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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH DIFFERENTIAL EQUATION AND SUBORDINATION

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Abstract

The main purpose of this paper is to further investigate the subclass $TS_m^l(\alpha, \beta, \gamma)$ by applying the differential subordination theorem. Also obtained are subordination results on a convex function and an incomplete beta function. Furthermore, we discussed certain subordination results for the function that belongs to the class $TS_m^l(\alpha, \beta, \gamma)$ with a Cauchy-Euler differential equation.

Keywords: *Subordination, Analytic functions, Convex function, Differential equation, Subordinating factor sequence, Hadamard product (or Convolution).*

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1 Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f^{'}(0) - 1 = 0$.

Also let S be the subclass of A consisting of univalent functions and K denotes the class of convex functions such that

$$K = \left\{ f \in A : Re\frac{zf^{''}(z)}{f^{'}(z)} + 1 > 0, z \in U \right\}.$$

Furthermore, we denote T as the subclass of S whose elements can be expressed in the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \tag{2}$$

introduced and studied by Silverman [1].

In 2010, Murugusundaramorthy and Magesh [2] introduced and studied the class of functions $f(z)$ defined by (2) satisfying the following conditions:

$$Re\left\{ 1+\frac{1}{\gamma}\left(\frac{z(H^l_m[\alpha_1]f(z))^{'}}{H^l_m[\alpha_1]f(z)} - \alpha\right)\right\} > \beta\left|1+\frac{1}{\gamma}\left(\frac{H^l_m([\alpha_1]f(z))^{'}}{H^l_m[\alpha_1]f(z)} - 1\right)\right|, z \in U \tag{3}$$

for $-1 \leq \alpha < 1, \beta \geq 0$, and γ is a non-zero complex.
This class is denoted by $TS^l_m(\alpha, \beta, \gamma)$.
We note that

$$H^l_m[\alpha_1]f(z) := z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n, \tag{4}$$

and

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \cdots, (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots, (\beta_m)_{n-1}} \frac{1}{(n-1)!}. \tag{5}$$

For this class, the authors obtained some geometric properties such as coefficient estimates, extreme points, the radii of close-to-convexity, starlikeness, convexity and neighbourhood results (for details see, [2]).

In this paper, motivated by the techniques in [3], we further investigate the class $TS^l_m(\alpha, \beta, \gamma)$ by applying the subordination differetial theorem.

2 Preliminaries and Definitions

We state some basic results which are relevant to our main results and also give certain fundamental definitions.

Definition 1: Let $f(z)$ and $g(z)$ be analytic in U, where $f(z)$ is as given in (1) and $g(z)$ is defined as

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{6}$$

The function $f(z)$ is subordinate to $g(z)$ (written as $f \prec g$) if there exists a function $\varphi(z)$ analytic (not necessarily univalent)in U and satisfying $\varphi(0) = 0, |\varphi(z)| < 1$ such that $f(z) = g(\varphi(z))$ for $z \in U$.

Definition 2: (Hadamard product or convolution)
Given two functions $f(z)$ and $g(z)$ where $f(z)$ is as defined in (1) and $g(z)$ is given by (6), then the Hadamard product (or convolution) $f * g$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{7}$$

The function $(f * g)(z)$ is also analytic in U.

Definition 3: (Subordinating factor sequence)
A sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1) is analytic, univalent and convex in U, the subordination is given by

$$\sum_{n=1}^{\infty} a_n c_n z^n \prec f(z), z \in U, a_1 = 1. \tag{8}$$

Theorem 1: [4]
The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1 + 2\sum_{n=1}^{\infty} c_n z^n\right\} > 0, z \in U. \tag{9}$$

Theorem 2: [5]
Let $0 < a \leq c$. If $c \geq 2$ or $a + c \geq 3$, then the function

$$h(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, (z \in U) \tag{10}$$

belongs to the class K of convex functions.

In [2], Murugusundaramorthy and Magesh proved the necessary and sufficient condition for functions $f \in T$ to be in the class $TS_m^l(\alpha, \beta, \gamma)$

which is equivalent to the following Theorem:

Theorem 3: [2]
A function $f \in T$ is in the class $TS_m^l(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n|a_n| \leq (1-\alpha)+|\gamma|(1-\beta), \tag{11}$$

where $-1 \leq \alpha < 1, \beta \geq 0, \gamma \in C\{0\}$ and Γ_n is as defined in (5).

Theorem 4: [6] If the function $f(z)$ and $g(z)$ are analytic in U with $g(z) \prec f(z)$, then for $p > 0$ and $z = re^{i\theta}, (0 < r < 1)$, we have

$$\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |g(z)|^p d\theta \tag{12}$$

3 Main Results

3.1 CERTAIN SUBORDINATION PROPERTIES OF THE CLASS $TS_m^l(\alpha, \beta, \gamma)$

We begin with the following theorem: **Theorem 5:**
Let $f \in TS_m^l(\alpha, \beta, \gamma)$, then

$$\frac{A}{2(A+B)}(f*g)(z) \prec g(z) \tag{13}$$

and

$$\begin{aligned}&\frac{A}{2(A+B)}\int_0^{2\pi} |(f*g)(re^{i\theta})|^p d\theta \\ &\leq 2\int_0^{2\pi} |g(re^{i\theta})|^p d\theta.\end{aligned} \tag{14}$$

where

$$A = [(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_2$$

, and

$$B = [(1-\alpha)+|\gamma|(1-\beta)],$$

for $p > 0$, $0 < |z| = r < 1$, and $g \in K$.
The constant factor

$$\frac{A}{2(A+B)} = \frac{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_2}{2[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_2+[(1-\alpha)+|\gamma|(1-\beta)]}$$

in the subordination result (13) cannot be replaced by a larger one.
Moreover the result is sharp for the function

$$f_0(z) = z - \frac{1}{A}\Big(Az - Bz^2\Big). \tag{15}$$

ProofLet $f(z) \in TS_m^l(\alpha,\beta,\gamma)$, then

$$\frac{A}{2(A+B)}(f*g)(z) = \frac{A}{2(A+B)}\left(z + \sum_{n=2}^{\infty} a_n b_n z^n\right) \tag{16}$$

where $g(z) = \sum_{n=2}^{\infty} b_n z^n \in K$.
By Theorem 1, it is sufficient to show that

$$\begin{aligned} &Re\left\{1 + 2\sum_{n=1}^{\infty} \frac{A}{2(A+B)} a_n z^n\right\} \\ &= Re\left\{1 + \sum_{n=1}^{\infty} \frac{A}{(A+B)} a_n z^n\right\} > 0, z \in U \end{aligned} \tag{17}$$

inorder that the subordination (13) should hold true.
Thus it implies that the sequence

$$\left\{\frac{A}{(A+B)} a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$.
Now,

$$\begin{aligned} &Re\left\{1 + \sum_{n=1}^{\infty} \left(\frac{A}{(A+B)}\right) a_n z^n\right\} \\ &= Re\left\{1 + \left(\frac{A}{A+B}\right) a_1 z + \frac{1}{A+B}\sum_{n=2}^{\infty} A a_n z^n\right\} \\ &\geq 1 - \frac{A}{A+B} r - \frac{1}{A+B}\sum_{n=2}^{\infty} A|a_n| r^n \end{aligned} \tag{18}$$

Since $\psi(n) = [(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n, (n = 2, 3, ...)$ is an increasing function of n, so $0 < A \leq \psi(n), (n = 2, 3, ...)$ where $\psi(n) = [(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n$; we have

$$\begin{aligned} & Re\left\{1+\sum_{n=1}^{\infty}\left(\frac{A}{(A+B)}\right)a_n z^n\right\} \\ & = Re\left\{1+\left(\frac{A}{A+B}\right)a_1 z+\frac{1}{A+B}\sum_{n=2}^{\infty}\psi(n)a_n z^n\right\} \\ & > 1-\left(\frac{A}{A+B}\right)r-\left(\frac{B}{A+B}\right)r \\ & = 1-\left(\frac{A+B}{A+B}\right)r = 1-r > 0 \end{aligned} \tag{19}$$

which establishes (17) and consequently establish (13).
Note that by (11)

$$\begin{aligned} & \sum_{n=2}^{\infty}\psi(n)|a_n| \\ & = \sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n|a_n| \leq (1-\alpha)+|\gamma|(1-\beta) = B \end{aligned}$$

The inequality (14) follows from (13) and Theorem 4.
In order to prove the sharpness of the constant factor

$$\frac{A}{2(A+B)} = \frac{2[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_2}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)\Gamma_2+[(1-\alpha)+|\gamma|(1-\beta)]},$$

we consider the function $f_0(z) \in TS_m^l(\alpha,\beta,\gamma)$ given by (15).
Thus from (13), we have

$$\frac{A}{2(A+B)}f_0(z) \prec \frac{z}{1-z}, (z \in U) \tag{20}$$

by taking $g(z) = \dfrac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$.
Moreover, it can easily be verified for the function $f_0(z)$ given by (15) that

$$Min\left\{Re\frac{A}{2(A+B)}f_0(z)\right\} = -\frac{1}{2}, \quad (z \in U). \tag{21}$$

This shows that the constant

$$\frac{A}{A+B} = \frac{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_2}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)\Gamma_2+[(1-\alpha)+|\gamma|(1-\beta)]},$$

is the best possible. This completes the proof of Theorem 5.

Corollary 6
Let $f(z) \in TS^l_m(\alpha, \beta, \gamma)$ and
$F(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$ where $(f * h(a, c; z))(z) = F(z)$ then

$$\frac{A}{A+B}(f * h(a, c; z))(z) \prec 2h(a, c; z). \tag{22}$$

and

$$Ref(z) > -\left(\frac{A+B}{A}\right) \tag{23}$$

where $h(a, c; z)$ is as defined in (10)

Proof.
Using Theorem 5, we have that

$$h(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \in K,$$

Since $0 < a \leq c, c \geq 2$ or $a + c \geq 3$.
Now, by taking $g(z) = h(a, c; z) = z + \sum_{n=2}^{\infty} z^n$ in theorem 1, respectively, the result (22) and (23) are obtained.

Corollary 6
Let $f(z) \in TS^l_m(\alpha, \beta, \gamma)$ in U and $p > 0, 0 < |z| = r < 1$, then for function $g \in K$

$$\frac{4\Gamma_n}{3 + 4\Gamma_n}(f * g)(z) \prec 2g(z) \tag{24}$$

and

$$\frac{4\Gamma_n}{3 + 4\Gamma_n} \int_0^{2\pi} |(f * g)(re^{i\theta})|^p d\theta \leq 2 \int_0^{2\pi} |g(re^{1\theta})|^p d\theta. \tag{25}$$

Proof.
By taking $\gamma = -1, \alpha = -1$ *and* $\beta = 0$ in Theorem 5, Corollary 6 is obtained.

Remark 3.1. *For $\gamma = \pm i, \alpha = -1$ and $\beta = 0$ in Theorem 5 we obtain the result in Corollary 6.*

Corollary 7
Let $f(z) \in TS^l_m(0, 0, \pm i)$ in U and $p > 0, 0 < |z| = r < 1$, then for function $g \in K$

$$\frac{3\Gamma_n}{2 + 3\Gamma_n}(f * g)(z) \prec 2g(z) \tag{26}$$

and

$$\frac{3\Gamma_n}{2+3\Gamma_n}\int_0^{2\pi}|(f*g)(re^{i\theta})|^p d\theta \leq 2\int_0^{2\pi}|g(re^{1\theta})|^p d\theta. \tag{27}$$

3.2 OTHER SUBORDINATION PROPERTIES OF THE CLASS $TS_m^l(\alpha,\beta,\gamma)$ WITH FIXED EQUATION

Our next results in this section are on the functions in the class $TS_m^l(\alpha,\beta,\gamma)$ which are associated with the following non-homogeneous Cauchy-Euler differential

$$\begin{aligned} &z^2\frac{d^2q}{dz^2}+2(\mu+1)z\frac{dq}{dz}+\mu(\mu+1)q \\ &=(\mu+1)(\mu+2)z+(\mu+2)(\mu+3)\sum_{n=2}^{\infty}c_nz^n \end{aligned} \tag{28}$$

for $q(z)\in T, f(z)\in TS_m^l(\alpha,\beta,\gamma), \mu+1>0, \mu\in\Re$.

Remark 3.2. *Equation* (28) *corrects the one in [3]*

O. Altintas, et al. [7] had earlier used Cauchy-Euler differential equation to study the distortion inequalities and neighbourhood problems of the other class of functions.

Theorem 8: Let the function $q(z)=z+\sum_{n=2}^{\infty}c_nz^n$ be in T and satisfy the equation (28) with $f(z)\in TS_m^l(\alpha,\beta,\gamma)$, then

$$\frac{A(\mu+3)}{B(\mu+1)+(\mu+3)A}(q*g)(z)\prec 2g(z) \tag{29}$$

and

$$\frac{A(\mu+3)}{B(\mu+1)+(\mu+3)A}\int_0^{2\pi}|(q*g)(re^{i\theta})|^p d\theta \leq 2\int_0^{2\pi}|g(re^{i\theta})|^p d\theta \tag{30}$$

for function $g\in K, 0<|z|=r<1, p>0.$ and $\psi(2)$ as defined earlier.

ProofSuppose $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in K$, then

$$\begin{aligned} &\frac{A(\mu+3)}{B(\mu+1)+(\mu+3)A}(q*g)(z)=\frac{A(\mu+3)}{2[A(\mu+3)+B(\mu+1)]}\left(z+\sum_{n=2}^{\infty}b_nc_nz^n\right) \\ &=\frac{A(\mu+3)}{2[A(\mu+3)+B(\mu+1)]}z+\frac{A(\mu+3)}{2[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}b_nc_nz^n, \end{aligned} \tag{31}$$

By Theorem 1, it is sufficient to show that

$$\begin{aligned} &Re\left\{1+2\sum_{n=1}^{\infty}\frac{A(\mu+3)}{2[A(\mu+3)+B(\mu+1)]}c_n z^n\right\} \\ &= Re\left\{1+\sum_{n=1}^{\infty}\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}c_n z^n\right\} > 0, \end{aligned} \tag{32}$$

$z \in U$ inorder that the subordination (29) should hold true.
Thus it implies that the sequence

$$\left\{\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}c_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$.

Now,

$$\begin{aligned} &Re\left\{1+\sum_{n=1}^{\infty}\left(\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}\right)a_n z^n\right\} \\ &= Re\left\{1+\left(\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}\right)a_1 z \right. \\ &\left. + \frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}\psi(2)c_n z^n\right\} \\ &\geq 1-\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}r-\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}A|c_n|r^n \end{aligned} \tag{33}$$

Because $L(z)$ satisfies the differential equation with the $f(z) \in TS_m^l(\alpha,\beta,\gamma)$, so

$$c_n = \frac{(\mu+1)(\mu+2)}{(n+\mu)(n+\mu+1)}a_n.$$

Following (33), we have

$$\begin{aligned}
&\geq 1-\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}r\\
&-\frac{(\mu+3)}{[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}A\frac{(\mu+1)(\mu+2)}{(n+\mu)(n+\mu+1)}|a_n|r^n\\
&\geq 1-\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}r\\
&-\frac{(\mu+3)}{[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}A\frac{(\mu+1)(\mu+2)}{(2+\mu)(\mu+3)}|a_n|r^n\\
&1-\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}r\\
&-\frac{(\mu+1)}{[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}A|a_n|r^n
\end{aligned} \tag{34}$$

Since $\psi(n)=[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n, (n=2,3,...)$ is an increasing function of n, so $0<A\leq\psi(n), (n=2,3,...)$ where $\psi(n)=[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n$.
Following (34)we have

$$\begin{aligned}
&> Re\left\{1-\frac{A(\mu+3)}{[A(\mu+3)+B(\mu+1)]}r-\frac{(\mu+1)}{[A(\mu+3)+B(\mu+1)]}\sum_{n=2}^{\infty}\psi(n)|a_n|r^n\right\}\\
&> 1-\left(\frac{A(\mu+3)+(\mu+1)[(1-\alpha)+|\gamma|(1-\beta)]}{(\mu+1)[(1-\alpha)+|\gamma|(1-\beta)]+A(\mu+3)}\right)r\\
&= 1-r>0 \ \textit{since}\ 0<r<1.
\end{aligned} \tag{35}$$

which thus establishes (32) and consequently establish (29).
Note that by (11)

$$\sum_{n=2}^{\infty}\psi(n)|a_n|=\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Gamma_n|a_n|\leq(1-\alpha)+|\gamma|(1-\beta)=B$$

The inequality (30) follows from (13) and Theorem 4.

Corollary 9
Let $q(z)=\sum_{n=2}^{\infty}c_nz^n\in T$ satisfy (28) with $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in TS_m^l(\alpha,\beta,\gamma)$ and $F(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^n$,
then

A$(\mu+3)+(\mu+1)[(1-\alpha)+|\gamma|(1-\beta)]F(z) \prec 2h(a,c;z)$(36)and

$$Req(z) > -\left(\frac{A(\mu+3)+(\mu+1)[(1-\alpha)+|\gamma|(1-\beta)]}{A(\mu+3)}\right). \tag{37}$$

where $h(a,c;z)$ is as defined in (10) with $0 < a \le c, c \ge 2$ or $a+c \ge 3$ and $|z| = r < 1, p > 0$.

Proof

Since $0 < a \le c, c \ge 2$ or $a+c \ge 3$, using Theorem 5, we have that

$$h(a,c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \in K.$$

Taking $g(z) = h(a,c;z) = z + \sum_{n=2}^{\infty} z^n$ in theorem 8, respectively, the results (36) and (37) are obtained.

By taking $\beta = 0, \gamma = \pm i$, in Theorem 8 we have the following:

Corollary 10

Let the function $q(z) = z+\sum_{n=2}^{\infty} c_n z^n$ satisfy (28) with $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in TS_m^l(\alpha, 0, \pm i)$, then for $g \in K$

$$\frac{(\mu+3)(3-\alpha)\Gamma_2}{(\mu+1)(2-\alpha)+(\mu+3)(3-\alpha)\Gamma_2}(q*g)(z) \prec 2g(z), z \in U \tag{38}$$

and

$$\frac{(\mu+3)(3-\alpha)\Gamma_2}{(\mu+1)(2-\alpha)+(\mu+3)(3-\alpha)\Gamma_2}\int_0^{2\pi} |(q*g)(re^{i\theta})|^p d\theta \le 2\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \tag{39}$$

By taking $\alpha = -1, \gamma = \pm i$, in Theorem 8 we have the following:

Corollary 11

Let $q(z)$ satisfy (28) with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in TS_m^l(\alpha, 0, \pm i)$, then for $g \in K$

$$\frac{2(\mu+3)(2-\beta)\Gamma_2}{(\mu+1)(2-\beta)+2(\mu+3)(2-\beta)\Gamma_2}(q*g)(z) \prec 2g(z), z \in U \tag{40}$$

and

$$\frac{2(\mu+3)(2-\beta)\Gamma_2}{(\mu+1)(2-\beta)+2(\mu+3)(2-\beta)\Gamma_2}\int_0^{2\pi} |(q*g)(re^{i\theta})|^p d\theta \le 2\int_0^{2\pi} |g(re^{i\theta})|^p d\theta. \tag{41}$$

4 Open Problem

Some of the results obtained in this paper (e.g. Theorem 8 and the Corollaries arisen from it) may be extended further.

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