Int. J. Open Problems Complex Analysis, Vol. 8, No. 1, March 2016
ISSN 2074-2827; Copyright ©ICSRS Publication, 2016
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Certain Subclass of Multivalent Functions with Higher Order Derivatives and Negative Coefficients

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Received 5 July 2015; Accepted 20 November 2015

Abstract: In this paper, we introduce new class of multivalent functions with higher order derivatives defined in the open unit disc. We obtain coefficient inequalities, distortion theorems, radii of convexity, clousre theorems and modified Hadamard products for functions in this class. Finally several applications involving an integeral operator and certain generalized fractional calculus operators are also considered for this class.

Keywords and phrases: *Analytic, multivalent functions, higher order derivatives, modified Hadamard products, generalized fractional calculus.*
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1. Introduction

Let $S_p(n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \ (p, n \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Let $T_p(n)$ denote the subclass of $S_p(n)$ of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k (a_k \geq 0). \tag{1.2}$$

We note that $T_p(1) = T_p$.

For each $f(z) \in S_p(n)$, we have

$$f^{(m)}(z) = \delta(p, m) z^{p-m} + \sum_{k=p+n}^{\infty} \delta(k, m) a_k z^{k-m}, \tag{1.3}$$

where

$$\delta(i, j) = \frac{i!}{(i-j)!} = \begin{cases} i(i-1)...(i-j+1) & (j \neq 0), \\ 1 & (j = 0). \end{cases} \tag{1.4}$$

Aouf [1] introduced and studied the class $T_p^*(\alpha, \beta)$ consisting of functions $f(z) \in S_p(n)$ which satisfies:

$$\left| \frac{\frac{f'(z)}{z^{p-1}} - p}{\frac{f'(z)}{z^{p-1}} + p - 2\alpha} \right| < \beta, \tag{1.5}$$

where $0 < \beta \leq 1,\ 0 \leq \alpha < p$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$.

Let $S_n(p, q; \alpha, \beta)$ be the subclass of $S_p(n)$ consisting of functions $f(z)$ of the form (1.1), and satisfying the analytic criterion:

$$\left| \frac{\frac{f^{(q+1)}(z)}{\delta(p-1,q) z^{p-q-1}} - p}{\frac{f^{(q+1)}(z)}{\delta(p-1,q) z^{p-q-1}} + p - 2\alpha} \right| < \beta \ (z \in \mathbb{U}), \tag{1.6}$$

where $0 < \beta \leq 1,\ 0 \leq \alpha < p,\ p \in \mathbb{N},\ q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $p > q$.

Further, let

$$T_n^*(p, q; \alpha, \beta) = S^*(p, q; \alpha, \beta) \cap T_p(n). \tag{1.7}$$

We note that for suitable choices of $n,\ p,\ q,\ \alpha$ and β, we obtain the following subclasses:
(i) $T_1^*(p, 0; \alpha, \beta) = P_p^*(\alpha, \beta)$ (Aouf [1]);
(ii) $T_1^*(1, 0; \alpha, \beta) = P^*(\alpha, \beta)$ (Gupta and Jain [2]);
(iii) $T_1^*(p, 0; \alpha, 1) = F_p(1, \alpha)$ (Lee et al .[3]).

Also, we note that:

$$T_n^*(p, q; \alpha, 1) = T_n^*(p, q; \alpha) = \left\{ f \in T_p(n) : Re\left(\frac{f^{(q+1)}(z)}{\delta\,(p-1, q)\, z^{p-q-1}} \right) > \alpha,\ 0 \leq \alpha < p \right\}.$$

2. Coefficient inequalities

Unless otherwise mentioned, we assume throughout this paper that $0 < \beta \leq 1,\ 0 \leq \alpha < p,\ n \in \mathbb{N},\ q \in \mathbb{N}_0,\ p > q$ and $\delta(i,j)\ (i > j)$ is defined by (1.4).

Theorem 1. A function $f(z)$ of the form (1.2) is in the class $T_n^*(p,q;\alpha,\beta)$ if and only if

$$ {}_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_k \leq 2\beta(p-\alpha)\delta(p-1,q). \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds true, then

$$\begin{aligned} & \left|f^{(q+1)}(z) - \delta(p-1,q)pz^{p-q-1}\right| \\ & -\beta\left|f^{(q+1)}(z) + \delta(p-1,q)(p-2\alpha)z^{p-q-1}\right| \\ = & \left|{}_{k=p+n}^{\infty} k\delta(k-1,q)a_k z^{k-q-1}\right| \\ & -\beta\left|2(p-\alpha)\delta(p-1,q)z^{p-q-1} + {}_{k=p+n}^{\infty} k\delta(k-1,q)a_k z^{k-q-p}\right| \\ \leq & {}_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_k - 2\beta(p-\alpha)\delta(p-1,q) \leq 0. \end{aligned}$$

Hence, by the maximum modulus theorem, $f(z) \in T_n^*(p,q;\alpha,\beta)$.

Conversely, assume that $f(z) \in T_n^*(p,q;\alpha,\beta)$. Thus

$$\begin{aligned} & \left|\frac{f^{(q+1)}(z) - \delta(p-1,q)pz^{p-q-1}}{f^{(q+1)}(z) + \delta(p-1,q)(p-2\alpha)z^{p-q-1}}\right| \\ = & \left|\frac{{}_{k=p+n}^{\infty}(k-p)\delta(k-1,q)a_k z^{k-q-1}}{2(p-\alpha)\delta(p-1,q)z^{p-q-1} + {}_{k=p+n}^{\infty} k\delta(k-1,q)a_k z^{k-q-p}}\right| < \beta. \end{aligned}$$

Since $Re(z) \leq |z|$ for all z, we have

$$Re\left[\frac{{}_{k=p+n}^{\infty}(k-p)\delta(k-1,q)a_k z^{k-q-1}}{2(p-\alpha)\delta(p-1,q)z^{p-q-1} + {}_{k=p+n}^{\infty} k\delta(k-1,q)a_k z^{k-q-p}}\right] < \beta. \tag{2.2}$$

Choose values of z on the real axis so that $\frac{f^{(q+1)}(z)}{\delta(p-1,q)z^{p-q-1}}$ is real. Then, upon clearing the denominator in (2.2) and letting $z \to 1^-$ through *real* values, we obtain the desired result. This completes the proof of Theorem 1.

Corollary 1. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p,q;\alpha,\beta)$. Then

$$a_k \leq \frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}(k \geq n+p,\ p,\ n \in \mathbb{N}). \tag{2.3}$$

The result is sharp for the function

$$f(z) = z^p - \frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}z^k(k \geq n+p,\ p,\ n \in \mathbb{N}). \tag{2.4}$$

3. Distortion theorems

Theorem 2. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p,q;\alpha,\beta)$. Then for $|z|=r<1$, we have

$$\left(\delta(p,m)-\frac{2\beta(p-\alpha)\delta(p-1,q)\delta(p+n,m)}{(1+\beta)\delta(p+n,q+1)}r^n\right)r^{p-m}$$
$$\leq \left|f^{(m)}(z)\right| \leq$$
$$\left(\delta(p,m)+\frac{2\beta(p-\alpha)\delta(p-1,q)\delta(p+n,m)}{(1+\beta)\delta(p+n,q+1)}r^n\right)r^{p-m}. \quad (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n,q+1)}z^{p+n}. \quad (3.2)$$

Proof. In view of Theorem 1, we have

$$(p+n)(1+\beta)\delta(p+n-1,q)_{k=p+n}^{\infty}a_k$$
$$\leq_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_k \leq 2\beta(p-\alpha)\delta(p-1,q), \quad (3.3)$$

that is,

$$\sum_{k=p+n}^{\infty} a_k \leq \frac{2\beta(p-\alpha)\delta(p-1,q)}{(p+n)(1+\beta)\delta(p+n-1,q)}. \quad (3.4)$$

From (1.3) and (3.4), we have

$$\begin{aligned}\left|f^{(m)}(z)\right| &\geq \delta(p,m)r^{p-m}-r^{p+n-m}\delta(p+n,m)_{k=p+n}^{\infty}a_k\\ &\geq \delta(p,m)r^{p-m}-r^{p+n-m}\delta(p+n,m)\frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n,q+1)}\\ &= \left(\delta(p,m)-\frac{2\beta(p-\alpha)\delta(p-1,q)\delta(p+n,m)}{(1+\beta)\delta(p+n,q+1)}r^n\right)r^{p-m}\\ &\quad 3.5 \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\left|f^{(m)}(z)\right| &\leq \delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)_{k=p+n}^{\infty}a_k\\ &\leq \delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n,q+1)}\\ &= \left(\delta(p,m)+\frac{2\beta(p-\alpha)\delta(p-1,q)\delta(p+n,m)}{(1+\beta)\delta(p+n,q+1)}r^n\right)r^{p-m}.\\ &\quad 3.6 \qquad (2)\end{aligned}$$

This completes the proof of Theorem 2.

Putting $m = 0$ in Theorem 2, we have the following corollary.
Corollary 2. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p, q; \alpha, \beta)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq \left[1 - \frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n,q+1)} r^n\right] r^p, \tag{3.7}$$

and

$$|f(z)| \leq \left[1 + \frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n,q+1)} r^n\right] r^p. \tag{3.8}$$

The result is sharp.

Putting $m = 1$ in Theorem 2, we have the following corollary.
Corollary 3. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p, q; \alpha, \beta)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq \left[p - \frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n-1,q)} r^n\right] r^{p-1}, \tag{3.9}$$

and

$$|f'(z)| \leq \left[p + \frac{2\beta(p-\alpha)\delta(p-1,q)}{(1+\beta)\delta(p+n-1,q)} r^n\right] r^{p-1}. \tag{3.10}$$

The result is sharp.

Remark 1. Putting $q = 0$ and $n = 1$ in Corollaries 2 and 3, we obtain the result obtained by Aouf [3, Theorem 2]

4. Radii of starlikeness and convexity

Theorem 3. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p, q; \alpha, \beta)$. Then $f(z)$ is p-valent close-to-convex of order η $(0 \leq \eta < p)$ *in* $|z| \leq r_1$, where

$$r_1 = \inf \left\{ \frac{(1+\beta)\delta(k,q+1)(p-\eta)}{2\beta(p-\alpha)\delta(p-1,q)} \right\}^{\frac{1}{k-p}} \quad (k \geq n+p, \; p, \; n \in \mathbb{N}). \tag{4.1}$$

The result is sharp, the extremal function given by (2.4).
Proof. We must show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \eta \; for \; |z| \leq r_1, \tag{4.2}$$

where r_1 is given by (4.1). Indeed we find from (1.2) that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq \sum_{k=n+p}^{\infty} ka_k \left|z\right|^{k-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\eta,$$

if

$$\sum_{k=p+n}^{\infty}\left(\frac{k}{p-\eta}\right) a_k \left|z\right|^{k-p} \leq 1. \tag{4.3}$$

But by using Theorem 1, (4.3) will be true if

$$\left(\frac{k}{p-\eta}\right)\left|z\right|^{k-p} \leq \left(\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right).$$

Then

$$|z| \leq \left\{\frac{(1+\beta)\delta(k-1,q)(p-\eta)}{2\beta(p-\alpha)\delta(p-1,q)}\right\}^{\frac{1}{k-p}}. \tag{4.4}$$

The result follows easily from (4.4).

Theorem 4. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p,q;\alpha,\beta)$. Then $f(z)$ is p-valent starlike of order η $(0 \leq \eta < p)$ *in* $|z| \leq r_2$, where

$$r_2 = \inf_{k \geq n+p}\left\{\frac{k(1+\beta)\delta(k-1,q)(p-\eta)}{2\beta(p-\alpha)\delta(p-1,q)(k-\eta)}\right\}^{\frac{1}{k-p}}. \tag{4.5}$$

The result is sharp, the extremal function given by (2.4).
Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq p-\eta \ for \ |z| \leq r_2, \tag{4.6}$$

where r_2 is given by (4.5). Indeed we find from the definition of (1.2) that

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)a_k\left|z\right|^{k-p}}{1-\sum_{k=n+p}^{\infty}a_k\left|z\right|^{k-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq p-\eta,$$

if

$$\sum_{k=p+n}^{\infty}\left(\frac{k-\eta}{p-\eta}\right) a_k \left|z\right|^{k-p} \leq 1. \tag{4.7}$$

But by using Theorem 1, (4.7) will be true if

$$\left(\frac{k-\eta}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right).$$

Then

$$|z| \leq \left\{\frac{k(1+\beta)\delta(k-1,q)(p-\eta)}{2\beta(p-\alpha)\delta(p-1,q)(k-\eta)}\right\}^{\frac{1}{k-p}} \ (k \geq n+p,\ p,\ n \in \mathbb{N}). \qquad (4.8)$$

The result follows easily from (4.8).

Corollary 4. Let the function $f(z)$ defined by (1.2) be in the class $T_n^*(p,q;\alpha,\beta)$. Then $f(z)$ is in p-valent convex of order η $(0 \leq \eta < p)$ in $|z| \leq r_3$, where

$$r_3 = \inf_{k \geq n+p} \left\{\frac{p(1+\beta)\delta(k-1,q)(p-\eta)}{2\beta(p-\alpha)\delta(p-1,q)(k-\eta)}\right\}^{\frac{1}{k-p}}. \qquad (4.9)$$

The result is sharp, with the extremal function given by (2.4).

5. Closure theorems

Body Math **Theorem 5.** Let $\mu_j \geq 0$ for $j = 1,2,...,m$ and ${}_{j=1}^{m}\mu_j \leq 1$. If the functions

Body Math $f_j(z)$ defined by

Body Math

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \ (a_{k,j} \geq 0;\ j = 1,2,...,m), \qquad (5.1)$$

are in the class $T_n^*(p,q;\alpha,\beta)$ for every $j = 1,2,...,m$, then the function $f(z)$ defined by

$$f(z) = z^p - \sum_{k=p+n}^{\infty}\left(\sum_{j=1}^{m}\mu_j a_{k,j}\right) z^k, \qquad (5.2)$$

is also in the class $T_n^*(p,q;\alpha,\beta)$.

Proof. Since $f_j(z)$ is in the class $T_n^*(p,q;\alpha,\beta)$, then by Theorem 1 that

$${}_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_{k,j} \leq 2\beta(p-\alpha)\delta(p-1,q). \qquad (5.3)$$

for every $j = 1, 2, ..., m$. Hence

$$\begin{aligned} & \sum_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)\left(\sum_{j=1}^{m}\mu_j a_{k,j}\right) \\ = & \sum_{j=1}^{m}\mu_j\left(\sum_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_{k,j}\right) \\ \leq & \sum_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_{k,j}\sum_{j=1}^{m}\mu_j = 2\beta(p-\alpha)\delta(p-1,q). \end{aligned}$$

From Theorem 1, it follows that $f(z) \in T_n^*(p,q;\alpha,\beta)$. This completes the proof of Theorem 5.

Corollary 5. The class $T_n^*(p,q;\alpha,\beta)$ is closed under conveq linear combination.
Proof. Let the function $f_j(z)$ $(j = 1, 2)$ be given by (5.1) be in the class $T_n^*(p,q;\alpha,\beta)$. It is sufficient to show that the function $f(z)$ defined by

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z)$$

is in the class $T_n^*(p,q;\alpha,\beta)$. But , taking $m = 2$, $c_1 = \mu$, $c_2 = 1-\mu$ in Theorem 5, we have the corollary.

Theorem 6. Let $f_{p+n-1}(z) = z^p$ and

$$f_k(z) = z^p - \frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}z^k, k \geq n+p. \tag{5.4}$$

Then $f(z)$ is in the class $T_n^*(p,q;\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=p+n-1}^{\infty} \mu_k f_k(z), \tag{5.5}$$

where $\mu_k \geq 0$ and $\sum_{k=p+n-1}^{\infty}\mu_k = 1$.
Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=p+n-1}^{\infty} \mu_k f_k(z) \\ &= z^p - \sum_{k=p+n}^{\infty} \frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}\mu_k z^k. \end{aligned}$$

5.6 (3)

Then it follows that

$$\begin{aligned}&\sum_{k=p+n}^{\infty}\left(\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right)\left(\frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}z^k\mu_k\right)\\ \leq\ &\sum_{k=p+n}^{\infty}\mu_k=(1-\mu_{p+n-1})\leq 1.\end{aligned}$$

Hence by Theorem 1, we have $f(z)\in T_n^*(p,q;\alpha,\beta)$.

Conversely, assume that the function $f(z)$ defined by (1.2) belongs to the class $T_n^*(p,q;\alpha,\beta)$. Then

$$a_k\leq\frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}z^k.$$

Setting

$$\mu_k=\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}a_k,$$

where

$$\mu_{p+n-1}=1-\sum_{k=p+n}^{\infty}\mu_k\ .$$

We can see that $f(z)$ can be expressed in the form (5.5). This completes the proof of Theorem 6.

Corollary 6. The extreme points of the class $T_n^*(p,q;\alpha,\beta)$ are the functions $f_p(z)=z^p$ and

$$f_k(z)=z^p-\frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}z^k\ (k\geq p+n).$$

6. Modified Hadamard products

Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z^p-\sum_{k=p+n}^{\infty}a_{k,1}a_{k,2}z^k=(f_2*f_1)(z). \tag{6.1}$$

Theorem 7. Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.1) be in the class $T_n^*(p,q;\alpha,\beta)$. Then $(f_1*f_2)(z)\in T_n^*(p,q;\gamma,\beta)$, where

$$\gamma=p-\frac{2\beta(p-\alpha)^2\delta(p-1,q)}{(p+n)(1+\beta)\delta(p+n-1,q)}(n\in\mathbb{N}). \tag{6.2}$$

The result is sharp for the function $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = z^p - \frac{2\beta(p-\alpha)\delta(p-1,q)}{(p+n)(1+\beta)\delta(p+n-1,q)} z^{p+n}. \tag{6.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [7], we need to find the largest γ such that

$$\sum_{k=p+n}^{\infty} \frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\gamma)\delta(p-1,q)} a_{k,1}a_{k,2} \leq 1. \tag{6.4}$$

Since $f_j(z) \in T_n^*(p,q;\alpha,\beta)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=p+n}^{\infty} \frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)} a_{k,1} \leq 1, \tag{6.5}$$

and

$$\sum_{k=p+n}^{\infty} \frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)} a_{k,2} \leq 1. \tag{6.6}$$

By the Cauchy Schwarz inequality we have

$$\sum_{k=p+n}^{\infty} \frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{6.7}$$

Thus it is sufficient to show that

$$\frac{k(1+\beta)}{2\beta(p-\gamma)} a_{k,1}a_{k,2} \leq \frac{k(1+\beta)}{2\beta(p-\alpha)} \sqrt{a_{k,1}a_{k,2}} \ (k \geq p+n), \tag{6.8}$$

or equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\gamma)}{(p-\alpha)} \ (k \geq p+n). \tag{6.9}$$

Hence, in light of the inequality (6.9), it is sufficient to prove that

$$\frac{2\beta(p-\alpha)\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)} \leq \frac{(p-\gamma)}{(p-\alpha)} (k \geq p+n). \tag{6.10}$$

It follows from (6.10) that

$$\gamma \leq p - \frac{2\beta(p-\alpha)^2\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}. \tag{6.11}$$

Now defining the function $G(k)$ by

$$G(k) = p - \frac{2\beta(p-\alpha)^2\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}. \tag{6.12}$$

We see that $G(k)$ is an increasing function of k $(k \geq p+n)$. Therefore, we conclude that

$$\gamma \leq G(p+n) = p - \frac{2\beta(p-\alpha)^2\delta(p-1,q)}{(p+n)(1+\beta)\delta(p+n-1,q)}, \tag{6.13}$$

which evidently completes the proof of Theorem 7.

Putting $\beta = 1$ in Theorem 7, we obtain the following corollary.
Corollary 7. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (1.2) be in the class $T_n^*(p, q; \alpha)$. Then $(f_1 * f_2)(z) \in T_n^*(p, q; \gamma)$, where

$$\gamma = p - \frac{(p-\alpha)^2\delta(p-1,q)}{(p+n)\delta(p+n-1,q)}.$$

The result is sharp.

Corollary 8. For $f_1(z)$ and $f_2(z)$ as in Theorem 7, the function

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}a_{k,2}}\, z^k,$$

belongs to the class $T_n^*(p, q; \alpha, \beta)$.
This result follows from the Cauchy-Schwarz inequality (6.7). It is sharp for the same functions as in Theorem 7.

Theorem 8. Let the functions $f_j\,(z)\,(j = 1, 2)$ defined by (5.1) be in the class $T_n^*(p, q; \alpha, \beta)$. Then the function

$$h\,(z) = z^p - \sum_{k=p+n}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2\right) z^k \ , \tag{6.14}$$

belongs to the class $T_n^*(p, q; \zeta, \beta)$, where

$$\zeta = p - \frac{4\beta(p-\alpha)^2\delta(p-1,q)}{(n+p)(1+\beta)\delta(n+p-1,q)}. \tag{6.15}$$

The result is sharp for the functions $f_j\,(z)$ $(j = 1, 2)$ defined by (6.3).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=p+n}^{\infty}\left[\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right]^2 a_{k,1}^2 \leq \left[\sum_{k=p+n}^{\infty}\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}a_{k,1}\right]^2 \leq 1, \tag{6.16}$$

and

$$\sum_{k=p+n}^{\infty}\left[\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right]^2 a_{k,2}^2 \leq \left[\sum_{k=p+n}^{\infty}\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}a_{k,2}\right]^2 \leq 1. \tag{6.17}$$

It follows from (6.16) and (6.17) that

$$\sum_{k=p+n}^{\infty}\frac{1}{2}\left[\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right]^2 \left(a_{k,1}^2+a_{k,2}^2\right) \leq 1. \tag{6.18}$$

Therefore, we need to find the largest ζ such that

$$\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\zeta)\delta(p-1,q)} \leq \frac{1}{2}\left[\frac{k(1+\beta)\delta(k-1,q)}{2\beta(p-\alpha)\delta(p-1,q)}\right]^2, \tag{6.19}$$

that is, that

$$\zeta \leq p-\frac{4\beta(p-\alpha)^2\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}. \tag{6.20}$$

Since

$$D\left(k\right) = p-\frac{4\beta(p-\alpha)^2\delta(p-1,q)}{k(1+\beta)\delta(k-1,q)}, \tag{6.21}$$

is an increasing function of k $(k \geq p+n)$, we readily have

$$\zeta \leq D(p+n) = p-\frac{4\beta(p-\alpha)^2\delta(p-1,q)}{(n+p)(1+\beta)\delta(n+p-1,q)}, \tag{6.22}$$

and Theorem 8 follows at once.

7. Definitions and Applications of Generalized Fractional Calculus

We recall some definitions which will be used in this section.
Let ${}_2F_1(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$${}_2F_1(a,b;c;z) =_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,$$

where $(z \in \mathbb{U}; a, b, c \in \mathbb{C},\ c \neq 0, -1, -2, ...)$ and

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1) & \text{if } n \in \mathbb{N}. \end{cases}$$

We note that ${}_2F_1(a, b; c; 1)$ converges for $Re(c-a-b) > 0$ and is related to Gamma functions by

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Definition 1.1 ([3], [6] and [8]). Assume that $0 \leq \lambda < 1$ and $\mu,\ \eta \in \mathbb{R}$. Then, in terms of the Gaussian hypergeometric function ${}_2F_1$, the generalized fractional derivative operator for a function $f(z) \in \mathcal{A}(p)$, is defined by

$$J_{0,z}^{\lambda,\mu,\eta,p} f(z) = \frac{d}{dz}\left[\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-\zeta)^{-\lambda} f(\zeta)_2F_1(\mu+\lambda, 1-\eta; 1-\lambda; 1-\frac{\zeta}{z})d\zeta\right]$$

$$(0 \leq \lambda < 1), \tag{7.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin with the order $f(z) = O(|z|^{\varepsilon}),\ z \to 0$ when $\varepsilon > \max\{0, \mu-\eta\} - 1$ and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 1.2 [3] (see also [6], [8], [10]). Assume that $\lambda > 0$ and $\mu, \eta \in \mathbb{R}$. Then, in terms of the Gaussian hypergeometric function ${}_2F_1$, the generalized fractional integral operator for a function $f(z) \in \mathcal{A}(p)$, is defined by

$$I_{0,z}^{\lambda,\mu,\eta,p} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^z (z-\zeta)^{\lambda-1} f(\zeta)_2F_1(\mu+\lambda, -\eta; \lambda; 1-\frac{\zeta}{z})d\zeta, \tag{7.2}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin with the order $f(z) = O(|z|^{\varepsilon})\ z \to 0$ when $\varepsilon > \max\{0, \mu-\eta\} - 1$ and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z - \zeta > 0$.

We note that,

$$\begin{aligned} I_{0,z}^{\lambda,-\lambda,\eta,p} f(z) &= D_z^{-\lambda} f(z)\ (\lambda > 0), \\ J_{0,z}^{\lambda,\lambda,\eta,p} f(z) &= D_z^{\lambda} f(z)\ (0 \leq \lambda > 1. \end{aligned}$$

where $D_z^{\lambda}(\lambda \in R)$ is the fractional operator considered by Owa [5] and (subsequently) by Srivastava and Owa [9]. Furthermore, in terms of Gamma functions, Definitions 1and 2 readily yield

Lemma 1. (Srivastava et al .[10]). The generalized fractional integral and the generalized fractional derivative of a power function are given by

$$I_{0,z}^{\lambda,\mu,\eta,p} z^p = \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p+\lambda+\mu+1)} z^{p-\mu}$$
$$(\lambda > 0;\ p > \max\{0,\mu-\eta\}-1),$$

and

$$J_{0,z}^{\lambda,\mu,\eta,p} z^p = \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)} z^{p-\mu}$$
$$(0 \le \lambda < 1;\ p > \max\{0,\mu-\eta\}-1).$$

With the aid of this Lemma, we prove
Theorem 9. Let the function $f(z)$ defined by (1.1) be in the class $T_n^*(p,q;\alpha,\beta)$. Then we have

$$\left|J_{0,z}^{\lambda,\mu,\eta,p} f(z)\right| \ge \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)} |z|^{p-\mu} .7.3$$
$$.\left\{1-\frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p-\lambda+\eta+1)_n}|z|\right\}, \quad (4)$$

and

$$\left|J_{0,z}^{\lambda,\mu,\eta,p} f(z)\right| \le \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)} |z|^{p-\mu} .$$
$$.\left\{1+\frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p-\lambda+\eta+1)_n}|z|\right\},$$
$$7.4 \quad (5)$$

for $z \in \mathbb{U}_0;\ 0 \le \lambda < 1;\ \max\{\mu,\mu-\eta,\lambda-\eta\} < 2;\ \gamma(\lambda-\eta) \ge 3\lambda.$

$$\mathbb{U}_0 = \begin{cases} \mathbb{U} & (\mu \le 1), \\ \mathbb{U}\backslash\{0\} & (\mu > 1). \end{cases}$$

The result is sharp.
Proof. Let

$$F(z) = \frac{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)}{\Gamma(p-\mu+\eta+1)\Gamma(p+1)} z^\mu J_{0,z}^{\lambda,\mu,\eta,p} f(z)$$
$$= z^p - \sum_{k=p+n}^{\infty} \frac{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(p-\mu+\eta+1)\Gamma(p+1)\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} a_k z^k.$$
$$7.5 \quad (6)$$

Then

$$F(z) = z^p - \sum_{k=p+n}^{\infty} \Psi(k) a_k z^k, \tag{7.6}$$

where

$$\Psi(k) = \frac{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(p-\mu+\eta+1)\Gamma(p+1)\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} \ (\mu > 0).$$

Since $\Psi(k)$ is an decreasing function of k ($k \in \mathbb{N}$), then

$$0 < \Psi(k) \leq \Psi(p+n) = \frac{(p+1)_n(p-\mu+\eta+1)_n}{(p-\mu+1)_n(p-\lambda+\eta+1)_n}. \tag{7.7}$$

Also, according to Theorem 1, we have

$(p+n)(1+\beta)\delta(p+n-1,q)_{k=p+n}^{\infty} a_k \leq_{k=p+n}^{\infty} k(1+\beta)\delta(k-1,q)a_k \leq 2\beta(p-\alpha)\delta(p-1,q),$

Then

$$\sum_{k=p+n}^{\infty} a_k \leq \frac{2\beta(p-\alpha)\delta(p-1,q)}{(p+n)(1+\beta)\delta(p+n-1,q)}. \tag{7.8}$$

From (7.6) and (7.7), we have

$$|F(z)| \geq |z|^p - \Psi(p+n)\,|z|^{p+n} \sum_{k=p+n}^{\infty} a_k. \tag{7.9}$$

In view of (7.8) and (7.9), we have

$$\begin{aligned} |F(z)| &= \left| \frac{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)}{\Gamma(p-\mu+\eta+1)\Gamma(p+1)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta,p} f(z) \right| \\ &\geq |z|^p - \frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p-\lambda+\eta+1)_n} |z|^{p+n}, \end{aligned}$$

and

$$\begin{aligned} |F(z)| &= \left| \frac{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)}{\Gamma(p-\mu+\eta+1)\Gamma(p+1)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta,p} f(z) \right| \\ &\leq |z|^p + \frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p-\lambda+\eta+1)_n} |z|^{p+1}. \end{aligned}$$

which proves the inequalities of Theorem 9. Further equalities are attained for the function

$$\begin{aligned} J_{0,z}^{\lambda,\mu,\eta,p} f(z) &= \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p-\lambda+\mu+1)} z^{p-\mu}.7.10 \\ &\cdot \left\{ 1 - \frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p-\lambda+\eta+1)_n} z \right\}, \end{aligned} \tag{7}$$

or the function $f(z)$ given by(3.2).

Using arguments similiar to those in the proof of Theorem 9, we obtain the following theorem.
Theorem 10. Let the function $f(z)$ defined by (1.1) be in the class $T_n^*(p,q;\alpha,\beta)$. Then we have

$$\left|I_{0,z}^{\lambda,\mu,\eta,p}f(z)\right| \geq \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p+\lambda+\mu+1)}\,|z|^{p-\mu}\,.7.11 \quad (8)$$
$$.\left\{1-\frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p+\lambda+\eta+1)_n}\,|z|\right\},$$

and

$$\left|I_{0,z}^{\lambda,\mu,\eta,p}f(z)\right| \leq \frac{\Gamma(p-\mu+\eta+1)\Gamma(p+1)}{\Gamma(p-\mu+1)\Gamma(p+\lambda+\mu+1)}\,|z|^{p-\mu}\,.7.12 \quad (9)$$
$$.\left\{1+\frac{2\beta(p-\alpha)\delta(p-1,q)(p+1)_n(p-\mu+\eta+1)_n}{(p+n)(1+\beta)\delta(p+n-1,q)(p-\mu+1)_n(p+\lambda+\eta+1)_n}\,|z|\right\}$$

ffor $z \in \mathbb{U}_{0;}$ $\lambda > 0$; $\max\{\mu, \mu-\eta, -\lambda-\eta\} < 2$; $\gamma(\lambda+\eta) \leq 3\lambda$.

$$\mathbb{U}_0 = \begin{cases} \mathbb{U} & (\mu \leq 1), \\ \mathbb{U}\backslash\{0\} & (\mu > 1). \end{cases}$$

The result is sharp for the function $f(z)$ given by (3.2).

Remark 2.
(1) Putting $\mu = \lambda = \eta = 0$ in Theorem 10, we obtain the result of Corollary 2;
(2) Putting $q = 0$ and $n = 1$ in our results, we obtain the results obtained by Aouf [1].

8. Open Problem

The authors suggest to study the properties of the same class $T_n^*(p,q;\alpha,\beta)$ by replacing f by $(f*g)$.

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