*Int. J. Open Problems Complex Analysis, Vol. 7, No. 3, July 2015*
*ISSN 2074-2827; Copyright ©ICSRS Publication, 2015*
*www.i-csrs.org*

# Some Characterizations for Subclasses of Meromorphic Bazilevic Functions Associated with Cho-Kwon-Srivastava Operator

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Received 5 January 2015; Accepted 25 February 2015

**Abstract**

***The purpose of this paper is to introduce two subclasses of meromorphic functions by using Cho-Kwon-Srivastava operator and to investigate various characterizations for these subclasses.***

**Keywords and phrases:** *Meromorphic, Bazilevic functions, Hadamard product (or convolution), Cho-Kwon-Srivastava operator.*
**2000 Mathematical Subject Classification:** 30C45, 30C50.

## 1. Introduction

Let $\Sigma$ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}$.

Let $\mathcal{P}_k(\rho)$ be the class of functions $p(z)$ analytic in $\mathbb{U}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{Rep(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \tag{2}$$

where $k \geq 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan and Parvatham [15]. For $\rho = 0$, the class $\mathcal{P}_k(0) = \mathcal{P}_k$ introduced by Pinchuk [17]. Also, $\mathcal{P}_2(\rho) = \mathcal{P}(\rho)$, where $\mathcal{P}(\rho)$ is the class of functions with positive real part greater than $\rho$ and $\mathcal{P}_2(0) = \mathcal{P}$, is the class of functions with positive real part. From (2), we have $p(z) \in \mathcal{P}_k(\rho)$ if and only if there exist $p_1, p_2 \in \mathcal{P}(\rho)$ such that

$$p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z) \ (z \in \mathbb{U}). \tag{3}$$

It is known that the class $\mathcal{P}_k(\rho)$ is a convex set (see [10]).

The Hadamard product (or convolution) $f(z) * g(z)$ of $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \tag{4}$$

is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z). \tag{5}$$

Define the function

$$f_\alpha(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left( \frac{n+1+\lambda}{\lambda} \right)^\alpha z^k \ (\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \mathbb{N} = \{1, 2, 3, ...\};\ \lambda > 0),$$

and let the associated function $f_{\alpha,\mu}^*(z)$ defined by the Hadamard product (or convolution):

$$f_\alpha(z) * f_{\alpha,\mu}^*(z) = \frac{1}{z(1-z)^\mu} \ (\mu > 0;\ z \in \mathbb{U}^*).$$

Then, we have

$$\mathcal{I}_{\lambda,\mu}^\alpha f(z) = f_{\alpha,\mu}^*(z) * f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left( \frac{\lambda}{n+1+\lambda} \right)^\alpha \frac{(\mu)_{n+1}}{(1)_{n+1}} a_k z^k. \tag{6}$$

We note that

$$\mathcal{I}_{1,2}^{0}f(z)=zf'(z)+2f(z) \text{ and } \mathcal{I}_{1,2}^{1}f(z)=f(z).$$

It is easy to verify from (6) that

$$z\left(\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)\right)'=\mu\mathcal{I}_{\lambda,\mu+1}^{\alpha}f(z)-(\mu+1)\mathcal{I}_{\lambda,\mu}^{\alpha}f(z), \tag{7}$$

and

$$z\left(\mathcal{I}_{\lambda,\mu}^{\alpha+1}f(z)\right)'=\lambda\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)-(\lambda+1)\mathcal{I}_{\lambda,\mu}^{\alpha+1}f(z). \tag{8}$$

The operator $\mathcal{I}_{\lambda,\mu}^{\alpha}$ was introduced by Cho–kwon–Srivastava [3] (see also, Piejko and Sokol [16]). The definition of the operator $\mathcal{I}_{\lambda,\mu}^{\alpha}$ was motivated essentially by the Choi–Saigo–Srivastava operator [4] for analytic functions which includes a simpler integral operator studied earlier by others (see [6, 7] and [11, 12]).

Also, we note that:
$\mathcal{I}_{\delta,1}^{\alpha}f(z)=P_{\delta}^{\alpha}f(z)$ (see Lashin [5], see also Bulboacă et al. [2, with $\lambda=1$]).

Next, by using the operator $\mathcal{I}_{\lambda,\mu}^{\alpha}$, we introduce two subclasses of meromorphic Bazilevic functions of $\Sigma$ as follows:
**Definition 1.** A function $f(z)\in\Sigma$ is said to be in the class $\mathcal{M}_{\lambda,\mu}^{\alpha}(\beta,\gamma,\rho,k)$ if it satisfies the condition:

$$\left[(1-\gamma)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)\right)^{\beta}+\gamma\left(\frac{\mathcal{I}_{\lambda,\mu+1}^{\alpha}f(z)}{\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)}\right)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)\right)^{\beta}\right]\in\mathcal{P}_k(\rho),$$

$$\left(k\geq 2;\ \alpha\in\mathbb{N}_0;\ \beta,\gamma,\mu,\lambda>0;\ 0\leq\rho<1\right). \tag{9}$$

**Definition 2.** A function $f(z)\in\Sigma$ is said to be in the class $\mathcal{N}_{\lambda,\mu}^{\alpha}(\beta,\gamma,\rho,k)$ if it satisfies the condition:

$$\left[(1-\gamma)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha+1}f(z)\right)^{\beta}+\gamma\left(\frac{\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)}{\mathcal{I}_{\lambda,\mu}^{\alpha+1}f(z)}\right)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha+1}f(z)\right)^{\beta}\right]\in\mathcal{P}_k(\rho),$$

$$\left(k\geq 2;\ \alpha\in\mathbb{N}_0;\ \beta,\gamma,\mu,\lambda>0;\ 0\leq\rho<1\right). \tag{10}$$

## 2. Main results

Unless otherwise mentioned, we assume throughout this paper that $k\geq 2,\ \alpha\in\mathbb{N}_0,\ \beta,\gamma,\mu,\lambda>0$ and $0\leq\rho<1$.

To establish our results, we need the following lemma due to Miller and Mocanu [8].

**Lemma 2.1** [8]. Let $\phi(u,v)$ be a complex valued function $\phi : D \to \mathbb{C},\ D \subset \mathbb{C}^2$ and let $u = u_1 + iu_2,\ v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies
(i) $\phi(u,v)$ is continuous in $D$;
(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{n}{2}(1+u_2^2),\ Re\{\phi(iu_2, v_1)\} \le 0$.
Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$ be regular in $\mathbb{U}$ such that $(p(z), zp'(z)) \in D$ for all $z \in \mathbb{U}$. If $Re\{\phi(p(z), zp'(z))\} > 0$ for all $z \in \mathbb{U}$, then $Re p(z) > 0$.

Employing the techniques used by Owa [14] for univalent functions, Noor and Muhammad [13], Aouf and Seoudy [1] for multivalent functions and Mostafa et al. [9] for meromorphic multivalent functions, we prove the following theorems.
**Theorem 2.1.** If $f(z) \in \mathcal{M}_{\lambda,\mu}^{\alpha}(\beta,\gamma,\rho,k)$, then

$$\left(z\mathcal{I}_{\lambda,\mu}^{\alpha} f(z)\right)^{\beta} \in \mathcal{P}_k(\rho_1), \tag{11}$$

where $\rho_1$ is given by

$$\rho_1 = \frac{2\mu\beta\rho + n\gamma}{2\mu\beta + n\gamma}\ (0 \le \rho_1 < 1). \tag{12}$$

**Proof.** Let

$$\left(z\mathcal{I}_{\lambda,\mu}^{\alpha} f(z)\right)^{\beta} = (1-\rho_1)p(z) + \rho_1$$
$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left[(1-\rho_1)p_1(z) + \rho_1\right] - \left(\frac{k}{4} - \frac{1}{2}\right)\left[(1-\rho_1)p_2(z) + \rho_1\right], \tag{13}$$

where $p_i(z)$ is analytic in $\mathbb{U}$ with $p_i(0) = 1$ for $i = 1,2$. Differentiating (13) with respect to $z$, and using identity (7) in the resulting equation, we get

$$\begin{aligned} &\left[(1-\gamma)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha} f(z)\right)^{\beta} + \gamma\left(\frac{\mathcal{I}_{\lambda,\mu+1}^{\alpha} f(z)}{\mathcal{I}_{\lambda,\mu}^{\alpha} f(z)}\right)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha} f(z)\right)^{\beta}\right] \\ =\ &\left[(1-\rho_1)p(z) + \rho_1\right] + \frac{\gamma(1-\rho_1)zp'(z)}{\mu\beta} \in \mathcal{P}_k(\rho). \end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{\left[(1-\rho_1)p_i(z) + \rho_1\right] - \rho + \frac{\gamma(1-\rho_1)zp_i'(z)}{\mu\beta}\right\} \in \mathcal{P}\ (i = 1,2).$$

Defining the function

$$\phi(u,v) = \left[(1-\rho_1)u + \rho_1\right] - \rho + \frac{\gamma(1-\rho_1)v}{\mu\beta}$$

where $u = p_i(z) = u_1 + iu_2,\ v = zp_i'(z) = v_1 + iv_2$, we have

(i) $\phi(u,v)$ is continuous in $D=\mathbb{C}^2$;
(ii) $(1,0)\in D$ and $Re\{\phi(1,0)\}=1-\rho>0$;
(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq -\frac{n}{2}(1+u_2^2)$,

$$\begin{aligned} Re\{\phi(iu_2,v_1)\} &= \rho_1-\rho+\frac{\gamma(1-\rho_1)v_1}{\mu\beta} \\ &\leq \rho_1-\rho-\frac{n\gamma(1-\rho_1)(1+u_2^2)}{2\mu\beta} \\ &= \frac{A+Bu_2^2}{2C}, \end{aligned}$$

where $A=2(\rho_1-\rho)\mu\beta-n\gamma(1-\rho_1),\ B=-n\gamma(1-\rho_1)$ and $C=\mu\beta>0$. We note that $Re\{\phi(iu_2,v_1)\}<0$ if and only if $A=0,\ B<0$, this is true from (12). Therefore, by applying Lemma 2.1, $p_i(z)\in\mathcal{P}$ $(i=1,2)$ and consequently $p(z)\in\mathcal{P}_k$ for $z\in\mathbb{U}$. This completes the proof of Theorem 2.1.

Using similar arguments to those in the proof of Theorem 2.1 and the identity (8) instead of (7), we obtain the following theorem for the subclass $\mathcal{N}^{\alpha}_{\lambda,\mu}(\beta,\gamma,\rho,k)$.
**Theorem 2.2.** If $f(z)\in\mathcal{N}^{\alpha}_{\lambda,\mu}(\beta,\gamma,\rho,k)$, then

$$\left(z\mathcal{I}^{\alpha+1}_{\lambda,\mu}f(z)\right)^{\beta}\in\mathcal{P}_k(\rho_2), \tag{14}$$

where $\rho_2$ is given by

$$\rho_2=\frac{2\lambda\rho\beta+n\gamma}{2\lambda\beta+n\gamma}\ (0\leq\rho_2<1). \tag{15}$$

**Theorem 2.3.** If $f(z)\in\mathcal{M}^{\alpha}_{\lambda,\mu}(\beta,\gamma,\rho,k)$, then

$$\left(z\mathcal{I}^{\alpha}_{\lambda,\mu}f(z)\right)^{\beta/2}\in\mathcal{P}_k(\rho_3), \tag{16}$$

where $\rho_3$ is given by

$$\rho_3=\frac{n\gamma+\sqrt{(n\gamma)^2+4(\mu\beta+n\gamma)\mu\beta\rho}}{2(\mu\beta+n\gamma)}\ (0\leq\rho_3<1). \tag{17}$$

**Proof.** Let

$$\left(z\mathcal{I}^{\alpha}_{\lambda,\mu}f(z)\right)^{\beta/2}=(1-\rho_3)p(z)+\rho_3$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)[(1-\rho_3)p_1(z)+\rho_3]-\left(\frac{k}{4}-\frac{1}{2}\right)[(1-\rho_3)p_2(z)+\rho_3], \tag{18}$$

where $p_i(z)$ is analytic in $\mathbb{U}$ with $p_i(0) = 1$ for $i = 1, 2$. Differentiating (18) with respect to $z$, and using identity (7) in the resulting equation, we get

$$\begin{aligned} &\left[(1-\gamma)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)\right)^{\beta}+\gamma\left(\frac{\mathcal{I}_{\lambda,\mu+1}^{\alpha}f(z)}{\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)}\right)\left(z\mathcal{I}_{\lambda,\mu}^{\alpha}f(z)\right)^{\beta}\right] \\ = \;& \left[(1-\rho_3)p(z)+\rho_3\right]^2+\frac{2\gamma(1-\rho_3)\left[(1-\rho_3)p(z)+\rho_3\right]zp'(z)}{\mu\beta}\in\mathcal{P}_k(\rho), \end{aligned}$$

this implies that

$$\frac{1}{1-\rho}\left\{\left[(1-\rho_3)p_i(z)+\rho_3\right]^2-\rho+\frac{2\gamma(1-\rho_3)\left[(1-\rho_3)p_i(z)+\rho_3\right]zp_i'(z)}{\mu\beta}\right\}\in\mathcal{P}\ (i=1,2).$$

Defining the function

$$\phi(u,v)=\left[(1-\rho_3)u+\rho_3\right]^2-\rho+\frac{2\gamma(1-\rho_3)\left[(1-\rho_3)u+\rho_3\right]v}{\mu\beta}$$

where $u = p_i(z) = u_1 + iu_2,\ v = zp_i'(z) = v_1 + iv_2$, we have
(i) $\phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
(ii) $(1, 0) \in D$ and $Re\{\phi(1, 0)\} = 1 - \rho > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{n}{2}(1 + u_2^2)$,

$$\begin{aligned} Re\{\phi(iu_2,v_1)\} &= -(1-\rho_3)^2u_2^2+\rho_3^2-\rho+\frac{2\gamma\rho_3(1-\rho_3)v_1}{\mu\beta} \\ &\le -(1-\rho_3)^2u_2^2+\rho_3^2-\rho-\frac{n\gamma\rho_3(1-\rho_3)(1+u_2^2)}{\mu\beta} \\ &= \frac{A+Bu_2^2}{C}, \end{aligned}$$

where $A = \mu\beta\rho_3^2-\mu\beta\rho-n\gamma\rho_3(1-\rho_3),\ B = -(1-\rho_3)\left[\mu\beta(1-\rho_3)+\gamma n\rho_3\right]$ and $C = \mu\beta > 0$. We note that $Re\{\phi(iu_2, v_1)\} < 0$ if and only if $A = 0,\ B < 0$, this is true from (17). Therefore, by applying Lemma 2.1, $p_i(z) \in \mathcal{P}\ (i = 1, 2)$ and consequently $p(z) \in \mathcal{P}_k$ for $z \in \mathbb{U}$. This completes the proof of Theorem 2.3.

Using similar arguments to those in the proof of Theorem 2.3 and the identity (8) instead of (7), we obtain the following theorem for the subclass $\mathcal{N}_{\lambda,\mu}^{\alpha}(\beta, \gamma, \rho, k)$.
**Theorem 2.4.** If $f(z) \in \mathcal{N}_{\lambda,\mu}^{\alpha}(\beta, \gamma, \rho, k)$, then

$$\left(z\mathcal{I}_{\lambda,\mu}^{\alpha+1}f(z)\right)^{\beta/2}\in\mathcal{P}_k(\rho_4), \tag{19}$$

where $\rho_4$ is given by

$$\rho_4=\frac{n\gamma+\sqrt{(n\gamma)^2+4(\lambda\beta+n\gamma)\lambda\beta\rho}}{2(\lambda\beta+n\gamma)}\ (0\le\rho_4<1). \tag{20}$$

# 3. Open Problem

The authors suggest to study the idea of this paper on the class $\Sigma_p$ of $p-$valent meromorphic functions, where $\Sigma_p$ denotes the class of analytic and $p-$valent meromorphic functions in the punctured disc $\mathbb{U}^*$ of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \ (p \in \mathbb{N}). \tag{21}$$

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