delta})(\lambda)-\mathcal{H}^W(f)(\lambda)K_{\epsilon,\delta}(\lambda))\psi(\lambda)\mathcal{C}_k^W(\lambda)d\lambda = 0.$$

Thus

$$\mathcal{H}^W(f^{\epsilon,\delta})(\lambda) = \mathcal{H}^W(f)(\lambda)K_{\epsilon,\delta}(\lambda), \quad \lambda\in\mathbb{R}^d.$$

Proof of Theorem 3.2

From Lemma 3, the function $f^{\epsilon,\delta}$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. By using the Plancherel formula (2.24) and Lemma 3, we obtain

$$\begin{aligned}||f^{\epsilon,\delta}-f||_{\mathcal{A}_k,2} &= \int_{\mathbb{R}^d}|\mathcal{H}^W(f^{\epsilon,\delta}-f)(\lambda)|^2\mathcal{C}_k^W(\lambda)d\lambda\\ &= \int_{\mathbb{R}^d}|\mathcal{H}^W(f)(\lambda)(K_{\epsilon,\delta}(\lambda)-1)|^2\mathcal{C}_k^W(\lambda)d\lambda\\ &= \int_{\mathbb{R}^d}|\mathcal{H}^W(f)(\lambda)|^2|1-K_{\epsilon,\delta}(\lambda)|^2\mathcal{C}_k^W(\lambda)d\lambda.\end{aligned}$$

But from Lemma 2, for almost all $\lambda\in\mathbb{R}^d$, we have

$$\lim_{\epsilon\to0,\delta\to+\infty}|\mathcal{H}^W(f)(\lambda)|^2|1-K_{\epsilon,\delta}(\lambda)|^2 = 0,$$

and

$$|\mathcal{H}^W(f)(\lambda)|^2|1-K_{\epsilon,\delta}(\lambda)|^2 \leq 4|\mathcal{H}^W(f)(\lambda)|^2,$$

with $|\mathcal{H}^W(f)(\lambda)|^2$ in $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$. So, the relation (3.13) follows from the dominated convergence theorem.

4 Open Problem

The purpose of the future work is to generalize the Calderón's reproducing formula for the generalized wavelet on $\mathbb{R}^d$ associated to the Heckman-Opdam theory on functions spaces other than $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

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