New results on the $D_{L^p}$-type spaces associated with a singular second order differential operator

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Abstract

In this work, we consider a second order differential operator $\Delta_A$ defined on $(0, +\infty)$, where $A$ is a non negative function satisfying some conditions. To $\Delta_A$ we associate $D_{L^p}$-type spaces denoted by $D^p_A$. Some results, related to the spaces $D^p_A$, are proved. Moreover a generalization of Titchmarsh’s theorem for the Chébli-Trimèche transform in $D^2_A$ is established.

Keywords: second order differential operator $\Delta_A$, $D^p_A$ spaces, Titchmarsh’s theorem

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1 Introduction

L. Schwartz has introduced in [18] the space $D_{L^p}$, $1 \leq p \leq \infty$, of all $C^\infty$-functions $\psi$ on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, $D^n\psi$ is in $L^p(\mathbb{R})$ and the map $\psi \mapsto D^n\psi$ from $D_{L^p}$ into $L^p(\mathbb{R})$ is continuous. These spaces are studied by many authors (see [1], [2], [5], [17]) among others.

In [12] the authors define new function spaces similar to $D_{L^p}$ but replacing the usual derivative $D$ by the generalized Laplace operator $\Delta_A$ defined on $(0, \infty)$ by

$$\Delta_A = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx} + \rho^2, \quad \rho \geq 0,$$
where $A$ is the Chébli-Trimèche function (cf. [6], Section 3.5) defined on $[0, \infty)$ and satisfies the following conditions:

i) There exists a positive even infinitely differentiable function $B$ on $\mathbb{R}$, with $B(0) = 1$, $x \in \mathbb{R}_+$, such that $A(x) = x^{2\alpha+1}B(x)$, $\alpha > -\frac{1}{2}$.

ii) $A$ is increasing on $\mathbb{R}_+$ and $\lim_{x \to \infty} A(x) = \infty$.

iii) $\frac{A'}{A}$ is decreasing on $(0, \infty)$, and $\lim_{x \to \infty} \frac{A'(x)}{A(x)} = 2\rho$.

iv) There exists a constant $\sigma > 0$, such that for all $x \in [x_0, \infty)$, $x_0 > 0$, we have

$$
\frac{A'(x)}{A(x)} = \begin{cases}
2\rho + e^{-\sigma x}F(x), & \text{if } \rho > 0 \\
\frac{2\alpha + 1}{x} + e^{-\sigma x}F(x), & \text{if } \rho = 0
\end{cases}
$$

where $F$ is $C^\infty$ on $(0, \infty)$, bounded together with its derivatives.

For $A(x) = x^{2\alpha+1}$, $\alpha > -\frac{1}{2}$ and $\rho = 0$ we regain the Bessel operator

$$
l_\alpha f = \frac{d^2 f}{dx^2} + \left(\frac{2\alpha + 1}{x}\right) \frac{df}{dx}.
$$

For $A(x) = \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x)$, $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$ and $\rho = \alpha + \beta + 1$ we regain the Jacobi operator

$$
l_{\alpha,\beta} f = \frac{d^2 f}{dx^2} + \left[(2\alpha + 1) \coth x + (2\beta + 1) \tanh x\right] \frac{f(x)}{x} + \rho^2.
$$

In this paper, these spaces denoted by $D^p_A$, $1 \leq p \leq \infty$, are moreover considered as subspaces of $E_\ast(\mathbb{R})$ (the space of even $C^\infty$-functions on $\mathbb{R}$). Some properties, related to the spaces $D^p_A$, are given.

The contents of the paper is as follows:

In §2 we recall some basic facts about the harmonic analysis results related to the operator $\Delta_A$, in §3 we introduce the space $D^p_A$ and we show some results. In particular, we study the continuity of the Chébli-Trimèche transform on $D^p_A$, $1 \leq p \leq 2$. In §4 a generalization of Titchmarsh’s theorem for the Chébli-Trimèche transform $F$ for functions satisfying the Chébli-Trimèche-Lipschitz condition in $D^2_A$ is established.

2 Preliminaries

In this section, we collect some harmonic analysis results related to the operator $\Delta_A$. For details we refer the reader to [6], [8], [12], [14], [21], and [22].
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2.1 Eigenfunctions of the operator $\triangle_A$

In the following we denote by $C^0_\ast(\mathbb{R})$ the space of even continuous functions $f$ on $\mathbb{R}$ such that

$$\lim_{|x| \to +\infty} |f(x)| = 0.$$ 

$S_\ast(\mathbb{R})$ the subspace of $E_\ast(\mathbb{R})$, consisting of functions $f$ rapidly decreasing together with their derivatives.

$S^2_\ast(\mathbb{R}) = \varphi_0 S_\ast(\mathbb{R})$, where $\varphi_0$ is the eigenfunction of the operator $\triangle_A$ associated with the value $\lambda = 0$.

$S'_\ast(\mathbb{R})$ the dual topological space of $S_\ast(\mathbb{R})$.

$(S^2_\ast)'(\mathbb{R})$ the dual topological space of $S^2_\ast(\mathbb{R})$.

$E_\ast(\mathbb{R}+, \mathbb{R})$ the dual topological space of $E_\ast(\mathbb{R})$.

$H_{\ast}(\mathbb{C})$ the space of even entire functions on $\mathbb{C}$ which are of exponential type and slowly increasing.

$H_{\ast,a}(\mathbb{C})$ the subspace of $H_{\ast}(\mathbb{C})$ satisfying

$$\exists m \in \mathbb{N}, P_m(f) = \sup_{\lambda \in \mathbb{C}} (1 + \lambda^2)^{-m} |f(\lambda)| \exp(-a|\text{Im}\lambda|) < +\infty$$

we have $H_{\ast}(\mathbb{C}) = \cup_{a \geq 0} H_{\ast,a}(\mathbb{C})$.

For every $\lambda \in \mathbb{C}$, let us denote by $\varphi_\lambda$ the unique solution of the eigenvalue problem

$$\begin{cases} 
\triangle_A f(x) = -\lambda^2 f(x), \\
f(0) = 1, \quad f'(0) = 0.
\end{cases} \tag{1}$$

**Remark 1** This function satisfies the following properties.

- $\forall x \geq 0$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$.

- Product formula

$$\forall x, y \geq 0; \quad \varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)w(x, y, z)A(z)dz \tag{2}$$

where $w(x, y, \cdot)$ is a measurable positive function on $[0, \infty)$, with support in $[|x - y|, x + y]$.

- $\forall \lambda \geq 0$ and $x \in \mathbb{R}$, $|\varphi_\lambda(x)| \leq 1$. \tag{3}

- For $\rho > 0$, we have

$$\forall x \geq 0, \forall \lambda \in \mathbb{R}, \quad |\varphi_\lambda(x)| \leq \varphi_0(x) \leq m(1 + x)\exp(-\rho x), \tag{4}$$

where $m$ is a positive constant.
• For \( \rho = 0 \), we have
\[
\forall x \geq 0, \quad \varphi_0(x) = 1,
\]

• \( \forall x \geq 0, \forall \lambda \in \mathbb{R} \),
\[
|\varphi'_\lambda(x)| \leq c(\lambda^2 + \rho^2)(1 + x)\exp(-\rho x),
\]
where \( c \) is a positive constant.

• We have the following integral representation of Mehler type,
\[
\forall x > 0, \forall \lambda \in \mathbb{C}, \quad \varphi_\lambda(x) = \int_0^x k(x, t) \cos(\lambda t) dt
\]
where, \( k(x, \cdot) \) is an even positive \( C^\infty \) function on \( ]-x, x[ \) with support in \( ]-x, x[ \).

2.2 Generalized Fourier transform

For a Borel positive measure \( \mu \) on \( \mathbb{R} \), and \( 1 \leq p \leq \infty \), we write \( L_p^\mu(\mathbb{R}^+) \) for the Lebesgue space equipped with the norm \( \| f \|_{L_p^\mu(\mathbb{R}^+)} \) defined by
\[
\| f \|_{L_p^\mu(\mathbb{R}^+)} = \left( \int_{\mathbb{R}} |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad \text{if } p < \infty,
\]
and \( \| f \|_{L_\infty^\mu(\mathbb{R}^+)} = \text{ess sup}_{x \in \mathbb{R}^+} |f(x)| \). When \( \mu(x) = w(x) \, dx \), with \( w \) a nonnegative function on \( \mathbb{R}^+ \), we replace the \( \mu \) in the norms by \( w \).

For \( f \in L_1^A(\mathbb{R}^+) \), the generalized Fourier transform, called also Chébli-Triméche transform, is defined by
\[
\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}^+} f(x) \varphi_\lambda(x) A(x) \, dx, \quad \forall \lambda \in \mathbb{R}.
\]

The inverse generalized Fourier transform of a suitable function \( g \) on \( \mathbb{R}^+ \) is given by:
\[
\mathcal{J} g(x) = \mathcal{F}^{-1} g(x) = \int_{\mathbb{R}^+} g(\lambda) \varphi_\lambda(x) d\gamma(\lambda)
\]
where \( d\gamma(\lambda) \) is the spectral measure given by
\[
d\gamma(\lambda) = \frac{d\lambda}{|c_A(\lambda)|^2}.
\]

**Remark 2** The function \( \lambda \mapsto c_A(\lambda) \) satisfies the following properties.

• For \( \lambda \in \mathbb{R} \), we have \( c_A(-\lambda) = \overline{c_A(\lambda)} \).
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- The function $|c_A(\lambda)|^{-2}$ is continuous on $[0, \infty]$.  
- There exist positive constants $k_1$, $k_2$, and $k_3$, such that  
  If $\rho \geq 0$: \forall \lambda \in \mathbb{C}, \Im \lambda \leq 0, |\lambda| > k_3;  
  \[ k_1 |\lambda|^{2\alpha+1} \leq |c_A(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}. \]
  If $\rho = 0$, $\alpha > 0$; \forall \lambda \in \mathbb{C}, |\lambda| \leq k_3;  
  \[ k_1 |\lambda|^{2\alpha+1} \leq |c_A(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}. \]
  If $\rho > 0$; \forall \lambda \in \mathbb{C}, |\lambda| \leq k_3;  
  \[ k_1 |\lambda|^2 \leq |c_A(\lambda)|^{-2} \leq k_2 |\lambda|^2. \]

**Proposition 1** ([10]). i) The generalized transform $F$ and its inverse $J$ are topological isomorphisms between the generalized Schwartz space $S^2_\ast(\mathbb{R})$ and the Schwartz space $S(\mathbb{R}^\ast)$.  

ii) The transform $F$ is a topological isomorphism from $E^\ast(\mathbb{R}^+)\ast$ onto $H_\ast(\mathbb{C})$. Moreover, for all $T \in E^\ast(\mathbb{R}^+)$, we have: $\text{supp}(T) \subseteq [-a,a]$ if and only if $F(T) \in H_\ast,a(\mathbb{C})$.  

Next, we give some properties of this transform.  

i) For $f$ in $L^1_A(\mathbb{R}^+)$ we have  
  \[ ||F(f)||_{L^\infty(\mathbb{R}^+)} \leq ||f||_{L^1_A(\mathbb{R}^+)} \]. \tag{10}  

ii) For $f$ in $S^2_\ast(\mathbb{R})$ we have  
  \[ F(\triangle_A f)(y) = -y^2 F(f)(y), \quad \text{for all } y \in \mathbb{R}^+. \] \tag{11}  

**Proposition 2** ([10]). Plancherel formula for $F$. For all $f$ in $S^2_\ast(\mathbb{R})$ we have  
  \[ \int_{\mathbb{R}^+} |f(x)|^2 A(x) \, dx = \int_{\mathbb{R}^+} |F(f)(\xi)|^2 d\gamma(\xi). \] \tag{12}  

ii) Plancherel theorem.  
  The transform $F$ extends uniquely to an isomorphism from $L^2_A(\mathbb{R}^+)$ onto $L^2_\gamma(\mathbb{R}^+)$.  

iii) for all $f, g \in L^2_A(\mathbb{R}^+)$, we have  
  \[ \int_{\mathbb{R}^+} f(x)g(x)A(x) \, dx = \int_{\mathbb{R}^+} F(f)(\xi)\overline{F(g)(\xi)}d\gamma(\xi). \] \tag{13}  

**Remark 3** We have $S^2_\ast(\mathbb{R}) \subset L^p_A(\mathbb{R}^+)$ for all $2 \leq p \leq \infty$, but $S^2_\ast(\mathbb{R}) \hookrightarrow L^p_A(\mathbb{R}^+)$ for all $0 < p < 2$.  


Proposition 3 Let \( 1 \leq p \leq 2 \). The Fourier transform \( F \) (resp. \( J \)) can be extended as a continuous mapping from \( L^p_A(\mathbb{R}+) \) onto \( L^{p'}_\gamma(\mathbb{R}+) \) (resp. from \( L^p_\gamma(\mathbb{R}+) \) onto \( L^{p'}_A(\mathbb{R}+) \)) and we have

\[
\|Ff\|_{L^{p'}_\gamma(\mathbb{R}+)} \leq \|f\|_{L^p_A(\mathbb{R}+)}; \quad \|Jg\|_{L^{p'}_A(\mathbb{R}+)} \leq \|g\|_{L^p_\gamma(\mathbb{R}+)}
\]  

(14)

with \( \frac{1}{p'} + \frac{1}{p} = 1 \).

2.3 Generalized convolution

Definition 1 ([9]). The translation operator associated with the operator \( \triangle_A \) is defined on \( L^1_A(\mathbb{R}+) \), by

\[
\forall x, y \geq 0; \quad \tau^A_x f(y) = \int_0^\infty f(z) w(x, y, z) A(z) dz
\]  

(15)

where \( w \) is the function defined in the relation (2).

Proposition 4 ([9]). For a suitable function \( f \) on \( \mathbb{R}+ \), we have

i) \( \tau^A_x f(y) = \tau^A_y f(x) \).

ii) \( \tau^A_0 f(y) = f(y) \).

iii) \( \tau^A_x \tau^A_y = \tau^A_y \tau^A_x \).

iv) \( \tau^A_x \varphi_A(y) = \varphi_A(x) \varphi_A(y) \).

v) \( F(\tau^A_x f)(\lambda) = \varphi_A(x) F(f)(\lambda) \).

vi) \( \triangle_A(\tau^A_x f) = \tau^A_x(\triangle_A f) \).

vii) \( \forall x \geq 0; \quad \|\tau^A_x f\|_{L^p_A(\mathbb{R}+)} \leq \|f\|_{L^p_A(\mathbb{R}+)} \), \( p \in [1, \infty] \).

Definition 2 ([9]). For suitable functions \( f \) and \( g \), we define the convolution product \( f \ast_A g \) by

\[
f \ast_A g(x) = \int_{\mathbb{R}+} \tau^A_x f(y) g(y) A(y) dy.
\]  

(16)

Remark 4 It is clear that this convolution product is both commutative and associative:

i) \( f \ast_A g = g \ast_A f \).

ii) \( (f \ast_A g) \ast_A h = f \ast_A (g \ast_A h) \).

Proposition 5 ([9]).

i) Assume that \( 1 \leq p, q, r \leq \infty \) satisfy \( \frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r} \). Then, for every \( f \in L^p_A(\mathbb{R}+) \) and \( g \in L^q_\gamma(\mathbb{R}+) \), we have \( f \ast_A g \in L^r_A(\mathbb{R}+) \), and

\[
\|f \ast_A g\|_{L^r_A(\mathbb{R}+)} \leq C \|f\|_{L^p_A(\mathbb{R}+)} \|g\|_{L^q_\gamma(\mathbb{R}+)}.
\]  

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ii) If $\rho > 0$ and $1 \leq p < q \leq 2$. Then

$$L^p_A(\mathbb{R}_+) \ast_A L^q_A(\mathbb{R}_+) \hookrightarrow L^q_A(\mathbb{R}_+). \quad (18)$$

iii) If $\rho > 0$ and $2 < p, q < \infty$ such that $\frac{q}{2} \leq p < q$. Then

$$L^p_A(\mathbb{R}_+) \ast_A L^q_A(\mathbb{R}_+) \hookrightarrow L^q_A(\mathbb{R}_+) \quad (19)$$

where $q'$ is the conjugate exponent of $q$.

iv) If $\rho > 0$ and $1 < p < 2$ such that $p < q \leq \frac{2p}{p-2}$. Then

$$L^p_A(\mathbb{R}_+) \ast_A L^q_A(\mathbb{R}_+) \hookrightarrow L^q_A(\mathbb{R}_+). \quad (20)$$

v) $L^1_A(\mathbb{R}_+) \ast_A C^0_A(\mathbb{R}) \hookrightarrow C^0_A(\mathbb{R}). \quad (21)$

Proposition 6 If $\rho > 0$, then for $f \in L^2_A(\mathbb{R}_+)$ and $g \in L^p_A(\mathbb{R}_+)$, with $1 \leq p < 2$ we have

$$\mathcal{F}(f \ast_A g) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda). \quad (22)$$

Proposition 7 ([21]) Let $f, g \in L^2_A(\mathbb{R}_+)$. Then $f \ast_A g \in L^2_A(\mathbb{R}_+)$ if and only if $\mathcal{F}(f)\mathcal{F}(g)$ belongs to $L^2_A(\mathbb{R}_+)$, and in this case we have

$$\mathcal{F}(f \ast_A g) = \mathcal{F}(f)\mathcal{F}(g) \quad (22)$$

Proposition 8 ([21]) Let $f$ be locally integrable function on $[0, +\infty)$, and $g$ a measurable function on $[0, +\infty)$ satisfying the condition:

$$\exists r \in \mathbb{N} \text{ such that } (1 + \lambda^2)^{-r} g \in L^1_A(\mathbb{R}_+). \quad (23)$$

We suppose that for all $\psi \in D_*(\mathbb{R})$,

$$\int_0^\infty f(x)\psi(x)A(x)dx = \int_0^\infty g(\lambda)\mathcal{F}(\psi)(\lambda)d\gamma(\lambda).$$

Then the function $f$ belongs to $L^2_A(\mathbb{R}_+)$ if and only if the function $g$ belongs to $L^2_\gamma(\mathbb{R}_+)$ and we have

$$\mathcal{F}(f) = g.$$  

Definition 3 The generalized Fourier transform of a distribution $\tau$ in $(\mathcal{S}_2^2)'(\mathbb{R})$ is defined by

$$\langle \mathcal{F}(\tau), \phi \rangle = \langle \tau, \mathcal{F}^{-1}(\phi) \rangle, \quad \text{for all } \phi \in \mathcal{S}_*(\mathbb{R}). \quad (24)$$

Proposition 9 The generalized Fourier transform $\mathcal{F}$ is a topological isomorphism from $(\mathcal{S}_2^2)'(\mathbb{R})$ onto $\mathcal{S}_*(\mathbb{R})$.  

Let \( \tau \) be in \( (S^2)_a^*(\mathbb{R}_+) \). We define the distribution \( \Delta A\tau \), by
\[
\langle \Delta A\tau, \psi \rangle = \langle \tau, \Delta A\psi \rangle, \text{ for all } \psi \in S^2_*(\mathbb{R}_+).
\]
This distribution satisfy the following property
\[
\mathcal{F}(\Delta A\tau) = -y^2\mathcal{F}(\tau). \tag{25}
\]

**Remark 5** (see [20])

i) The generalized convolution product of a distribution \( S \) in \( D'_*(\mathbb{R}) \) and a function \( \psi \) in \( D_*(\mathbb{R}) \) is the function \( S *_A \psi \) defined by
\[
\forall x \in \mathbb{R}_+, \quad S *_A \psi(x) = \langle S_y, \tau^A_x \psi(y) \rangle \tag{26}
\]

ii) Let \( U \) be a distribution in \( D'_*(\mathbb{R}) \) and \( S \) a distribution in \( E'_*(\mathbb{R}) \). The generalized convolution product of \( U \) and \( S \) is the distribution in \( D'_*(\mathbb{R}) \) defined for all \( \psi \) in \( D_*(\mathbb{R}) \) by
\[
\langle U *_A S, \psi \rangle = \langle U_x, S *_A \psi(x) \rangle = \langle S_y, U *_A \psi(y) \rangle \tag{27}
\]

iii) Let \( k \in \mathbb{N}^* \). Then, for all \( U \) in \( D'_*(\mathbb{R}) \) and \( S \) in \( E'_*(\mathbb{R}) \), we have
\[
\Delta^k A(U *_A S) = U *_A \Delta^k A(S) = (\Delta^k A U) *_A S \tag{28}
\]

iv) let \( U \) and \( S \) be two distributions in \( E'_*(\mathbb{R}) \). Then the function \( U *_A S \) belongs to \( E'_*(\mathbb{R}) \) and we have
\[
\mathcal{F}(U *_A S) = \mathcal{F}(U)\mathcal{F}(S). \tag{29}
\]

3 The space \( D^p_A \)

Now, we start with the definition of the spaces of \( D^p_A \).

**Definition 4** If \( 1 \leq p < \infty \), the space \( D^p_A \) is the set of all of \( C^\infty \) and even functions \( f \) on \( \mathbb{R} \) such that, for all \( k \in \mathbb{N} \), \( \Delta^k A \phi \) is in \( L^p_A(\mathbb{R}_+) \) which is equipped with the topology generated by the countable norms
\[
\gamma^A_{m,p}(f) = \max_{0 \leq k \leq m} \| \Delta^k A f \|_{L^p_A(\mathbb{R}_+)}, \quad m \in \mathbb{N}.
\]

A function \( f \in E_*(\mathbb{R}) \) is in \( B^\infty_A \) when \( \gamma^A_{m,\infty}(f) < \infty \) for each \( m \in \mathbb{N} \), where
\[
\gamma^A_{m,\infty}(f) = \max_{0 \leq k \leq m} \| \Delta^k A f \|_{L^\infty_A(\mathbb{R}_+)}, \quad m \in \mathbb{N}.
\]

We denote by \( D^\infty_A \) the subspace of \( B^\infty_A \) that consists of all those functions \( f \in B^\infty_A \) for which \( \lim_{|x| \to \infty} \Delta^m A f(x) = 0 \) for each \( m \in \mathbb{N} \).
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The space $B^\infty_A$ is endowed with the topology generated by the system $\{\gamma_{m,\infty}^A\}_{m \in \mathbb{N}}$.

**Remark 6** i) Let $1 \leq p < \infty$. A function $\varphi \in L^p_A(\mathbb{R}_+)$ is in $\mathcal{D}^p_A$ if and only if $(I - \Delta_A)^m \varphi \in L^p_A(\mathbb{R}_+)$ for every $m \in \mathbb{N}$.

ii) A function $\phi \in L^\infty_A(\mathbb{R}_+)$ is in $B^\infty_A$ if and only if $(I - \Delta_A)^m \phi \in L^\infty_A(\mathbb{R}_+)$ for every $m \in \mathbb{N}$.

iii) For $0 < p \leq 2$, we define the generalized Schwartz space $S^p_A(\mathbb{R})$ by

$$S^p_A(\mathbb{R}) = \left\{ f \in \mathcal{E}_*(\mathbb{R}) / \forall k, l \in \mathbb{N}, \sup_{x \geq 0} (1 + x)^l \varphi_0^{-2/p}(x)|f^k(x)| < \infty \right\}.$$  

Then, for $q \in [\max\{1, p\}, +\infty[$, $S^p_A(\mathbb{R}) \subset \mathcal{D}^q_A$. In particular, when $p = 0$ or $0 < p \leq 1$, for all $q \in [1, +\infty[$, $S^p_A(\mathbb{R}) \subset \mathcal{D}^q_A$.

**Proposition 10** ([12]) For every $p \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for any $m \in \mathbb{N}$, $m \geq m_0$, we can find two functions $\chi_m \in D_{*,\varepsilon}(\mathbb{R})$ (the subspace of $\mathcal{D}_*(\mathbb{R})$ consisting of function $f$ such that $\text{supp}f \subset [-\varepsilon, \varepsilon]$) and $\Gamma_m \in \mathcal{W}^p_A(\mathbb{R})$ (the space of function $f : \mathbb{R} \to \mathbb{C}$ of class $C^{2p}$ on $\mathbb{R}$, even and with support in $[-\varepsilon, \varepsilon]$) such that

$$\delta = (I - \Delta_A)^m \Gamma_m + \chi_m.$$  

We start with some topological properties of the spaces $\mathcal{D}^p_A$.

**Proposition 11** i) $\mathcal{D}^p_A$, $1 \leq p \leq \infty$ and $B^\infty_A$ are Fréchet spaces.

ii) $\mathcal{D}^p_A$ is continuously contained in $\mathcal{D}^q_A$, when $1 \leq p \leq q \leq \infty$.

iii) If $1 < p < \infty$ then $\mathcal{D}^p_A$ is a reflexive space.

**Proof.** In Proposition 2.1 [12], the result is proved for the spaces $\mathcal{D}^p_A$, $1 \leq p < \infty$ and $B^\infty_A$. Let $v_0$ be a Cauchy sequence in $\mathcal{D}^\infty_A$. Since $C^0_A(\mathbb{R})$ is a Banach space, then there exists $v_m \in C^0_A(\mathbb{R})$ such that $\Delta^m_A v_n \to v_m$, as $n \to \infty$, in $C^0_A(\mathbb{R})$, for each $m \in \mathbb{N}$. On the other hand by a simple calculation we see that, $\Delta^m_A v_0 = v_m$, $m \in \mathbb{N}$. Which implies that $(u_n)_{n \in \mathbb{N}}$ converge to $v_0$ in $\mathcal{D}^\infty_A$. Thus the proof of i) is finished.

ii) Let $\varphi \in \mathcal{D}^p_A$. Then, using Proposition 10, for $a > 0$ and $n \in \mathbb{N}$, there exist two functions $\chi \in D_A(\mathbb{R})$ and $\Gamma \in \mathcal{W}^p_A(\mathbb{R})$ such that

$$\Delta^k_A \varphi = \delta \ast_A \Delta^k_A \varphi = \Gamma \ast_A (I - \Delta_A)^n \Delta^k_A \varphi + \chi \ast_A \Delta^k_A \varphi, \ k \in \mathbb{N} \tag{30}$$

Therefore, from proposition 5 i),v) we deduce the result. Moreover, for $1 \leq p \leq q \leq \infty$, there exists $c > 0$ such that

$$\forall m \in \mathbb{N}, \exists m_1 \in \mathbb{N} \text{ satisfying } \gamma^A_{m,q}(\varphi) \leq c \gamma^A_{m_1,\varphi}(\varphi). \tag{31}$$
To see iii) it is sufficient to argue like in [18].

It is well known (see [21]) that for all \( f \in L^p_A(\mathbb{R}_+) \), \( p \in [1, \infty) \),

\[
\lim_{x \to 0} \| \tau_x A f - f \|_{L^p_A(\mathbb{R}_+)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \| f * A v_\varepsilon - f \|_{L^p_A(\mathbb{R}_+)} = 0.
\] (32)

where

\[
v_\varepsilon = (\varepsilon A(x))^{-1} A(\frac{x}{\varepsilon}) v(\frac{x}{\varepsilon})
\] (33)

with \( v \) is a positive function in \( L^1_A(\mathbb{R}_+) \) such that \( \| v \|_{L^1_A(\mathbb{R}_+)} = 1 \).

The case \( p = \infty \) is given by the following Lemma which we need in the sequel to study the density of the space \( D^*_A(\mathbb{R}) \) in \( D^p_A, \ p \in [1, \infty] \).

**Lemma 1** Let \( f \in L^\infty_A(\mathbb{R}_+) \) such that there exists a continuous function \( g \) in \( C_0^0(\mathbb{R}) \) satisfying \( f = g \) a.e. Then

i) \( \lim_{x \to 0} \| \tau_x A f - f \|_{L^\infty_A(\mathbb{R}_+)} = 0. \)

ii) \( \lim_{\varepsilon \to 0} \| f * A v_\varepsilon - f \|_{L^\infty_A(\mathbb{R}_+)} = 0. \)

where \( v_\varepsilon \) is given by (33).

**Proof.** i) Suppose that \( f \in D_A(\mathbb{R}) \), then from inversion formula (8) and Proposition 4 v), we deduce that for \( x, y \geq 0 \)

\[
| \tau_x A f(y) - f(y) | \leq \int_0^\infty | \varphi_\lambda(y) F(f)(\lambda) | | \varphi_\lambda(x) - 1 | d\gamma(\lambda)
\] (34)

Now, using mean value theorem and the fact that for \( x \geq 0 \) and \( \lambda \in \mathbb{R} \)

\[
| \varphi_\lambda(x) | \leq C(\lambda^2 + \rho^2)(1 + x)e^{-\rho x}
\] (35)

where \( C \) is a positive constant (see proposition II.2 [4]), it follows from (3) and (34) that for \( x \geq 0 \)

\[
\| \tau_x A f - f \|_{L^\infty_A(\mathbb{R}_+)} \leq C(1 + x)x^2 \| (\lambda^2 + \rho^2) F(f) \|_{L^1_A(\mathbb{R}_+)}
\] (36)

and this completes the proof for \( f \in D_A(\mathbb{R}). \)

Now, suppose that \( f \in L^\infty_A(\mathbb{R}_+) \) such that there exists a continuous function \( g \) in \( C_0^0(\mathbb{R}) \) satisfying \( f = g \) a.e. Then, there exists a sequence \( (f_n)_n \) in \( D_A(\mathbb{R}) \) such that

\[
\lim_{n \to \infty} \| f_n - f \|_{L^\infty_A(\mathbb{R}_+)} = 0.
\] (37)
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According to proposition 4 vii), we deduce that for $n \in \mathbb{N}$,
$$
\|\tau_x^A f - f\|_{L^\infty_A((\mathbb{R}_+))} \leq \|\tau_x^A(f - f_n)\|_{L^\infty_A((\mathbb{R}_+))} + \|\tau_x^A f_n - f\|_{L^\infty_A((\mathbb{R}_+))} + \|f_n - f\|_{L^\infty_A((\mathbb{R}_+))}
$$
$$
\leq 2\|f_n - f\|_{L^\infty_A((\mathbb{R}_+))} + \|\tau_x^A f_n - f\|_{L^\infty_A((\mathbb{R}_+))} \tag{38}
$$
and the result follows by applying Lemma 1 i) for $f_n$.

ii) Let $f \in L^\infty_A((\mathbb{R}_+))$ such that there exists a continuous function $g$ in $C^0_\text{c}((\mathbb{R}))$ satisfying $f = g$ a.e. Then, for $x \geq 0$, we have
$$
|f * A v_\varepsilon(x) - f(x)| \leq \int_0^\infty v_\varepsilon(y)|\tau_x^A f(y) - f(x)|A(y)dy \tag{39}
$$
which implies, by putting $t = \frac{y}{\varepsilon}$, that
$$
\|f * A v_\varepsilon - f\|_{L^\infty_A((\mathbb{R}_+))} \leq \int_0^\infty v(t)|\tau_x^A f - f(A)(t)|dt \tag{40}
$$
But, from Lemma 1 i),
$$
\lim_{\varepsilon \to 0}\|\tau_x^A f - f\|_{L^\infty_A((\mathbb{R}_+))} = 0 \text{ and } \|\tau_x^A f - f\|_{L^\infty_A((\mathbb{R}_+))}v(t) \leq 2\|f\|_{L^\infty_A((\mathbb{R}_+))}v(t) \in L^1_A((\mathbb{R}_+)).
$$
Hence, from (40) and by dominated convergence theorem, we deduce the result.

**Proposition 12** $D_\ast((\mathbb{R}))$ is dense in $D^p_A$, $p \in [1, \infty]$.

**Proof.** Let $f \in D^p_A$, $p \in [1, \infty]$. Then, from (32) and Lemma 1 ii), we have for all $k \in \mathbb{N}$,
$$
\lim_{\varepsilon \to 0}\|\Delta_A^k f * A v_\varepsilon - \Delta_A^k f\|_{L^p_A((\mathbb{R}_+))} = 0. \tag{41}
$$
On the other hand, from the density of $D_\ast((\mathbb{R}))$ respectively in $L^p_A((\mathbb{R}_+))$ and $C^0_\text{c}((\mathbb{R}))$, there exists a sequence $(f_n)_n$ in $D_\ast((\mathbb{R}))$ such that
$$
\lim_{n \to \infty}\|f_n - f\|_{L^p_A((\mathbb{R}_+))} = 0. \tag{42}
$$
Let $k \in \mathbb{N}$ and $\delta > 0$, there exist $\varepsilon > 0$ and $n \in \mathbb{N}$ such that
$$
\|\Delta_A^k f * A v_\varepsilon - \Delta_A^k f\|_{L^p_A((\mathbb{R}_+))} < \delta/2 \text{ and } \|f_n - f\|_{L^p_A((\mathbb{R}_+))} < \frac{\delta}{2\|\Delta_A^k v_\varepsilon\|_{L^1_A((\mathbb{R}_+))}}. \tag{43}
$$
Thus, by virtue of remark 5 iii) and using Proposition 5, it follows that
$$
\|\Delta_A^k (f_n * A v_\varepsilon - f)\|_{L^p_A((\mathbb{R}_+))} \leq \|f_n * A \Delta_A^k v_\varepsilon - f * A \Delta_A^k v_\varepsilon\|_{L^p_A((\mathbb{R}_+))} + \|\Delta_A^k f * A v_\varepsilon - \Delta_A^k f\|_{L^p_A((\mathbb{R}_+))}
$$
$$
\leq \|f_n - f\|_{L^p_A((\mathbb{R}_+))}\|\Delta_A^k v_\varepsilon\|_{L^1_A((\mathbb{R}_+))} + \|\Delta_A^k f * A v_\varepsilon - \Delta_A^k f\|_{L^p_A((\mathbb{R}_+))} \leq \delta \tag{44}
$$
Choosing the function \( v \) in \( D_+(\mathbb{R}) \), it follows that for all \( \varepsilon > 0 \) and all \( n \in \mathbb{N} \), \( f_n * A v_n \in D_+ (\mathbb{R}) \). And this achieves the proof of the proposition.

In the sequel, we give some result concerning the continuity of the Fourier transform \( \mathcal{F} \) and its inverse. We start with the following Lemma deduced from the hypothesis of the function \( A \)

**Lemma 2**

i) For any real \( a > 0 \) there exist positive constants \( C_1(a), C_2(a) \) such that for all \( x \in [0, a] \),

\[
C_1(a) x^{2a+1} \leq A(x) \leq C_2(a) x^{2a+1}.
\]

ii) For \( \rho > 0 \),

\[
A(x) \sim e^{2\rho x}, \quad (x \to +\infty)
\]

iii) For \( \rho = 0 \),

\[
A(x) \sim x^{2a+1}, \quad (x \to +\infty)
\]

**Theorem 1** The inverse of the Fourier transform \( \mathcal{F}^{-1} \) defines a continuous linear map from \( D_+ (\mathbb{R}) \) into \( D_A^p \) if \( p \in \left\{ \begin{array}{ll} [1, +\infty[ & \text{for } \rho = 0 \\ [2, \infty[ & \text{for } \rho > 0 \end{array} \right. \).

**Proof.** According to lemma 2 and using relation (4), it is not hard to see that \( \mathcal{S}^2_+ (\mathbb{R}) \hookrightarrow D_A^p \), \( p \in \left\{ \begin{array}{ll} [1, +\infty[ & \text{for } \rho = 0 \\ [2, \infty[ & \text{for } \rho > 0 \end{array} \right. \).

Thus, the result follows from the fact that, for all \( k \in \mathbb{N} \), \( \triangle_A^k \mathcal{F}^{-1} \) is continuous from \( D_+ (\mathbb{R}) \) into \( \mathcal{S}^2_+ (\mathbb{R}) \).

Let \( E_0 = \{ f \in C_+^0 (\mathbb{R}) / x^{2k} f \in C_+^0 (\mathbb{R}), k \in \mathbb{N} \} \) and \( E_1 = \{ f \in L_+^2 (\mathbb{R}+) / x^{2k} f \in L_+^2 (\mathbb{R}+), k \in \mathbb{N} \} \) equipped respectively with the topology generated by the countable norms

\[
\mu_m^\infty (f) = \max_{0 \leq k \leq m} \| \lambda^{2k} f \|_{L_+^\infty (\mathbb{R}+)}, \quad m \in \mathbb{N}.
\]

and

\[
\mu_m^2 (f) = \max_{0 \leq k \leq m} \| \lambda^{2k} f \|_{L_+^2 (\mathbb{R}+)}, \quad m \in \mathbb{N}.
\]

Thus, \( E_0 \) and \( E_1 \) are Fréchet spaces and we have

**Theorem 2** 1) The Fourier transform \( \mathcal{F} \) is a continuous from \( D_A^1 \) into \( E_0 \).

2) The Fourier transform \( \mathcal{F} \) is an isomorphism from \( D_A^2 \) onto \( E_1 \).

3) Let \( p \in [1, 2] \). Then, for \( r \in [1, p] \) and \( q \in [1, p'] \) with \( p' \) is the conjugate exponent of \( p \), there exist \( c > 0 \) and \( m \in \mathbb{N} \) such that for all \( f \in D_A^p \),

\[
\| \mathcal{F} (f) \|_{L_+^q (\mathbb{R}+)} \leq c \gamma_{m,r}^A (f),
\]

\( \gamma_{m,r}^A (f) \) is finite or infinite. In particular, the Fourier transform \( \mathcal{F} \) is a continuous from \( D_A^p \) into \( L_+^q (\mathbb{R}+), q \in [1, p'] \).
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**Proof.** For all $f \in D^1_A$ (resp. $f \in D^2_A$), we have

$$F(\triangle_A f) = -\lambda^2 F(f)$$  \hspace{1cm} (45)

Then 1) and 2) follows respectively from (10) and Plancherel Theorem.

3) According to Proposition 3 and using inequality (31), we deduce that, for $r \in [1, p]$, there exist $c_1 > 0$ and $k \in \mathbb{N}$ such that for all $f \in D^p_A$,

$$\|Ff\|_{L^p_\gamma'(\mathbb{R}^+)} \leq \|f\|_{L^p_A(\mathbb{R}^+)} \leq c_1 \gamma^A_{k,r}(f),$$  \hspace{1cm} (46)

$\gamma^A_{m,r}(f)$ is finite or infinite.

On the other hand, using Holder inequality, it follows that for $q \in [1, p']$ there exist $c_2 > 0$ and $n \in \mathbb{N}$ such that for all $g \in D^p_A$,

$$\|Fg\|_{L^q_\gamma(\mathbb{R}^+)} \leq c_2 \|F((I - \triangle_A)^n g)\|_{L^q_\gamma'(\mathbb{R}^+)}.$$  \hspace{1cm} (47)

Hence, the result follows by combining (46) and (47).

4 Titchmarsh’s theorem for the Chébli-Trimèche transfrom $F$ in $D^2_A$

In [19], Titchmarsh characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms. More precisely, we have:

**Theorem 3** [19] Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalent:

1) $\|f(t + h) - f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ as $h \to 0$.

2) $\int_{|\lambda| \geq r} |g(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \to \infty$. Where $g$ stands for the Fourier transform of $f$.

In this section, we prove a generalization of Titchmarsh’s theorem for the Chébli-Trimèche transfrom $F$ for functions satisfying the Chébli-Trimèche-Lipschitz condition in $D^2_A$.

Putting $V_\lambda(x) = \sqrt{A(x)} \varphi_\lambda(x)$, we see that $V_\lambda$ satisfies

$$(L_\alpha + \chi(x) + \lambda^2) V_\lambda(x) = 0,$$

where $L_\alpha$ and $\chi(x)$ are defined by

$$(L_\alpha u)(x) = u''(x) - \frac{\alpha^2 - \frac{1}{2}}{x^2} u(x)$$
and
\[ \chi(x) = \rho^2 - (2\alpha + 1) \frac{B'(x)}{2xB(x)} - \frac{1}{2} \left( \frac{B'(x)}{B(x)} \right)' - \frac{1}{4} \left( \frac{B'(x)}{B(x)} \right)^2. \]

Thus, we have
\[ V_{\lambda}(x) \sim x^{\alpha + \frac{1}{2}}, \quad V_{\lambda}'(x) \sim (\alpha + \frac{1}{2})x^{\alpha - \frac{1}{2}} \quad (x \to 0). \]

We assume in this section that \( \chi \) is holomorphic in a disc \( D(0, 2b) = \{ z \in \mathbb{C}, |z| < 2b \}, b > 0 \). Therefore, from [13], we have:

**Theorem 4** The eigenfunction \( \varphi_{\lambda} \) can be expanded in a Bessel-function series as follows
\[ \varphi_{\lambda}(x) = x^{\alpha + \frac{1}{2}} \frac{\sum_{p=0}^{\infty} x^{2p} B_p(x)}{2^\alpha \sqrt{A(x)}} \frac{\Gamma(\alpha + 1)}{2p\Gamma(\alpha + p + 1)} j_{\alpha + p}(\lambda x) \quad (48) \]

where \( B_p \) are functions defined by a recursive relation and satisfy:
\[ |B_p^{(q)}(x)| \leq \left( \frac{c}{2} \right)^p d_p \cdot b^{1-p-q} (p + q - 1)! \quad (49) \]

for all \( x \in [0, b], p = 1, 2, ... \) and \( q = 0, 1, 2, ..., \) where
\[ c = \max\{1 + |2\alpha - 1|, 2 \sup(|\chi(z)|, |z| \leq 2b)\}, \quad d = (b + b^{-1}) \]
and \( B_0(x) = B_0 > 0 \).

**Corollary 1** There exist \( \eta > 0, N > 1 \) and \( c > 0 \) such that

for all \( \lambda > N \) and \( \lambda x \in [\eta^2, \eta], 1 - \varphi_{\lambda}(x) > c. \)

**Proof.** Firstly it can be observed from inequality (3) and the integral representation (6) that, for \( \lambda x \in [0, \frac{\pi}{2}], 0 \leq \varphi_{\lambda}(x) \leq 1. \)

Since \( \varphi_{\lambda}(0) = 1 \), then by virtue of Theorem 4 we deduce that
\[ \lim_{x \to 0} \frac{B_0 x^{\alpha + \frac{1}{2}}}{2^\alpha \Gamma(\alpha + 1) \sqrt{A(x)}} = 1. \]

And from the expansion of the normalized Bessel function
\[ j_{\alpha}(x) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\alpha + n + 1)} \left( \frac{x}{2} \right)^{2n}, \]
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which implies that

$$\lim_{x \to 0} \frac{j_\alpha(x) - 1}{x^2} \neq 0$$

and therefore, there exist $\kappa > 0$ and $\sigma > 0$ such that for every $\eta \in (0, \sigma]$, we have

$$\forall x \in \left[\frac{\eta}{2}, \eta\right], \ |1 - j_\alpha(x)| = 1 - j_\alpha(x) \geq \kappa x^2 \geq \frac{\kappa \eta^2}{4}. \quad (51)$$

Thus, there exists $\delta \in (0, 1)$ such that for all $\lambda, x \geq 0$ satisfying $\lambda x \in \left[\frac{\eta}{2}, \eta\right]$, we have

$$0 \leq j_\alpha(\lambda x) < \delta. \quad (52)$$

and from (50) one can easily see that for $\delta_1 \in (\delta, 1)$ there exist $N > 1$ such that for $x \leq \frac{\eta}{N}$

$$0 \leq \frac{B_0 x^{\alpha + \frac{1}{2}}}{2^{\alpha} \Gamma(\alpha + 1) \sqrt{A(x)}} < \frac{\delta_1}{\delta}. \quad (53)$$

Combining (52) and (53) we deduce that for $\lambda \geq N$ and $\lambda x \in \left[\frac{\eta}{2}, \eta\right]$

$$0 \leq \frac{B_0 x^{\alpha + \frac{1}{2}} j_\alpha(\lambda x)}{2^{\alpha} \Gamma(\alpha + 1) \sqrt{A(x)}} < \delta_1. \quad (54)$$

On the other hand, using the inequality (49), we have for all $p \in \mathbb{N} \setminus \{0\}$

$$\left| \frac{x^{2p} B_p(x) j_{\alpha + p}(\lambda x)}{2^{\alpha} \Gamma(\alpha + p + 1)} \right| \leq \frac{b}{d} \left( \frac{cdx^2}{4b} \right)^p \frac{\Gamma(p)}{\Gamma(p + \alpha + 1)} \leq \frac{1}{\Gamma(\alpha + 2)} \left( \frac{cdx^2}{4b} \right)^p, \ x \in [0, b].$$

Thus, by virtue of (53) and choosing $\eta \leq \min\{\sigma, b, \frac{1}{\sqrt{2}}, \sqrt{\frac{\delta B_0 (\delta_2 - \delta_1)}{4\delta_1}}\}$ with $\delta_2 \in (\delta_1, 1)$, and $N \geq \max\{1, \sqrt{\frac{cd}{4b}}\}$, we have for $x \in [0, \frac{\eta}{N}]$

$$\left| \frac{x^{\alpha + \frac{1}{2}}}{2^{\alpha} \sqrt{A(x)}} \sum_{p=1}^{\infty} \frac{x^{2p} B_p(x)}{2^{\alpha} \Gamma(\alpha + p + 1)} j_{\alpha + p}(\lambda x) \right| \leq \frac{x^{\alpha + \frac{1}{2}}}{2^{\alpha} \Gamma(\alpha + 2) \sqrt{A(x)}} \sum_{p=1}^{\infty} \left( \frac{cdx^2}{4b} \right)^p \leq \frac{2^{\alpha+\frac{1}{2}}}{2^{\alpha} \Gamma(\alpha + 1) \sqrt{A(x)}} \sum_{p=1}^{\infty} \eta^{2p} \leq \frac{\delta \left( \delta_2 - \delta_1 \right)}{\delta_1} \leq \delta_2 - \delta_1. \quad (55)$$
Therefore, according to the expansion of $\varphi_\lambda$ given by (48) and using inequalities (54) and (55), it follows that for $\lambda \geq N$ and $\lambda x \in [\eta^2, \eta)$

$$1 - \varphi_\lambda(x) \geq 1 - \delta_2 > 0,$$

and this achieves the proof of the Corollary.

**Definition 5** Let $\delta \in (0, 1)$. A function $f \in D^2_A$ is said to be in the Chébli Trimèche Lipschitz class, denoted by $\text{Lip}(\delta, 2)$, if

$$\forall k \in \mathbb{N}, \forall r \in \mathbb{N}, \|L^k_x \Delta^r_A f\|_{L^2(\mathbb{R}^+)} = O(x^\delta) \quad \text{as} \quad x \to 0,$$

where $L_x f = \tau^A_x f - f$.

To prove the main result of this section we need the following Lemma

**Lemma 3** Let $f \in D^2_A$. Then for every $k \in \mathbb{N}$ and $r \in \mathbb{N}$, we have

$$\|L^k_x \Delta^r_A f\|^2_{L^2(\mathbb{R}^+)} = \int_0^\infty \lambda^{4r}|1 - \varphi_\lambda(x)|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda).$$

**Proof.** It is easily to see that for all $k \in \mathbb{N}$

$$L^k_x f(y) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (\tau^A_x)^i f(y)$$

Thus, by virtue of Proposition 4 v) and using the fact that

$$\mathcal{F}(\Delta^r_A f)(\lambda) = (-\lambda^2)^r \mathcal{F}(f)(\lambda)$$

it follows that

$$\mathcal{F}(L^k_x \Delta^r_A f)(\lambda) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \mathcal{F}((\tau^A_x)^i \Delta^r_A f)(\lambda)$$

$$= (-\lambda^2)^r \mathcal{F}(f)(\lambda) \sum_{i=0}^{k} (-1)^{k-i} \varphi_\lambda^i(x)$$

$$= (-\lambda^2)^r \mathcal{F}(f)(\lambda)(\varphi_\lambda(x) - 1)^k.$$ 

Hence, from Plancherel formula for $\mathcal{F}$, we deduce the result.

**Theorem 5** Let $f \in D^2_A$. Then the following are equivalents

i) $f \in \text{Lip}(\delta, 2)$.

ii) $\forall r \in \mathbb{N}, \int_s^\infty \lambda^{4r}|\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(s^{-2\delta}) \quad \text{as} \quad s \to +\infty.$
**Proof.** Suppose that \( f \in \text{Lip}(\delta, 2) \). According to Corollary 1 we deduce that there exist \( \eta > 0 \), \( a > 0 \) and \( c > 0 \) such that for \( x \in (0, a) \) and for all \( r \in \mathbb{N} \), we have

\[
\int_{\frac{r}{2}}^{2} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq \frac{1}{C^{2k}} \int_{0}^{\infty} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda), \; k \in \mathbb{N}
\]

which implies, from Lemma 3, that

\[
\int_{\frac{r}{2}}^{2} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(x^{2\delta}) \quad \text{as} \quad x \to 0.
\]

Therefore, there exists \( C > 0 \) such that

\[
\int_{s}^{2s} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq C s^{-2\delta}.
\]

Thus, there exists \( C_1 > 0 \) satisfying

\[
\int_{s}^{\infty} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = \sum_{i=0}^{\infty} \int_{2^i s}^{2^{i+1} s} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq C_1 s^{-2\delta}
\]

and consequently ii) is proved. Let's prove now that ii) \( \Rightarrow \) i). We have

\[
\int_{0}^{\infty} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = \int_{0}^{2} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda)
\]

\[
+ \int_{2}^{\infty} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda).
\]

By virtue of i), it follows that

\[
\int_{\frac{r}{2}}^{2} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq 4^k \int_{\frac{r}{2}}^{2} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(x^{2\delta}). \quad (56)
\]

On the other hand, estimate \( 1 - \varphi_\lambda(x) \) using the mean value theorem and inequality (5), we get for \( x \leq 1 \)

\[
\int_{0}^{\infty} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq 4^k \int_{0}^{2} \lambda^{4r} |1 - \varphi_\lambda(x)||\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda)
\]

\[
\leq C 4^k x^2 \left\{ \int_{0}^{2} \lambda^{4r+2} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \right\}
\]

\[
+ \rho^2 \int_{0}^{2} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda).
\]
But $\delta < 1$, then we have
\[
4^k x^2 \rho^2 \int_0^2 \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq 4^k x^2 \rho^2 \int_0^\infty \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda)
\]
\[
= 4^k x^2 \rho^2 \|\mathcal{A}_\lambda f\|_{L_2^A(\mathbb{R}_+)}^2
\]
\[
= O(x^{2\delta}).
\]

Now, by putting $\psi(s) = \int_s^\infty \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda)$ and using integration by parts, we obtain
\[
4^k x^2 \int_0^2 \lambda^{4r+2} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = 4^k x^2 \int_0^2 -s^2 \psi'(s) ds
\]
\[
= 4^k (-\psi(x) + 2x^2 \int_0^2 s \psi(s) ds).
\]
Therefore, using the fact that $\psi(s) = O(s^{-2\delta})$, it is not hard to see that
\[
4^k x^2 \int_0^2 \lambda^{4r+2} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(x^{2\delta})
\]
and the theorem is proved.

## 5 Open Problem

The purpose of the future work is to generalize the Titchmarsh’s theorem for the Chèbli-Trimèche transform $\mathcal{F}$ for functions satisfying the Chèbli-Trimèche-Lipschitz condition in $\mathcal{D}'_{L^p}$ with $1 \leq p \leq 2$.

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## References


New results on the $D_{L_p}$-type spaces


