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New results on the D_{Lp}-type spaces associated with a singular second order differential operator

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Abstract

In this work, we consider a second order differential operator $\triangle_A$ defined on $(0, +\infty)$, where A is a non negative function satisfying some conditions. To $\triangle_A$ we associate D_{Lp}-type spaces denoted by $\mathcal{D}_A^p$. Some results, related to the spaces $\mathcal{D}_A^p$, are proved. Moreover A generalization of Titchmarsh's theorem for the Chébli-Trimèche transfrom in $\mathcal{D}_A^2$ is established.

Keywords: *second order differential operator $\triangle_A$, $\mathcal{D}_A^p$ spaces, Titchmarsh's theorem*

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1 Introduction

L. Schwartz has introduced in [18] the space D_{L^p}, $1 \leq p \leq \infty$, of all C^∞-functions ψ on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, $D^n\psi$ is in $L^p(\mathbb{R})$ and the map $\psi \mapsto D^n\psi$ from D_{L^p} into $L^p(\mathbb{R})$ is continuous. These spaces are studied by many authors (see [1], [2], [5], [17]) among others.

In [12] the authors define new function spaces similar to D_{L^p} but replacing the usual derivative D by the generalized Laplace operator $\triangle_A$ defined on $(0, \infty)$ by

$$\triangle_A = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx} + \rho^2, \quad \rho \geq 0,$$

where A is the Chébli-Trimèche function (cf. [6], Section 3.5) defined on $[0,\infty)$ and satisfies the following conditions:

i) There exists a positive even infinitely differentiable function B on $\mathbb{R}$, with $B(0)=1$,
$x\in\mathbb{R}_+$, such that $A(x)=x^{2\alpha+1}B(x)$, $\alpha>\frac{-1}{2}$.

ii) A is increasing on $\mathbb{R}_+$ and $\lim\limits_{x\to\infty}A(x)=\infty$.

iii) $\dfrac{A'}{A}$ is decreasing on $(0,\infty)$, and $\lim\limits_{x\to\infty}\dfrac{A'(x)}{A(x)}=2\rho$.

iv) There exists a constant $\sigma>0$, such that for all $x\in[x_0,\infty)$, $x_0>0$, we have

$$\frac{A'(x)}{A(x)}=\begin{cases}2\rho+e^{-\sigma x}F(x), \text{ if}\,\rho>0\\ \dfrac{2\alpha+1}{x}+e^{-\sigma x}F(x)\text{ , if}\,\rho=0\end{cases}$$

where F is C^∞ on $(0,\infty)$, bounded together with its derivatives.

For $A(x)=x^{2\alpha+1}$, $\alpha>-\frac{1}{2}$ and $\rho=0$ we regain the Bessel operator

$$l_\alpha f=\frac{d^2f}{dx^2}+\left(\frac{2\alpha+1}{x}\right)\frac{df}{dx}.$$

For $A(x)=\sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, $\alpha\geq\beta\geq-\frac{1}{2}$, $\alpha\neq-\frac{1}{2}$ and $\rho=\alpha+\beta+1$ we regain the Jacobi operator

$$l_{\alpha,\beta}f=\frac{d^2f}{dx^2}+\Big[(2\alpha+1)\coth x+(2\beta+1)\tanh x\Big]\frac{f(x)}{x}+\rho^2.$$

In this paper, these spaces denoted by $\mathcal{D}^p_A$, $1\leq p\leq\infty$, are moreover considered as subspaces of $\mathcal{E}_*(\mathbb{R})$ (the space of even C^∞-functions on $\mathbb{R}$). Some properties, related to the spaces $\mathcal{D}^p_A$, are given.

The contents of the paper is as follows :

In §2 we recall some basic facts about the harmonic analysis results related to the operator $\triangle_A$. in §3 we introduce the space $\mathcal{D}^p_A$ and we show some results. In particular, we study the continuity of the Chébli-Trimèche transform on $\mathcal{D}^p_A$, $1\leq p\leq 2$. In §4 a generalization of Titchmarsh's theorem for the Chébli-Trimèche transfrom $\mathcal{F}$ for functions satisfying the Chébli-Trimèche-Lipschitz condition in $\mathcal{D}^2_A$ is established.

2 Preliminaries

In this section, we collect some harmonic analysis results related to the operator $\triangle_A$. For details we refer the reader to [6], [8], [12], [14], [21], and [22].

2.1 Eigenfunctions of the operator $\triangle_A$

In the following we denote by
$C^0_*(\mathbb{R})$ the space of even continuous functions f on $\mathbb{R}$ such that

$$\lim_{|x|\to+\infty} |f(x)| = 0.$$

$\mathcal{S}_*(\mathbb{R})$ the subspace of $\mathcal{E}_*(\mathbb{R})$, consisting of functions f rapidly decreasing together with their derivatives.
$\mathcal{S}^2_*(\mathbb{R}) = \varphi_0\mathcal{S}_*(\mathbb{R})$, where φ_0 is the eigenfunction of the operator $\triangle_A$ associated with the value $\lambda = 0$.
$\mathcal{S}'_*(\mathbb{R})$ the dual topological space of $\mathcal{S}_*(\mathbb{R})$.
$(\mathcal{S}^2_*)'(\mathbb{R})$ the dual topological space of $\mathcal{S}^2_*(\mathbb{R})$.
$\mathcal{E}'_*(\mathbb{R}_+)$ the dual topological space of $\mathcal{E}_*(\mathbb{R})$.
$\mathcal{H}_*(\mathbb{C})$ the space of even entire functions on $\mathbb{C}$ which are of exponential type and slowly increasing.
$\mathcal{H}_{*,a}(\mathbb{C})$ the subspace of $\mathcal{H}_*(\mathbb{C})$ satisfying

$$\exists m \in \mathbb{N}, P_m(f) = \sup_{\lambda\in\mathbb{C}}(1+\lambda^2)^{-m}|f(\lambda)|\exp(-a|Im\lambda|) < +\infty$$

we have $\mathcal{H}_*(\mathbb{C}) = \cup_{a\geq 0}\mathcal{H}_{*,a}(\mathbb{C})$.

For every $\lambda \in \mathbb{C}$, let us denote by φ_λ the unique solution of the eigenvalue problem

$$\begin{cases} \triangle_A f(x) = -\lambda^2 f(x), \\ f(0) = 1, \quad f'(0) = 0. \end{cases} \tag{1}$$

Remark 1 *This function satisfies the following properties.*

- $\forall x \geq 0$, *the function* $\lambda \longmapsto \varphi_\lambda(x)$ *is analytic on* $\mathbb{C}$.
- *Product formula*

$$\forall x, y \geq 0;\ \varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)w(x,y,z)A(z)dz \tag{2}$$

where $w(x,y,.)$ *is a measurable positive function on* $[0,\infty)$, *with support in* $[|x-y|, x+y]$.

- $\forall\ \lambda \geq 0$ *and* $x \in \mathbb{R}$, $|\varphi_\lambda(x)| \leq 1.$ (3)
- *For* $\rho > 0$, *we have*

$$\forall x \geq 0, \forall\lambda \in \mathbb{R},\ |\varphi_\lambda(x)| \leq \varphi_0(x) \leq m(1+x)exp(-\rho x), \tag{4}$$

where m *is a positive constant.*

- *For $\rho = 0$, we have*

$$\forall x \geq 0, \quad \varphi_0(x) = 1,$$

- $\forall x \geq 0, \forall \lambda \in \mathbb{R}, \quad |\varphi'_\lambda(x)| \leq c(\lambda^2 + \rho^2)(1+x)xexp(-\rho x), \qquad (5)$

  where c is a positive constant.

- *We have the following integral representation of Mehler type,*

$$\forall x > 0, \forall \lambda \in \mathbb{C}, \quad \varphi_\lambda(x) = \int_0^x k(x,t)\cos(\lambda t)dt \tag{6}$$

  where, $k(x,.)$ is an even positive C^∞ function on $]-x,x[$ with support in $[-x,x]$.

2.2 Generalized Fourier transform

For a Borel positive measure μ on $\mathbb{R}$, and $1 \leq p \leq \infty$, we write $L^p_\mu(\mathbb{R}_+)$ for the Lebesgue space equipped with the norm $\|\cdot\|_{L^p_\mu(\mathbb{R}_+)}$ defined by

$$\|f\|_{L^p_\mu(\mathbb{R}_+)} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu(x)\right)^{1/p}, \quad \text{if } p < \infty,$$

and $\|f\|_{L^\infty_\mu(\mathbb{R}_+)} = \text{ ess sup}_{x\in\mathbb{R}_+}|f(x)|$. When $\mu(x) = w(x)dx$, with w a nonnegative function on $\mathbb{R}_+$, we replace the μ in the norms by w.

For $f \in L^1_A(\mathbb{R}_+)$, the generalized Fourier transform, called also Chébli-Trimèche transform, is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}_+} f(x)\varphi_\lambda(x)A(x)dx, \quad \forall \lambda \in \mathbb{R}. \tag{7}$$

The inverse generalized Fourier transform of a suitable function g on $\mathbb{R}_+$ is given by:

$$\mathcal{J}g(x) = \mathcal{F}^{-1}g(x) = \int_{\mathbb{R}_+} g(\lambda)\varphi_\lambda(x)d\gamma(\lambda) \tag{8}$$

where $d\gamma(\lambda)$ is the spectral measure given by

$$d\gamma(\lambda) = \frac{d\lambda}{|c_A(\lambda)|^2}. \tag{9}$$

Remark 2 *The function $\lambda \longmapsto c_A(\lambda)$ satisfies the following properties.*

- *For $\lambda \in \mathbb{R}$,we have $c_A(-\lambda) = \overline{c_A(\lambda)}$.*

- *The function* $|c_A(\lambda)|^{-2}$ *is continuous on* $[0, \infty[$.
- *There exist positive constants* k_1, k_2, *and* k_3, *such that*
  If $\rho \geq 0 : \forall \lambda \in \mathbb{C},\ Im\lambda \leq 0,\ |\lambda| > k_3;$

$$k_1 \, |\lambda|^{2\alpha+1} \leq |c_A(\lambda)|^{-2} \leq k_2 \, |\lambda|^{2\alpha+1} .$$

If $\rho = 0,\ \alpha > 0 : \forall \lambda \in \mathbb{C},\ |\lambda| \leq k_3;$

$$k_1 \, |\lambda|^{2\alpha+1} \leq |c_A(\lambda)|^{-2} \leq k_2 \, |\lambda|^{2\alpha+1} .$$

If $\rho > 0 : \forall \lambda \in \mathbb{C},\ |\lambda| \leq k_3;$

$$k_1 \, |\lambda|^{2} \leq |c_A(\lambda)|^{-2} \leq k_2 \, |\lambda|^{2} .$$

Proposition 1 *([10]). i) The generalized transform* $\mathcal{F}$ *and its inverse* $\mathcal{J}$ *are topological isomorphisms between the generalized Schwartz space* $\mathcal{S}^2_*(\mathbb{R})$ *and the Schwartz space* $\mathcal{S}(\mathbb{R}_*)$.

ii) The transform $\mathcal{F}$ *is a topological isomorphism from* $\mathcal{E}'_*(\mathbb{R}_+)$ *onto* $\mathcal{H}_*(\mathbb{C})$. *Moreover, for all* $T \in \mathcal{E}'_*(\mathbb{R}_+)$, *we have:* $\operatorname{supp}(T) \subseteq [-a, a]$ *if and only if* $\mathcal{F}(T) \in \mathcal{H}_{*,a}(\mathbb{C})$.

Next, we give some properties of this transform.

i) For f in $L^1_A(\mathbb{R}_+)$ we have

$$||\mathcal{F}(f)||_{L^\infty_\gamma(\mathbb{R}_+)} \leq ||f||_{L^1_A(\mathbb{R}_+)}. \tag{10}$$

ii) For f in $\mathcal{S}^2_*(\mathbb{R})$ we have

$$\mathcal{F}(\triangle_A f)(y) = -y^2 \mathcal{F}(f)(y), \quad \text{for all } y \in \mathbb{R}_+. \tag{11}$$

Proposition 2 *([10]).* **Plancherel formula for** $\mathcal{F}$. *For all* f *in* $\mathcal{S}^2_*(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}_+} |f(x)|^2 A(x)\, dx = \int_{\mathbb{R}_+} |\mathcal{F}(f)(\xi)|^2 d\gamma(\xi). \tag{12}$$

ii) *Plancherel theorem.*

The transform $\mathcal{F}$ *extends uniquely to an isomorphism from* $L^2_A(\mathbb{R}_+)$ *onto* $L^2_\gamma(\mathbb{R}_+)$.

iii) for all f, $g \in L^2_A(\mathbb{R}_+)$, *we have*

$$\int_{\mathbb{R}_+} f(x)\overline{g(x)} A(x)\, dx = \int_{\mathbb{R}_+} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)} d\gamma(\xi). \tag{13}$$

Remark 3 *We have* $\mathcal{S}^2_*(\mathbb{R}) \subset L^p_A(\mathbb{R}_+)$ *for all* $2 \leq p \leq \infty$, *but* $S^2_*(\mathbb{R}) \hookrightarrow L^p_A(\mathbb{R}_+)$ *for all* $0 < p < 2$.

Proposition 3 *Let $1 \leq p \leq 2$. The Fourier transform $\mathcal{F}$, (resp. $\mathcal{J}$) can be extended as a continuous mapping from $L^p_A(\mathbb{R}_+)$ onto $L^{p'}_\gamma(\mathbb{R}_+)$ (resp. from $L^p_\gamma(\mathbb{R}_+)$ onto $L^{p'}_A(\mathbb{R}_+)$) and we have*

$$\|\mathcal{F}f\|_{L^{p'}_\gamma(\mathbb{R}_+)} \leq \|f\|_{L^p_A(\mathbb{R}_+)}\,;\ \|\mathcal{J}g\|_{L^{p'}_A(\mathbb{R}_+)} \leq \|g\|_{L^p_\gamma(\mathbb{R}_+)} \tag{14}$$

with $\dfrac{1}{p'} + \dfrac{1}{p} = 1.$

2.3 Generalized convolution

Definition 1 *([9]). The translation operator associated with the operator $\triangle_A$ is defined on $L^1_A(\mathbb{R}_+)$, by*

$$\forall x, y \geq 0;\ \tau^A_x f(y) = \int_0^\infty f(z)w(x,y,z)A(z)dz \tag{15}$$

where w is the function defined in the relation (2).

Proposition 4 *([9]). For a suitable function f on $\mathbb{R}_+$, we have*
i) $\tau^A_x f(y) = \tau^A_y f(x)$.
ii) $\tau^A_0 f(y) = f(y)$.
iii) $\tau^A_x \tau^A_y = \tau^A_y \tau^A_x$.
iv) $\tau^A_x \varphi_\lambda(y) = \varphi_\lambda(x)\varphi_\lambda(y)$.
v) $\mathcal{F}(\tau^A_x f)(\lambda) = \varphi_\lambda(x)\mathcal{F}(f)(\lambda)$.
vi) $\triangle_A(\tau^A_x)f = \tau^A_x(\triangle_A f)$
vii) $\forall\, x \geq 0\ \ \|\tau^A_x f\|_{L^p_A(\mathbb{R}_+)} \leq \|f\|_{L^p_A(\mathbb{R}_+)},\ p \in [1, \infty]$.

Definition 2 *([9]). For suitable functions f and g, we define the convolution product $f *_A g$ by*

$$f *_A g(x) = \int_{\mathbb{R}_+} \tau^A_x f(y)g(y)A(y)dy. \tag{16}$$

Remark 4 *It is clear that this convolution product is both commutative and associative:*
i) $f *_A g = g *_A f$.
ii) $(f *_A g) *_A h = f *_A (g *_A h)$.

Proposition 5 *([9]).*
i) *Assume that $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every $f \in L^p_A(\mathbb{R}_+)$ and $g \in L^q_A(\mathbb{R}_+)$, we have $f *_A g \in L^r_A(\mathbb{R}_+)$, and*

$$\|f *_A g\|_{L^r_A(\mathbb{R}_+)} \leq C\|f\|_{L^p_A(\mathbb{R}_+)}\|g\|_{L^q_A(\mathbb{R}_+)}. \tag{17}$$

ii) *If* $\rho > 0$ *and* $1 \leq p < q \leq 2$. *Then*

$$L^p_A(\mathbb{R}_+) *_A L^q_A(\mathbb{R}_+) \hookrightarrow L^q_A(\mathbb{R}_+). \tag{18}$$

iii) *If* $\rho > 0$ *and* $2 < p, q < \infty$ *such that* $\frac{q}{2} \leq p < q$. *Then*

$$L^p_A(\mathbb{R}_+) *_A L^{q'}_A(\mathbb{R}_+) \hookrightarrow L^q_A(\mathbb{R}_+) \tag{19}$$

where q' *is the conjugate exponent of* q.

iv) *If* $\rho > 0$ *and* $1 < p < 2$ *such that* $p < q \leq \frac{p}{2-p}$. *Then*

$$L^p_A(\mathbb{R}_+) *_A L^p_A(\mathbb{R}_+) \hookrightarrow L^q_A(\mathbb{R}_+). \tag{20}$$

v)

$$L^1_A(\mathbb{R}_+) *_A C^0_*(\mathbb{R}) \hookrightarrow C^0_*(\mathbb{R}). \tag{21}$$

Proposition 6 *If* $\rho > 0$, *then for* $f \in L^2_A(\mathbb{R}_+)$ *and* $g \in L^p_A(\mathbb{R}_+)$, *with* $1 \leq p < 2$ *we have*

$$\mathcal{F}(f *_A g) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda). \tag{22}$$

Proposition 7 *([21]) Let* $f, g \in L^2_A(\mathbb{R}_+)$. *Then* $f *_A g \in L^2_A(\mathbb{R}_+)$ *if and only if* $\mathcal{F}(f)\mathcal{F}(g)$ *belongs to* $L^2_\gamma(\mathbb{R}_+)$, *and in this case we have*

$$\mathcal{F}(f *_A g) = \mathcal{F}(f)\mathcal{F}(g).$$

Proposition 8 *([21])Let* f *be locally integrable function on* $[0, +\infty)$, *and* g *a measurable function on* $[0, +\infty)$ *satisfying the condition:*

$$\exists\, r \in \mathbb{N} \quad \text{such that} \quad (1+\lambda^2)^{-r} g \in L^1_\gamma(\mathbb{R}_+). \tag{23}$$

We suppose that for all $\psi \in D_*(\mathbb{R})$,

$$\int_0^\infty f(x)\psi(x)A(x)dx = \int_0^\infty g(\lambda)\mathcal{F}(\psi)(\lambda)d\gamma(\lambda).$$

Then the function f *belongs to* $L^2_A(\mathbb{R}_+)$ *if and only if the function* g *belongs to* $L^2_\gamma(\mathbb{R}_+)$ *and we have*

$$\mathcal{F}(f) = g.$$

Definition 3 *The generalized Fourier transform of a distribution* τ *in* $(\mathcal{S}^2_*)'(\mathbb{R})$ *is defined by*

$$\langle \mathcal{F}(\tau), \phi \rangle = \langle \tau, \mathcal{F}^{-1}(\phi) \rangle, \quad \text{for all } \phi \in \mathcal{S}_*(\mathbb{R}). \tag{24}$$

Proposition 9 *The generalized Fourier transform* $\mathcal{F}$ *is a topological isomorphism from* $(\mathcal{S}^2_*)'(\mathbb{R})$ *onto* $\mathcal{S}'_*(\mathbb{R})$.

Let τ be in $(\mathcal{S}_*^2)'(\mathbb{R}_+)$. We define the distribution $\triangle_A\tau$, by

$$\langle \triangle_A\tau, \psi\rangle \;=\; \langle \tau, \triangle_A\psi\rangle, \;\text{ for all } \psi \in \mathcal{S}_*^2(\mathbb{R}_+).$$

This distribution satisfy the following property

$$\mathcal{F}(\triangle_A\tau) \;=\; -y^2\mathcal{F}(\tau). \tag{25}$$

Remark 5 (*see [20]*)

i) The generalized convolution product of a distribution S in $D'_*(\mathbb{R})$ and a function ψ in $D_*(\mathbb{R})$ is the function $S *_A \psi$ defined by

$$\forall\ x \in \mathbb{R}_+,\;\; S *_A \psi(x) = \langle S_y, \tau_x^A\psi(y)\rangle \tag{26}$$

ii) Let U be a distribution in $D'_*(\mathbb{R})$ and S a distribution in $\mathcal{E}'_*(\mathbb{R})$. The generalized convolution product of U and S is the distribution in $D'_*(\mathbb{R})$ defined for all ψ in $D_*(\mathbb{R})$ by

$$\langle U *_A S,\, \psi\rangle = \langle U_x, S *_A \psi(x)\rangle = \langle S_y, U *_A \psi(y)\rangle \tag{27}$$

iii) Let $k \in \mathbb{N}^*$. Then, for all U in $D'_*(\mathbb{R})$ and S in $\mathcal{E}'_*(\mathbb{R})$, we have

$$\triangle_A^k(U *_A S) = U *_A \triangle_A^k(S) = (\triangle_A^k U) *_A S \tag{28}$$

iv) let U and S be two distributions in $\mathcal{E}'_*(\mathbb{R})$. Then the function $U *_A S$ belongs to $\mathcal{E}'_*(\mathbb{R})$ and we have

$$\mathcal{F}(U *_A S) = \mathcal{F}(U)\mathcal{F}(S). \tag{29}$$

3 The space $\mathcal{D}_A^p$

Now, we start with the definition of the spaces of $\mathcal{D}_A^p$.

Definition 4 *If $1 \le p < \infty$, the space $\mathcal{D}_A^p$ is the set of all of C^∞ and even functions f on $\mathbb{R}$ such that, for all $k \in \mathbb{N}$, $\triangle_A^k\phi$ is in $L_A^p(\mathbb{R}_+)$ which is equipped with the topology generated by the countable norms*

$$\gamma_{m,p}^A(f) = \max_{0\le k\le m} \|\triangle_A^k f\|_{L_A^p(\mathbb{R}_+)},\;\; m \in \mathbb{N}.$$

A function $f \in \mathcal{E}_*(\mathbb{R})$ is in $\mathcal{B}_A^\infty$ when $\gamma_{m,\infty}^A(f) < \infty$ for each $m \in \mathbb{N}$, where

$$\gamma_{m,\infty}^A(f) = \max_{0\le k\le m} \|\triangle_A^k f\|_{L_A^\infty(\mathbb{R}_+)},\;\; m \in \mathbb{N}.$$

We denote by $\mathcal{D}_A^\infty$ the subspace of $\mathcal{B}_A^\infty$ that consists of all those functions $f \in \mathcal{B}_A^\infty$ for which $\lim\limits_{|x|\to\infty} \triangle_A^m f(x) = 0$ for each $m \in \mathbb{N}$.

The space $\mathcal{B}_A^\infty$ is endowed with the topology generated by the system $\{\gamma_{m,\infty}^A\}_{m\in\mathbb{N}}$.

Remark 6 *i) Let $1 \le p < \infty$. A function $\varphi \in L_A^p(\mathbb{R}_+)$ is in $\mathcal{D}_A^p$ if and only if $(I-\triangle_A)^m\varphi \in L_A^p(\mathbb{R}_+)$ for every $m \in \mathbb{N}$.*

ii) A function $\phi \in L_A^\infty(\mathbb{R}_+)$ is in $\mathcal{B}_A^\infty$ if and only if $(I-\triangle_A)^m\phi \in L_A^\infty(\mathbb{R}_+)$ for every $m \in \mathbb{N}$.

iii) For $0 < p \le 2$, we define the generalized Schwartz space $S_*^p(\mathbb{R})$ by

$$S_*^p(\mathbb{R}) = \left\{ f \in \mathcal{E}_*(\mathbb{R}) \,/\, \forall\, k, l \in \mathbb{N},\ \ \sup_{x\ge 0}\ (1+x)^l \varphi_0^{-2/p}(x)|f^k(x)| < \infty \right\}.$$

Then, for $q \in [\max\{1,p\},\, +\infty]$, $S_*^p(\mathbb{R}) \subset \mathcal{D}_A^q$. In particular, when $\rho = 0$ or $0 < p \le 1$, for all $q \in [1,\, +\infty]$, $S_*^p(\mathbb{R}) \subset \mathcal{D}_A^q$.

Proposition 10 *([12]) For every $p \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for any $m \in \mathbb{N}$, $m \ge m_0$, we can find two functions $\chi_m \in D_{*,\varepsilon}(\mathbb{R})$(the subspace of $\mathcal{D}_*(\mathbb{R})$ consisting of function f such that supp$f \subset [-\varepsilon,\varepsilon]$) and $\Gamma_m \in \mathcal{W}_\varepsilon^p(\mathbb{R})$ (the space of function $f : \mathbb{R} \to \mathbb{C}$ of class C^{2p} on $\mathbb{R}$, even and with support in $[-\varepsilon,\varepsilon]$) such that*

$$\delta = (I-\triangle_A)^m\Gamma_m + \chi_m.$$

We start with some topological properties of the spaces $\mathcal{D}_A^p$.

Proposition 11 *i) $\mathcal{D}_A^p$, $1 \le p \le \infty$ and $\mathcal{B}_A^\infty$ are Fréchet spaces.*
ii) $\mathcal{D}_A^p$ is continuously contained in $\mathcal{D}_A^q$, when $1 \le p \le q \le \infty$.
iii) If $1 < p < \infty$ then $\mathcal{D}_A^p$ is a reflexive space.

Proof. In Proposition 2.1 [12], the result is proved for the spaces $\mathcal{D}_A^p$, $1 \le p < \infty$ and $\mathcal{B}_A^\infty$. Lets prove that $\mathcal{D}_A^\infty$ is a Fréchet space. let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}_A^\infty$. Since $C_*^0(\mathbb{R})$ is a Banach space, then there exists $v_m \in C_*^0(\mathbb{R})$ such that $\triangle_A^m u_n \to v_m$, as $n \to \infty$, in $C_*^0(\mathbb{R})$, for each $m \in \mathbb{N}$. On the other hand by a simple calculation we see that, $\triangle_A^m v_0 = v_m$, $m \in \mathbb{N}$. Which implies that $(u_n)_{n\in\mathbb{N}}$ converge to v_0 in $\mathcal{D}_A^\infty$. Thus the proof of i) is finished.
ii) Let $\varphi \in \mathcal{D}_A^p$. Then, using Proposition 10, for $a > 0$ and $n \in \mathbb{N}$, there exist two functions $\chi \in D_*(\mathbb{R})$ and $\Gamma \in \mathcal{W}_a^n(\mathbb{R})$ such that

$$\triangle_A^k\varphi = \delta *_A \triangle_A^k\varphi = \Gamma *_A (I-\triangle_A)^n\triangle_A^k\varphi + \chi *_A \triangle_A^k\varphi,\ k \in \mathbb{N} \tag{30}$$

Therefore, from proposition 5 i),v) we deduce the result. Moreover, for $1 \le p \le q \le \infty$, there exists $c > 0$ such that

$$\forall\, m \in \mathbb{N},\ \exists\, m_1 \in \mathbb{N} \text{ satisfying } \gamma_{m,q}^A(\varphi) \le c\gamma_{m_1,p}^A(\varphi). \tag{31}$$

To see iii) it is sufficient to argue like in [18].

It is well known (see [21]) that for all $f \in L^p_A(\mathbb{R}_+)$, $p \in [1, \infty)$,

$$\lim_{x \to 0} \|\tau^A_x f - f\|_{L^p_A(\mathbb{R}_+)} = 0 \text{ and } \lim_{\varepsilon \to 0} \|f *_A v_\varepsilon - f\|_{L^p_A(\mathbb{R}_+)} = 0. \tag{32}$$

where

$$v_\varepsilon = (\varepsilon A(x))^{-1} A(\frac{x}{\varepsilon}) v(\frac{x}{\varepsilon}) \tag{33}$$

with v is a positive function in $L^1_A(\mathbb{R}_+)$ such that $\|v\|_{L^1_A(\mathbb{R}_+)} = 1$.

The case $p = \infty$ is given by the following Lemma which we need in the sequel to study the density of the space $D_*(\mathbb{R})$ in $\mathcal{D}^p_A,\ p \in [1, \infty]$.

Lemma 1 *Let $f \in L^\infty_A(\mathbb{R}_+)$ such that there exists a continuous function g in $C^0_*(\mathbb{R})$ satisfying $f = g$ a.e. Then*

i) $\lim_{x \to 0} \|\tau^A_x f - f\|_{L^\infty_A(\mathbb{R}_+)} = 0.$

ii) $\lim_{\varepsilon \to 0} \|f *_A v_\varepsilon - f\|_{L^\infty_A(\mathbb{R}_+)} = 0.$

where v_ε is given by (33).

Proof. i) Suppose that $f \in D_*(\mathbb{R})$, then from inversion formula (8) and Proposition 4 v), we deduce that for $x, y \geq 0$

$$|\tau^A_x f(y) - f(y)| \ \leq \ \int_0^\infty |\varphi_\lambda(y)\mathcal{F}(f)(\lambda)||\varphi_\lambda(x) - 1| d\gamma(\lambda) \tag{34}$$

Now, using mean value theorem and the fact that for $x \geq 0$ and $\lambda \in \mathbb{R}$

$$|\varphi'_\lambda(x)| \leq C(\lambda^2 + \rho^2)(1 + x)xe^{-\rho x} \tag{35}$$

where C is a positive constant (see proposition II.2 [4]), it follows from (3) and (34) that for $x \geq 0$

$$\|\tau^A_x f - f\|_{L^\infty_A(\mathbb{R}_+)} \ \leq \ C(1 + x)x^2 \|(\lambda^2 + \rho^2)\mathcal{F}(f)\|_{L^1_\gamma(\mathbb{R}_+)} \tag{36}$$

and this completes the proof for $f \in D_*(\mathbb{R})$.

Now, suppose that $f \in L^\infty_A(\mathbb{R}_+)$ such that there exists a continuous function g in $C^0_*(\mathbb{R})$ satisfying $f = g$ *a.e.* Then, there exists a sequence $(f_n)_n$ in $D_*(\mathbb{R})$ such that

$$\lim_{n \to \infty} \|f_n - f\|_{L^\infty_A(\mathbb{R}_+)} = 0. \tag{37}$$

According to proposition 4 vii), we deduce that for $n \in \mathbb{N}$,

$$\begin{aligned}\|\tau_x^A f - f\|_{L_A^\infty(\mathbb{R}_+)} &\leq \|\tau_x^A (f - f_n)\|_{L_A^\infty(\mathbb{R}_+)} + \|\tau_x^A f_n - f_n\|_{L_A^\infty(\mathbb{R}_+)} + \|f_n - f\|_{L_A^\infty(\mathbb{R}_+)} \\ &\leq 2\|f_n - f\|_{L_A^\infty(\mathbb{R}_+)} + \|\tau_x^A f_n - f_n\|_{L_A^\infty(\mathbb{R}_+)} \qquad (38)\end{aligned}$$

and the result follows by applying Lemma 1 i) for f_n.

ii) Let $f \in L_A^\infty(\mathbb{R}_+)$ such that there exists a continuous function g in $C_*^0(\mathbb{R})$ satisfying $f = g$ *a.e.* Then, for $x \geq 0$, we have

$$|f *_A v_\varepsilon(x) - f(x)| \leq \int_0^\infty v_\varepsilon(y) |\tau_x^A f(y) - f(x)| A(y) dy \qquad (39)$$

which implies, by putting $t = \dfrac{y}{\varepsilon}$, that

$$\|f *_A v_\varepsilon - f\|_{L_A^\infty(\mathbb{R}_+)} \leq \int_0^\infty v(t) \|\tau_{t\varepsilon}^A f - f\|_{L_A^\infty(\mathbb{R}_+)} A(t) dt \qquad (40)$$

But, from Lemma 1 i),

$$\lim_{\varepsilon \to 0} \|\tau_{t\varepsilon}^A f - f\|_{L_A^\infty(\mathbb{R}_+)} = 0 \text{ and } \|\tau_{t\varepsilon}^A f - f\|_{L_A^\infty(\mathbb{R}_+)} v(t) \leq 2\|f\|_{L_A^\infty(\mathbb{R}_+)} v(t) \in L_A^1(\mathbb{R}_+).$$

Hence, from (40) and by dominated convergence theorem, we deduce the result.

Proposition 12 *$D_*(\mathbb{R})$ is dense in $\mathcal{D}_A^p$, $p \in [1, \infty]$.*

Proof. Let $f \in \mathcal{D}_A^p$, $p \in [1, \infty]$. Then, from (32) and Lemma 1 ii), we have for all $k \in \mathbb{N}$,

$$\lim_{\varepsilon \to 0} \|\triangle_A^k f *_A v_\varepsilon - \triangle_A^k f\|_{L_A^p(\mathbb{R}_+)} = 0. \qquad (41)$$

On the other hand, from the density of $D_*(\mathbb{R})$ respectively in $L_A^p(\mathbb{R}_+)$ and $C_*^0(\mathbb{R})$, there exists a sequence $(f_n)_n$ in $D_*(\mathbb{R})$ such that

$$\lim_{n \to \infty} \|f_n - f\|_{L_A^p(\mathbb{R}_+)} = 0. \qquad (42)$$

Let $k \in \mathbb{N}$ and $\delta > 0$, there exist $\varepsilon > 0$ and $n \in \mathbb{N}$ such that

$$\|\triangle_A^k f *_A v_\varepsilon - \triangle_A^k f\|_{L_A^p(\mathbb{R}_+)} < \delta/2 \text{ and } \|f_n - f\|_{L_A^p(\mathbb{R}_+)} < \frac{\delta}{2\|\triangle_A^k v_\varepsilon\|_{L_A^1(\mathbb{R}_+)}}. \qquad (43)$$

Thus, by virtue of remark 5 iii) and using Proposition 5, it follows that

$$\begin{aligned}\|\triangle_A^k (f_n *_A v_\varepsilon - f)\|_{L_A^p(\mathbb{R}_+)} &\leq \|f_n *_A \triangle_A^k v_\varepsilon - f *_A \triangle_A^k v_\varepsilon\|_{L_A^p(\mathbb{R}_+)} + \|\triangle_A^k f *_A v_\varepsilon - \triangle_A^k f\|_{L_A^p(\mathbb{R}_+)} \\ &\leq \|f_n - f\|_{L_A^p(\mathbb{R}_+)} \|\triangle_A^k v_\varepsilon\|_{L_A^1(\mathbb{R}_+)} + \|\triangle_A^k f *_A v_\varepsilon - \triangle_A^k f\|_{L_A^p(\mathbb{R}_+)} \\ &\leq \delta \qquad (44)\end{aligned}$$

Choosing the function v in $D_*(\mathbb{R})$, it follows that for all $\varepsilon > 0$ and all $n \in \mathbb{N}$, $f_n *_A v_\varepsilon \in D_*(\mathbb{R})$. And this achieves the proof of the proposition.

In the sequel, we give some result concerning the continuity of the Fourier transform $\mathcal{F}$ and its inverse. We start with the following Lemma deduced from the hypothesis of the function A

Lemma 2
i) For any real $a > 0$ there exist positive constants $C_1(a), C_2(a)$ such that for all $x \in [0, a]$,

$$C_1(a)x^{2\alpha+1} \leq A(x) \leq C_2(a)x^{2\alpha+1}.$$

ii) For $\rho > 0$,

$$A(x) \sim e^{2\rho x}, \ \ (x \longrightarrow +\infty)$$

iii) For $\rho = 0$,

$$A(x) \sim x^{2\alpha+1}, \ \ (x \longrightarrow +\infty)$$

Theorem 1 *The inverse of the Fourier transform $\mathcal{F}^{-1}$ defines a continuous linear map from $\mathcal{D}_*(\mathbb{R})$ into $\mathcal{D}_A^p$* if $p \in \left\{ \begin{array}{ll} [1, +\infty[, & \text{for } \rho = 0 \\ [2, \infty[, & \text{for } \rho > 0 \end{array} \right.$.

Proof. According to lemma 2 and using relation (4), it is not hard to see that $S_*^2(\mathbb{R}) \hookrightarrow D_A^p$, $p \in \left\{ \begin{array}{ll} [1, +\infty[, & \text{for } \rho = 0 \\ [2, \infty[, & \text{for } \rho > 0 \end{array} \right.$.
Thus, the result follows from the fact that, for all $k \in \mathbb{N}$, $\triangle_A^k \mathcal{F}^{-1}$ is continuous from $\mathcal{D}_*(\mathbb{R})$ into $\mathcal{S}_*^2(\mathbb{R})$.

Let $E_0 = \{f \in C_*^0(\mathbb{R})/x^{2k} f \in C_*^0(\mathbb{R}), k \in \mathbb{N}\}$ and $E_1 = \{f \in L_\gamma^2(\mathbb{R}_+)/x^{2k} f \in L_\gamma^2(\mathbb{R}_+), k \in \mathbb{N}\}$ equipped respectively with the topology generated by the countable norms

$$\mu_{m,\infty}^\gamma(f) = \max_{0 \leq k \leq m} \|\lambda^{2k} f\|_{L_\gamma^\infty(\mathbb{R}_+)}, \ \ m \in \mathbb{N}.$$

and

$$\mu_{m,2}^\gamma(f) = \max_{0 \leq k \leq m} \|\lambda^{2k} f\|_{L_\gamma^2(\mathbb{R}_+)}, \ \ m \in \mathbb{N}.$$

Thus, E_0 and E_1 are Fréchet spaces and we have

Theorem 2 *1)The Fourier transform $\mathcal{F}$ is a continuous from $\mathcal{D}_A^1$ into E_0.*
2) The Fourier transform $\mathcal{F}$ is an isomorphism from $\mathcal{D}_A^2$ onto E_1.
3) Let $p \in [1, 2]$. Then, for $r \in [1, p]$ and $q \in [1, p']$ with p' is the conjugate exponent of p, there exist $c > 0$ and $m \in \mathbb{N}$ such that for all $f \in \mathcal{D}_A^p$,

$$\|\mathcal{F}(f)\|_{L_\gamma^q(\mathbb{R}_+)} \leq c\, \gamma_{m,r}^A(f),$$

$\gamma_{m,r}^A(f)$ is finite or infinite. In particular, the Fourier transform $\mathcal{F}$ is a continuous from $\mathcal{D}_A^p$ into $L_\gamma^q(\mathbb{R}_+)$, $q \in [1, p']$.

Proof. For all $f \in \mathcal{D}_A^1$ (resp. $f \in \mathcal{D}_A^2$), we have

$$\mathcal{F}(\triangle_A f) = -\lambda^2 \mathcal{F}(f) \tag{45}$$

Then 1) and 2) follows respectively from (10) and Plancherel Theorem.
3) According to Proposition 3 and using inequality (31), we deduce that, for $r \in [1, p]$, there exist $c_1 > 0$ and $k \in \mathbb{N}$ such that for all $f \in \mathcal{D}_A^p$,

$$\|\mathcal{F}f\|_{L_\gamma^{p'}(\mathbb{R}_+)} \leq \|f\|_{L_A^p(\mathbb{R}_+)} \leq c_1 \gamma_{k,r}^A(f), \tag{46}$$

$\gamma_{m,r}^A(f)$ is finite or infinite.

On the other hand, using Holder inequality, it follows that for $q \in [1, p']$ there exist $c_2 > 0$ and $n \in \mathbb{N}$ such that for all $g \in \mathcal{D}_A^p$,

$$\|\mathcal{F}g\|_{L_\gamma^q(\mathbb{R}_+)} \leq c_2 \|\mathcal{F}((I - \triangle_A)^n g)\|_{L_\gamma^{p'}(\mathbb{R}_+)}. \tag{47}$$

Hence, the result follows by combining (46) and (47).

4 Titchmarsh's theorem for the Chébli-Trimèche transfrom $\mathcal{F}$ in $\mathcal{D}_A^2$

In [19], Titchmarsh characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms. More precisely, we have:

Theorem 3 *[19] Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalent:*

1) $\|f(t+h) - f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ as $h \longrightarrow 0$.

2) $\displaystyle\int_{|\lambda| \geq r} |g(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \longrightarrow \infty$. *Where g stands for the Fourier transform of f.*

In this section, we prove a generalization of Titchmarsh's theorem for the Chébli-Trimèche transfrom $\mathcal{F}$ for functions satisfying the Chébli-Trimèche-Lipschitz condition in $\mathcal{D}_A^2$.

Putting $V_\lambda(x) = \sqrt{A(x)}\varphi_\lambda(x)$, we see that V_λ satisfies

$$(L_\alpha + \chi(x) + \lambda^2)V_\lambda(x) = 0,$$

where L_α and $\chi(x)$ are defined by

$$(L_\alpha u)(x) = u"(x) - \frac{\alpha^2 - \frac{1}{4}}{x^2} u(x)$$

and

$$\chi(x) = \rho^2 - (2\alpha+1)\frac{B'(x)}{2xB(x)} - \frac{1}{2}\left(\frac{B'(x)}{B(x)}\right)' - \frac{1}{4}\left(\frac{B'(x)}{B(x)}\right)^2.$$

Thus, we have

$$V_\lambda(x) \sim x^{\alpha+\frac{1}{2}},\ V_\lambda'(x) \sim (\alpha+\frac{1}{2})x^{\alpha-\frac{1}{2}} \qquad (x \to 0).$$

We assume in this section that χ is holomorphic in a disc $D(0, 2b) = \{z \in \mathbb{C}, |z| < 2b\}$, $b > 0$. Therefore, from [13], we have:

Theorem 4 *The eigenfunction φ_λ can be expanded in a Bessel-function series as follows*

$$\varphi_\lambda(x) = \frac{x^{\alpha+\frac{1}{2}}}{2^\alpha\sqrt{A(x)}}\sum_{p=0}^{\infty}\frac{x^{2p}B_p(x)}{2^p\Gamma(\alpha+p+1)}j_{\alpha+p}(\lambda x) \tag{48}$$

where B_p are functions defined by a recursive relation and satisfy:

$$|B_p^{(q)}(x)| \leq (c/2)^p d^{p-1} b^{1-p-q}(p+q-1)! \tag{49}$$

for all $x \in [0,\ b]$, $p = 1, 2, ...$ and $q = 0, 1, 2, ...$, where

$$c = \max\{1 + |2\alpha - 1|,\ 2\sup(|\chi(z)|,\ |z| \leq 2b)\},\ d = (b + b^{-1})$$

and $B_0(x) = B_0 > 0$,

Corollary 1 *There exist $\eta > 0$, $N > 1$ and $c > 0$ such that*

$$\text{for all } \lambda > N \text{ and } \lambda x \in [\frac{\eta}{2},\ \eta],\ 1 - \varphi_\lambda(x) > c.$$

Proof. Firstly it can be observed from inequality (3) and the integral representation (6) that, for $\lambda x \in [0,\ \frac{\pi}{2}]$, $0 \leq \varphi_\lambda(x) \leq 1$.

Since $\varphi_\lambda(0) = 1$, then by virtue of Theorem 4 we deduce that

$$\lim_{x\to 0}\frac{B_0\, x^{\alpha+\frac{1}{2}}}{2^\alpha\Gamma(\alpha+1)\sqrt{A(x)}} = 1. \tag{50}$$

And from the expansion of the normalized Bessel function

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\alpha+n+1)}(\frac{x}{2})^{2n},$$

which implies that

$$\lim_{x\to 0}\frac{j_\alpha(x)-1}{x^2}\neq 0$$

and therefore, there exist $\kappa>0$ and $\sigma>0$ such that for every $\eta\in(0,\,\sigma]$, we have

$$\forall\, x\in[\frac{\eta}{2},\,\eta],\ |1-j_\alpha(x)|=1-j_\alpha(x)\geq\kappa x^2\geq\kappa\frac{\eta^2}{4}. \tag{51}$$

Thus, there exists $\delta\in(0,\,1)$ such that for all $\lambda,\,x\geq 0$ satisfying $\lambda x\in[\frac{\eta}{2},\,\eta]$ we have

$$0\leq j_\alpha(\lambda x)<\delta. \tag{52}$$

and from (50) one can easily see that for $\delta_1\in(\delta,\,1)$ there exist $N>1$ such that for $x\leq\dfrac{\eta}{N}$

$$0\leq\frac{B_0\,x^{\alpha+\frac12}}{2^\alpha\Gamma(\alpha+1)\sqrt{A(x)}}<\frac{\delta_1}{\delta}. \tag{53}$$

Combining (52) and (53) we deduce that for $\lambda\geq N$ and $\lambda x\in[\dfrac{\eta}{2},\,\eta]$

$$0\leq\frac{B_0\,x^{\alpha+\frac12}j_\alpha(\lambda x)}{2^\alpha\Gamma(\alpha+1)\sqrt{A(x)}}<\delta_1. \tag{54}$$

On the other hand, using the inequality (49), we have for all $p\in\mathbb{N}\backslash\{0\}$

$$|\frac{x^{2p}B_p(x)j_{\alpha+p}(\lambda x)}{2^p\Gamma(\alpha+p+1)}|\leq\frac{b}{d}(\frac{cdx^2}{4b})^p\frac{\Gamma(p)}{\Gamma(p+\alpha+1)}\leq\frac{1}{\Gamma(\alpha+2)}(\frac{cdx^2}{4b})^p,\ x\in[0,\,b].$$

Thus, by virtue of (53) and choosing $\eta\leq\min\{\sigma,\,b,\,\frac{1}{\sqrt2},\,\sqrt{\frac{\delta B_0(\delta_2-\delta_1)}{4\delta_1}}\}$ with $\delta_2\in(\delta_1,\,1)$, and $N\geq\max\{1,\,\sqrt{\frac{cd}{4b}}\}$, we have for $x\in[0,\,\dfrac{\eta}{N}]$

$$\begin{aligned}|\frac{x^{\alpha+\frac12}}{2^\alpha\sqrt{A(x)}}\sum_{p=1}^{\infty}\frac{x^{2p}B_p(x)}{2^p\Gamma(\alpha+p+1)}j_{\alpha+p}(\lambda x)| &\leq \frac{x^{\alpha+\frac12}}{2^\alpha\Gamma(\alpha+2)\sqrt{A(x)}}\sum_{p=1}^{\infty}(\frac{cdx^2}{4b})^p\\ &\leq \frac{2x^{\alpha+\frac12}}{2^\alpha\Gamma(\alpha+1)\sqrt{A(x)}}\sum_{p=1}^{\infty}\eta^{2p}\\ &\leq \frac{B_0\,x^{\alpha+\frac12}}{2^\alpha\Gamma(\alpha+1)\sqrt{A(x)}}\frac{\delta(\delta_2-\delta_1)}{\delta_1}\\ &\leq \delta_2-\delta_1.\end{aligned} \tag{55}$$

Therefore, according to the expansion of φ_λ given by (48) and using inequalities (54) and (55), it follows that for $\lambda \geq N$ and $\lambda x \in [\frac{\eta}{2}, \eta]$

$$1 - \varphi_\lambda(x) \geq 1 - \delta_2 > 0,$$

and this achieves the proof of the Corollary.

Definition 5 *Let $\delta \in (0, 1)$. A function $f \in \mathcal{D}^2_A$ is said to be in the Chébli Trimèche Lipschitz class, denoted by $Lip(\delta, 2)$, if*

$$\forall\ k \in \mathbb{N},\ \forall\ r \in \mathbb{N},\ \|L^k_x \triangle^r_A f\|_{L^2_A(\mathbb{R}_+)} = O(x^\delta) \quad \text{as} \quad x \to 0,$$

where $L_x f = \tau^A_x f - f$.

To prove the main result of this section we need the following Lemma

Lemma 3 *Let $f \in \mathcal{D}^2_A$. Then for every $k \in \mathbb{N}$ and $r \in \mathbb{N}$, we have*

$$\|L^k_x \triangle^r_A f\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda).$$

Proof. It is easily to see that for all $k \in \mathbb{N}$

$$L^k_x f(y) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (\tau^A_x)^i f(y)$$

Thus, by virtue of Proposition 4 v) and using the fact that

$$\mathcal{F}(\triangle^r_A f)(\lambda) = (-\lambda^2)^r \mathcal{F}(f)(\lambda)$$

it follows that

$$\begin{aligned}
\mathcal{F}(L^k_x \triangle^r_A f)(\lambda) &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \mathcal{F}((\tau^A_x)^i \triangle^r_A f)(\lambda) \\
&= (-\lambda^2)^r \mathcal{F}(f)(\lambda) \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \varphi^i_\lambda(x) \\
&= (-\lambda^2)^r \mathcal{F}(f)(\lambda) (\varphi_\lambda(x) - 1)^k.
\end{aligned}$$

Hence, from Plancherel formula for $\mathcal{F}$, we deduce the result.

Theorem 5 *Let $f \in \mathcal{D}^2_A$. Then the following are equivalents*

i) $f \in Lip(\delta, 2)$.

ii) $\forall\ r \in \mathbb{N},\ \displaystyle\int_s^\infty \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(s^{-2\delta}) \quad \text{as} \quad s \to +\infty.$

Proof. Suppose that $f \in Lip(\delta, 2)$. According to Corollary 1 we deduce that there exist $\eta > 0$, $a > 0$ and $c > 0$ such that for $x \in (0, a)$ and for all $r \in \mathbb{N}$, we have

$$\int_{\frac{\eta}{2x}}^{\frac{\eta}{x}} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq \frac{1}{c^{2k}} \int_0^\infty \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda), \ k \in \mathbb{N}$$

which implies, from Lemma 3, that

$$\int_{\frac{\eta}{2x}}^{\frac{\eta}{x}} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(x^{2\delta}) \quad \text{as} \quad x \to 0.$$

Therefore, there exists $C > 0$ such that

$$\int_s^{2s} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq C\, s^{-2\delta}.$$

Thus, there exists $C_1 > 0$ satisfying

$$\int_s^\infty \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = \sum_{i=0}^\infty \int_{2^i s}^{2^{i+1} s} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq C_1\, s^{-2\delta}$$

and consequently ii) is proved. Lets prove now that ii) $\Rightarrow$ i). We have

$$\begin{aligned} \int_0^\infty \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) &= \int_0^{\frac{\eta}{x}} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \\ &+ \int_{\frac{\eta}{x}}^\infty \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda). \end{aligned}$$

By virtue of i), it follows that

$$\int_{\frac{\eta}{x}}^\infty \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \leq 4^k \int_{\frac{\eta}{x}}^\infty \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(x^{2\delta}). \quad (56)$$

On the other hand, estimate $1 - \varphi_\lambda(x)$ using the mean value theorem and inequality (5), we get for $x \leq 1$

$$\begin{aligned} \int_0^{\frac{\eta}{x}} \lambda^{4r} |1 - \varphi_\lambda(x)|^{2k} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) &\leq 4^k \int_0^{\frac{\eta}{x}} \lambda^{4r} |1 - \varphi_\lambda(x)| |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \\ &\leq C 4^k x^2 \left\{ \int_0^{\frac{\eta}{x}} \lambda^{4r+2} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \right. \\ &+ \left. \rho^2 \int_0^{\frac{\eta}{x}} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda). \right. \end{aligned}$$

But $\delta < 1$, then we have

$$\begin{aligned} 4^k x^2 \rho^2 \int_0^{\frac{\eta}{x}} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) &\leq 4^k x^2 \rho^2 \int_0^{\infty} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) \\ &= 4^k x^2 \rho^2 \|\triangle_A^r f\|^2_{L^2_A(\mathbb{R}_+)} \\ &= O(x^{2\delta}). \end{aligned}$$

Now, by putting $\psi(s) = \displaystyle\int_s^{\infty} \lambda^{4r} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda)$ and using integration by parts, we obtain

$$\begin{aligned} 4^k x^2 \int_0^{\frac{\eta}{x}} \lambda^{4r+2} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) &= 4^k x^2 \int_0^{\frac{\eta}{x}} -s^2 \psi'(s) ds \\ &= 4^k (-\psi(x) + 2x^2 \int_0^{\frac{\eta}{x}} s\psi(s) ds). \end{aligned}$$

Therefore, using the fact that $\psi(s) = O(s^{-2\delta})$, it is not hard to see that

$$4^k x^2 \int_0^{\frac{\eta}{x}} \lambda^{4r+2} |\mathcal{F}(f)(\lambda)|^2 d\gamma(\lambda) = O(x^{2\delta})$$

and the theorem is proved.

5 Open Problem

The purpose of the future work is to generalize the Titchmarsh's theorem for the Chébli-Trimèche transfrom $\mathcal{F}$ for functions satisfying the Chébli-Trimèche-Lipschitz condition in $\mathcal{D}_A^p$ with $1 \leq p \leq 2$.

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