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On Quasi-Hadamard Products of Some Families of Uniformly Starlike and Convex Functions with Negative Coefficients

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Abstract

The objective of the present paper is to obtain quasi-Hadamard products of some families of uniformly starlike and convex functions with negative coefficients in the unit disc. Our results generalize results studied earlier.

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1 Introduction

Let $\mathcal{A}_j$ denote the class of the functions of the from

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. We note that $\mathcal{A}_1 = \mathcal{A}$. For a function $f(z) \in \mathcal{A}_j$, let

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^n f(z) &= D\left(D^{n-1} f(z)\right) \end{aligned}$$

$$= z + \sum_{k=j+1}^{\infty} k^n a_k z^k, \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.2}$$

The differential operator D^n was introduced by Sălăgean [13].

With the help of the differential operator D^n, for $0 \le \alpha < 1$, $0 \le \lambda \le 1$, $\beta \ge 0$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, let $S_j(n, m, \lambda, \alpha, \beta)$ denote the subclass of $\mathcal{A}_j$ consisting of functions $f(z)$ of the form (1.1) and satisfying the condition

$$Re\left\{\frac{(1-\lambda) z \ (D^n f \ (z))' + \lambda z \left(D^{n+m} f(z)\right)'}{(1-\lambda) D^n f(z) + \lambda D^{n+m} f(z)} - \alpha\right\} >$$

$$\beta\left|\frac{(1-\lambda) z \ (D^n f \ (z))' + \lambda z \left(D^{n+m} f(z)\right)'}{(1-\lambda) D^n f(z) + \lambda D^{n+m} f(z)} - 1\right|, z \in \mathbb{U}. \tag{1.3}$$

The operator D^{n+m} was studied by Sekine [14], (see also [8] and [6]). Denote by T_j the subclass of A_j consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k \ z^k \quad (a_k \ge 0, k \ge j+1; j \in \mathbb{N}). \tag{1.4}$$

Further, we define the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$ by

$$\mathbb{Q}_j(m, n, \lambda, \alpha, \beta) = S_j(m, n, \lambda, \alpha, \beta) \cap \ T_j$$

Specializing the parameters $\alpha, \beta, \lambda, n$ and m, one can obtain many subclasses studied earlier by various authors ex. see ([1], [2], [3], [4], [5], [7], [9], [10], [12] and [15]).

Let $f_\ell(z)\ (\ell = 1, 2, ..., h)$ be given by

$$f_\ell(z) = z - \sum_{k=j+1}^{\infty} a_{k,\ell} z^k \quad (a_{k,\ell} \ge 0). \tag{1.5}$$

Then the quasi-Hadamard product (or convolution) of these functions is defined by (see Kuang et al. [10] and Owa [11])

$$(f_1 * f_2 * ... * f_h)(z) = z - \sum_{k=j+1}^{\infty} (\prod_{\ell=1}^{h} a_{k,\ell}) z^k.$$

In this paper we obtain the quasi-Hadamard product results for functions in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$.

2 Quasi-Hadamard products

Theorem 1. Let the function $f(z)$ be defined by (1.4). Then $f(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty} k^n \left[1+\lambda\left(k^m-1\right)\right]\left[k(1+\beta)-(\alpha+\beta)\right] a_k \leq 1-\alpha. \tag{2.1}$$

Proof. Assume that (2.1) holds.Then we must show that

$$\beta\left|\frac{(1-\lambda)\, z\ \left(D^n f\ (z)\right)'+\lambda z\left(D^{n+m} f(z)\right)'}{(1-\lambda)\, D^n f(z)+\lambda D^{n+m} f(z)}-1\right|-$$

$$Re\left\{\frac{(1-\lambda)\, z\ \left(D^n f\ (z)\right)'+\lambda z\left(D^{n+m} f(z)\right)'}{(1-\lambda)\, D^n f(z)+\lambda D^{n+m} f(z)}-1\right\}$$
$$\leq\ 1-\alpha.$$

We have

$$\beta\left|\frac{(1-\lambda)\, z\ \left(D^n f\ (z)\right)'+\lambda z\left(D^{n+m} f(z)\right)'}{(1-\lambda)\, D^n f(z)+\lambda D^{n+m} f(z)}-1\right|-$$
$$Re\left\{\frac{(1-\lambda)\, z\ \left(D^n f\ (z)\right)'+\lambda z\left(D^{n+m} f(z)\right)'}{(1-\lambda)\, D^n f(z)+\lambda D^{n+m} f(z)}-1\right\}$$

$$\leq \frac{(1+\beta)\sum\limits_{k=j+1}^{\infty} k^n\left[1+\lambda\left(k^m-1\right)\right](k-1)\, a_k z^{k-1}}{1-\sum\limits_{k=j+1}^{\infty} k^n\left[1+\left(k^m-1\right)\lambda\right] a_k z^{k-1}}$$

$$\leq \frac{(1+\beta)\sum\limits_{k=j+1}^{\infty} k^n\left[1+\lambda\left(k^m-1\right)\right](k-1)\, a_k}{1-\sum\limits_{k=j+1}^{\infty} k^n\left[1+\left(k^m-1\right)\lambda\right] a_k} \leq 1-\alpha.$$

Hence, $f(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$.

Conversely, let $f(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then we have

$$Re\left\{\frac{1-\sum\limits_{k=j+1}^{\infty} k^{n+1}\left[1+\left(k^m-1\right)\lambda\right] a_k z^{k-1}}{1-\sum\limits_{k=j+1}^{\infty} k^n\left[1+\left(k^m-1\right)\lambda\right] a_k z^{k-1}}-\alpha\right\} \geq$$

$$\beta\left|\frac{\sum\limits_{k=j+1}^{\infty}k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left(k-1\right)a_{k}z^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}k^{n}\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}\right|.$$

Letting $z\to 1^{-}$ along the real axis, we obtain the desired inequality by (2.1). This completes the proof of Theorem 1.

Theorem 2. If $f_{\ell}(z)\in\mathbb{Q}_{j}\left(m,n,\lambda,\alpha_{\ell},\beta\right)$ for each $\ell=1,2,...,h,$ then $\left(f_{1}*f_{2}*...*f_{h}\right)(z)\in\mathbb{Q}_{j}\left(m,n,\lambda,\delta,\beta\right)$, where

$$\delta=1-\frac{j(1+\beta)\prod\limits_{\ell=1}^{h}(1-\alpha_{\ell})}{(j+1)^{n(h-1)}\{1+\lambda((j+1)^{m}-1)\}^{h-1}\prod\limits_{\ell=1}^{h}[(j+1)(1+\beta)-(\alpha_{\ell}+\beta)]-\Pi_{\ell=1}^{h}(1-\alpha_{\ell})}. \tag{2.3}$$

The result is sharp for the functions

$$f_{\ell}(z)=z-\frac{1-\alpha_{\ell}}{(j+1)^{n}\left[1+\lambda\left((j+1)^{m}-1\right)\right]\left[j\left(1+\beta\right)+\left(1-\alpha_{\ell}\right)\right]}z^{j+1}\quad(\ell=1,2,....,h). \tag{2.4}$$

Proof. For $h=1$, we have that $\delta=\alpha_{1}$. For $h=2$, Theorem 1 gives

$$\sum_{k=j+1}^{\infty}\frac{k^{n}\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]\left[1+\left(k^{m}-1\right)\lambda\right]}{1-\alpha_{\ell}}a_{k,\ell}\leq 1\ (\ell=1,2). \tag{2.5}$$

Note that, from (2.5), we have

$$\sum_{k=j+1}^{\infty}k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\sqrt{\prod_{\ell=1}^{2}\left(\frac{\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}\right)a_{k,\ell}}\leq 1\ (\ell=1,2). \tag{2.6}$$

To prove the case when $h=2$, we have to fined the largest δ such that

$$\sum_{k=j+1}^{\infty}\frac{k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left[k\left(1+\beta\right)-\left(\delta+\beta\right)\right]}{1-\delta}a_{k,1}a_{k,2}\leq 1 \tag{2.7}$$

or, such that

$$\frac{\left[k\left(1+\beta\right)-\left(\delta+\beta\right)\right]}{1-\delta}\sqrt{a_{k,1}a_{k,2}}\leq\sqrt{\prod_{\ell=1}^{2}\left(\frac{\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}\right)}\quad\left(k\geq j+1\right). \tag{2.8}$$

Further, by using (2.6), we need to find the largest δ such that

$$\frac{[k(1+\beta)-(\delta+\beta)]}{1-\delta}\leq k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\prod_{\ell=1}^{2}\left(\frac{[k(1+\beta)-(\alpha_{\ell}+\beta)]}{1-\alpha_{\ell}}\right)\quad\left(k\geq j+1\right), \tag{2.9}$$

which is equivalent to

$$\delta \leq \frac{k^n[1+\lambda(k^m-1)]\prod_{\ell=1}^{2}[k(1+\beta)-(\alpha_\ell+\beta)]-k(1+\beta)\prod_{\ell=1}^{2}(1-\alpha_\ell)+\beta\prod_{\ell=1}^{2}(1-\alpha_\ell)}{k^n[1+\lambda(k^m-1)]\prod_{\ell=1}^{2}[k(1+\beta)-(\alpha_\ell+\beta)]-\prod_{\ell=1}^{2}(1-\alpha_\ell)}$$

or, equivalently, that

$$\delta \leq 1-\frac{(k-1)(1+\beta)\prod_{\ell=1}^{2}(1-\alpha_\ell)}{k^n[1+\lambda(k^m-1)]\prod_{\ell=1}^{2}[k(1+\beta)-(\alpha_\ell+\beta)]-\prod_{\ell=1}^{2}(1-\alpha_\ell)}. \qquad (2.10)$$

Defining the function $\Psi(k)$ by

$$\Psi(k)=1-\frac{(k-1)(1+\beta)\prod_{\ell=1}^{2}(1-\alpha_\ell)}{k^n[1+\lambda(k^m-1)]\Pi_{\ell=1}^{2}[k(1+\beta)-(\alpha_\ell+\beta)]-\Pi_{\ell=1}^{2}(1-\alpha_\ell)} \quad (k \geq j+1), \qquad (2.11)$$

we see that $\Psi(k)\geq 0$ for $k\geq j+1$. This implies that

$$\delta \leq \Psi(j+1)=1-\frac{j(1+\beta)\prod_{\ell=1}^{2}(1-\alpha_\ell)}{(j+1)^n[1+\lambda((j+1)^m-1)]\prod_{\ell=1}^{2}[(j+1)(1+\beta)-(\alpha_\ell+\beta)]-\prod_{\ell=1}^{2}(1-\alpha_\ell)}. \qquad (2.12)$$

Therefore, the result is true for $h=2$. Next, suppose that the result is true for any positive integer h. Then we have

$$(f_1 * f_2 * ... * f_h * f_{h+1})(z) \in \mathbb{Q}_j(m,n,\lambda,\gamma,\beta),$$

where

$$\gamma = 1-\{j(1+\beta)(1-\delta)(1-\alpha_{h+1})\}.$$
$$.\{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+(1-\delta)][j(1+\beta)+(1-\alpha_{h+1})]-(1-\delta)(1-\alpha_{h+1})\}^{-1}, \qquad (2.13)$$

where δ is given by (2.3). It follows from (2.13) that

$$\gamma = 1-\frac{j(1+\beta)\prod_{\ell=1}^{h+1}(1-\alpha_\ell)}{(j+1)^{nh}[1+\lambda((j+1)^m-1)]^h\prod_{\ell=1}^{h+1}[j(1+\beta)+(1-\alpha_\ell)]-\prod_{\ell=1}^{h+1}(1-\alpha_\ell)}. \qquad (2.14)$$

Thus, the result is true for $h+1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer h. Finally, taking the functions $f_\ell(z)$ given by (2.4), we see that

$$\begin{aligned}(f_1 * f_2 * ... * f_h)(z) &= z-\left\{\prod_{\ell=1}^{h}\left(\frac{1-\alpha_\ell}{(j+1)^n[j(1+\beta)+(1-\alpha_\ell)][1+((j+1)^m-1)\lambda]}\right)\right\}z^{j+1} \\ &= z-B_{j+1}z^{j+1},\end{aligned}$$

where

$$B_{j+1}=\prod_{\ell=1}^{h}\left(\frac{1-\alpha_\ell}{(j+1)^n\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j\left(1+\beta\right)+\left(1-\alpha_\ell\right)\right]}\right).$$

Thus, we know that

$$\sum_{k=j+1}^{\infty}\frac{k^n\left[1+\lambda\left(k^m-1\right)\right]\left[k\left(1+\beta\right)-\left(\delta+\beta\right)\right]}{1-\delta}B_k$$

$$=\frac{(j+1)^n\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j((1+\beta)+(1-\delta)\right]}{1-\delta}.$$

$$.\left\{\prod_{\ell=1}^{h}\left(\frac{1-\alpha_\ell}{(j+1)^n\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j\left(1+\beta\right)+\left(1-\alpha_\ell\right)\right]}\right)\right\}=1.$$

Consequently, the result is sharp for the functions $f_\ell(z)$ given by (2.4).

Putting $h=2,\alpha_\ell=\alpha$, in Theorem 2, we have the following corollary

Corollary 1. If $f_\ell(z)\in\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)\,(\ell=1,2)$, be defined by (1.5). Then $(f_1*f_2)(z)\in\mathbb{Q}_j(m,n,\lambda,\delta,\beta)$, where

$$\delta=1-\frac{j\left(1+\beta\right)\left(1-\alpha\right)^2}{(j+1)^n[1+\lambda((j+1)^m-1)]\left[(j+1)\left(1+\beta\right)-\left(\alpha_\ell+\beta\right)\right]^2-\left(1-\alpha\right)^2}. \tag{2.15}$$

The result is sharp for the functions

$$f_\ell(z)=z-\frac{(1-\alpha)}{(j+1)^n\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]}z^{j+1}(\ell=1,2) \tag{2.16}$$

Remark 1. Putting $h=2$ and $m=1$ in Corollary 1 , we obtain the following corollary which corrects the result obtained by Shanmugam et al. [15, Theorem 5.1]

Corollary 2. Let the functions $f_\ell(z)\,(\ell=1,2)$ defined by (1.5) be in the class $\mathbb{Q}_j(1,n,\lambda,\alpha,\beta)$. Then $(f_1*f_2)(z)\in\mathbb{Q}_j(1,n,\lambda,\gamma,\beta)$, where

$$\gamma=1-\frac{j(1+\beta)(1-\alpha)^2}{(j+1)^n[(j+1)(1+\beta)-(\alpha+\beta)]^2(\lambda j+1)-(1-\alpha)^2}. \tag{2.17}$$

The result is sharp.

Putting $\alpha_\ell=\alpha(\ell=1,2,....,h)$, in Theorem 2 we have the following corollary.

Corollary 3. If $f_\ell(z)\in\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)\,(\ell=1,2,....,h)$, then $(f_1*f_2*....*f_h)(z)\in\mathbb{Q}_j(m,n,\lambda,\delta,\beta)$, where

$$\delta=1-\frac{j\left(1+\beta\right)\left(1-\alpha\right)^h}{(j+1)^{n(h-1)}[1+\lambda((j+1)^m-1)]^{h-1}\left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]^h-\left(1-\alpha\right)^h}. \tag{2.18}$$

The result is sharp for the functions

$$f_\ell(z) = z - \frac{1-\alpha}{(j+1)^n\left[1+\lambda\left((j+1)^m-1\right)\right]\left[j(1+\beta)+(1-\alpha)\right]} z^{j+1} \quad (\ell = 1, 2, ..., h). \tag{2.19}$$

Putting $j = 1$, in Theorem 2 we have the following corollary.

Corollary 4. If $f_\ell(z) \in \mathbb{Q}_1(m, n, \lambda, \alpha_\ell, \beta)$ $(\ell = 1, 2, ..., h)$, then $(f_1 * f_2 * .... * f_h)(z) \in \mathbb{Q}_1(m, n, \lambda, \delta, \beta)$, where

$$\delta = 1 - \frac{(1+\beta)\prod_{\ell=1}^{h}(1-\alpha_\ell)}{2^{n(h-1)}[1+\lambda(2^m-1)]^{h-1}\prod_{\ell=1}^{h}(2+\beta-\alpha_\ell) - \prod_{\ell=1}^{h}(1-\alpha_\ell)}. \tag{2.20}$$

The result is sharp for the functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{2^n\left[1+\lambda\left(2^m-1\right)\right](2+\beta-\alpha_\ell)} z^2 \quad (\ell = 1, 2, ..., h). \tag{2.21}$$

Putting $\lambda = 0$, in Theorem 2 we have the following corollary.

Corollary 5. If $f_\ell(z) \in \mathbb{Q}_j(m, n, 0, \alpha_\ell, \beta)$ $(\ell = 1, 2, ..., h)$, then $(f_1 * f_2 * ... * f_h)(z) \in \mathbb{Q}_j(m, n, \delta, \beta)$, where

$$\delta = 1 - \frac{j(1+\beta)\prod_{\ell=1}^{h}(1-\alpha_\ell)}{(j+1)^{n(h-1)}\prod_{\ell=1}^{h}\left[j(1+\beta)+(1-\alpha)\right] - \prod_{\ell=1}^{h}(1-\alpha_\ell)}. \tag{2.22}$$

The result is sharp for the functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{(j+1)^n\left[j(1+\beta)+(1-\alpha_\ell)\right]} z^{j+1} \, (\ell = 1, 2, ..., h). \tag{2.23}$$

Putting $\lambda = 1$, in Theorem 2 we have the following corollary.

Corollary 6. If $f_\ell(z) \in \mathbb{Q}_j(m, n, 1, \alpha_\ell, \beta) = \mathbb{Q}_j(m, n+1, \alpha_\ell, \beta)(\ell = 1, 2, ..., h)$, then $(f_1 * f_2 * ... * f_h)(z) \in \mathbb{Q}_j(m, n+1, \delta, \beta)$, where

$$\delta = 1 - \frac{j(1+\beta)\prod_{\ell=1}^{h}(1-\alpha_\ell)}{(j+1)^{(n+m)(h-1)}\prod_{\ell=1}^{h}\left[j(1+\beta)+(1-\alpha)\right] - \prod_{\ell=1}^{h}(1-\alpha_\ell)} \, (\ell = 1, 2, ..., h). \tag{2.24}$$

The result is sharp for the functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{(j+1)^{n+m}\left[j(1+\beta)+(1-\alpha_\ell)\right]} z^{j+1} \, (\ell = 1, 2, ..., h). \tag{2.25}$$

Theorem 3. Let $f_{\ell}(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta_{\ell})\,(\ell = 1,...,h)$. Then $(f_1 * f_2 * ... * f_h)(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\eta)$, where

$$\eta = \frac{(j+1)^{n(h-1)}\left[1+\lambda((j+1)^m-1)\right]^{h-1}\prod_{\ell=1}^{h}\left[j(1+\beta_{\ell})+(1-\alpha)\right]}{j(1-\alpha)^{h-1}} + [(\alpha-(j+1)], \tag{2.26}$$

the result is sharp for the functions $f_{\ell}(z)\,(\ell = 1,2,....,h)$ given by

$$f_{\ell}(z) = z - \frac{1-\alpha}{(j+1)^n\left[1+\lambda\left((j+)1^m-1\right)\right]\left[j\left(1+\beta_{\ell}\right)+(1-\alpha)\right]} z^{j+1} \quad (\ell = 1,2,....,h). \tag{2.27}$$

Putting $\beta_{\ell} = \beta(\ell = 1,2,...,h)$ in Theorem 3, we get the following corollary.

Corollary 7. Let $f_{\ell} \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)\,(\ell = 1,...,h)$. Then $(f_1 * f_2 * ... * f_h)(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\eta)$, where

$$\eta = \frac{(j+1)^{n(h-1)}\left\{1+\lambda((j+1)^m-1)\right]^{h-1}[j(1+\beta)+(1-\alpha)]^h}{j(1-\alpha)^{h-1}} + [(\alpha-(j+1)].$$

The result is sharp for the functions $f_{\ell}(z)\,(\ell = 1,2,...,h)$ given by (2.19).

Theorem 4. Let $f_{\ell}(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha_{\ell},\beta)\,(\ell = 1,...,h)$ and suppose that

$$F(z) = z - \sum_{k=j+1}^{\infty}\left(\sum_{\ell=1}^{h} a_{k,\ell}^t\right) z^k \quad (t > 1, z \in \mathbb{U}). \tag{2.28}$$

Then $F \in \mathbb{Q}_j(m,n,\lambda,\gamma_h,\beta)$, where

$$\gamma_h = 1 - \frac{hj(1-\alpha)^t(1+\beta)}{(j+1)^{n(t-1)}[1+\lambda((j+1)^m-1)]^{t-1}[(j+1)(1+\beta)-(\alpha+\beta)]^t - h(1-\alpha)^t} \left(\alpha = \min_{1\le\ell\le h}\{\alpha_{\ell}\}\right), \tag{2.29}$$

and

$$k^{n(t-1)}\left[1+\lambda\left(k^m-1\right)\right]^{t-1}\left[k(1+\beta)-(\alpha+\beta)\right]^t \geq$$
$$h\left(1-\alpha\right)^t\left(k+\beta(k-1)\right).$$

The result is sharp for the functions $f_{\ell}(\ell = 1,2,...,h)$ given by (2.4).

Proof. Sines $f_{\ell}(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha_{\ell},\beta)$, in view of (2.1), we obtain

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n\left[1+\lambda\left(k^m-1\right)\right]\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}\right] a_{k,\ell} \leq 1 \quad (\ell = 1,...,h).$$

By virtue of the Cauchy-Schwarz inequality, we get

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n\left[1+\lambda\left(k^m-1\right)\right]\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}\right]^t a_{k,\ell}^t \leq$$

$$\left(\sum_{k=j+1}^{\infty}\frac{k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}a_{k,\ell}\right)^{t}\leq 1. \tag{2.30}$$

It follows from (2.30) that

$$\sum_{k=j+1}^{\infty}\left(\frac{1}{h}\sum_{\ell=1}^{h}\left(\frac{k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}\right)^{t}a_{k,\ell}^{t}\right)\leq 1.$$

By setting

$$\alpha=\min_{1\leq\ell\leq h}\{\alpha_{\ell}\},$$

therefore, to prove our result we need to fined the largest γ_h such that

$$\sum_{k=j+1}^{\infty}\frac{k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left[k\left(1+\beta\right)-\left(\gamma_{h}+\beta\right)\right]}{1-\gamma_{h}}\left(\sum_{\ell=1}^{h}a_{k,\ell}^{t}\right)\leq 1,$$

that is that

$$\frac{k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left[k\left(1+\beta\right)-\left(\gamma_{h}+\beta\right)\right]}{1-\gamma_{h}}\leq$$

$$\frac{1}{h}\left(\frac{k^{n}\left[1+\lambda\left(k^{m}-1\right)\right]\left[k\left(1+\beta\right)-\left(\alpha_{\ell}+\beta\right)\right]}{1-\alpha_{\ell}}\right)^{t}$$

which leads to

$$\gamma_{h}\leq 1-\frac{h\left(1-\alpha_{\ell}\right)^{t}(k-1)(1+\beta)}{k^{n(t-1)}\left[1+\lambda\left(k^{m}-1\right)\right]^{t-1}\left[k(1+\beta)-\left(\alpha_{\ell}+\beta\right)\right]^{t}-h\left(1-\alpha_{\ell}\right)^{t}}.$$

Now let

$$G(k)=1-\frac{h\left(1-\alpha\right)^{t}(k-1)(1+\beta)}{k^{n(t-1)}\left[1+\lambda\left(k^{m}-1\right)\right]^{t-1}\left[k(1+\beta)-\left(\alpha+\beta\right)\right]^{t}-h\left(1-\alpha\right)^{t}}.$$

Since $G(k)$ is an increasing function of $(k\in\mathbb{N})$, we readily have

$$\gamma_{h}=G(j+1)=1-\frac{hj(1-\alpha)^{t}(1+\beta)}{(j+1)^{n(t-1)}\left[1+\lambda((j+1)^{m}-1)\right]^{t-1}\left[j(1+\beta)+(1+\alpha)\right]^{t}-h(1-\alpha)^{t}},$$

we can see that $0\leq\gamma_h<1$. The result is sharp for the functions $f_{\ell}(z)\,(\ell=1,2,...,h)$ given by (2.4). The proof of Theorem 4 is thus completed.

Putting $t=2$ and $\alpha_{\ell}=\alpha\,(\ell=1,2,...,h)$ in Theorem 4, we obtain the following result.

Corollary 8. Let $f_{\ell}\in\mathbb{Q}_{j}\left(m,n,\lambda,\alpha,\beta\right)(\ell=1,2,...,h)$ and suppose that

$$F(z)=z-\sum_{k=j+1}^{\infty}\left(\sum_{\ell=1}^{h}a_{k,\ell}^{2}\right)z^{k}\quad\left(z\in\mathbb{U}\right). \tag{2.31}$$

Then $F(z) \in \mathbb{Q}_j(m,n,\lambda,\gamma_h,\beta)$, where

$$\gamma_h = 1 - \frac{hj(1-\alpha)^2(1+\beta)}{(j+1)^n[1+\lambda((j+1)^m-1)][(j+1)(1+\beta)-(\alpha+\beta)]^2 - h(1-\alpha)^2},$$

and

$$(j+1)^n[1+\lambda((j+1)^m-1)][(j+1)(1+\beta)-(\alpha+\beta)]^2 \geq h(1-\alpha)^2[(j+1)+\beta j].$$

The result is sharp for the functions $f_\ell(z)$ $(\ell = 1,2,...,h)$ given by (2.4).

By similarly applying the method of proof of Theorem 4, we easily get the following Theorem 5.

Theorem 5. Let $f_\ell \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta_\ell)$ $(\ell = 1,...,h)$ and the function F be defined by (2.29). Then $F \in \mathbb{Q}_j(m,n,\lambda,\alpha,\delta_h)$, where

$$\delta_h = \frac{(j+1)^{n(t-1)}[1+\lambda((j+1)^m-1)]^{(t-1)}[(j+1)(1+\beta)-(\alpha+\beta)]^t}{hj(1-\alpha)^{(t-1)}} + (\alpha - j - 1)$$
$$\left(\beta = \min_{1\leq\ell\leq h}\{\beta_\ell\}\right),$$

and

$$(j+1)^{n(t-1)}[1+\lambda((j+1)^m-1)]^{(t-1)}[(j+1)(1+\beta)-(\alpha+\beta)]^t \geq hj(j+1-\alpha)(1-\alpha)^{(t-1)}.$$

The result is sharp for the functions $f_\ell(z)$ $(\ell = 1,2,...,h)$ given by (2.4).

Taking $t = 2$ and $\beta_\ell = \beta(\ell = 1,2,...,h)$ in Theorem 5, we get the following result.

Corollary 9. Let $f_\ell \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ $(\ell = 1,...,h)$ and the function F be defined by (2.31). Then $F \in \mathbb{Q}_j(m,n,\lambda,\alpha,\delta_h)$, where

$$\delta_h = \frac{(j+1)^n[1+\lambda((j+1)^m-1)][(j+1)(1+\beta)-(\alpha+\beta)]^2}{hj(1-\alpha)} + (\alpha - j - 1),$$

and

$$(j+1)^n[1+\lambda((j+1)^m-1)][(j+1)(1+\beta)-(\alpha+\beta)]^2 \geq h(k-1)(j+1-\alpha)(1-\alpha).$$

The result is sharp for the functions $f_\ell(z)$ $(\ell = 1,2,...,h)$ given by (2.4).

Finally, we derive some quasi-Hadamard product results for $f_\ell(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ and $g_s(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$.

Theorem 6. Let the functions $f_\ell(z)\,(\ell = 1, ..., h)$ defined by (1.4) be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha_\ell, \beta)\,(\ell = 1, ..., h)$ and let the functions $g_s(z)$ defined by

$$g_s(z) = z - \sum_{k=j+1}^{\infty} b_k z^k \quad (b_k \geq 0, k \geq j+1; j \in \mathbb{N}), \tag{2.32}$$

be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha_s, \beta)\,(s = 1, ..., t)$. Then

$$(f_1 * f_2 * ... * f_h * g_1 * g_2 * ... * g_t)(z) \in \mathbb{Q}_j(m, n, \lambda, \psi, \beta),$$

where

$$\psi = 1 - \{j(1+\beta) \prod_{\ell=1}^{h}(1-\alpha_\ell) \prod_{s=1}^{t}(1-\alpha_s)\}.$$

$$.\Big\{\{(j+1)^{n(h+t-1)}[1+\lambda((j+1)^m-1)]^{(h+t-1)} \prod_{s=1}^{t}[j(1+\beta)+(1-\alpha_s)]$$

$$\prod_{\ell=1}^{h}[j(1+\beta)+(1-\alpha_\ell)] - \prod_{s=1}^{t}(1-\alpha_s)\prod_{\ell=1}^{h}(1-\alpha_\ell)\Big\} \tag{2.33}$$

The result is sharp for the functions $f_\ell(z)\,(\ell = 1, 2, ..., h)$ given by (2.5). and the functions $g_s(z)$ given by

$$g_s(z) = z - \frac{1-\alpha_s}{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+(1-\alpha_s)]} z^{j+1} \quad (s = 1, 2, .., t). \tag{2.34}$$

Proof. From Theorem 2 we note that, if $f(z) \in \mathbb{Q}_j(m, n, \lambda, \delta, \beta)$ and $g(z) \in \mathbb{Q}_j(m, n, \lambda, \mu, \beta)$, then $(f * g)(z) \in \mathbb{Q}_j(m, n, \lambda, \psi, \beta)$, where

$$\psi = 1 - \{j(1+\beta)(1-\delta)(1-\mu)\}.$$

$$.\{(j+1)^n[1+\lambda((j+1)^m-1)]\,[j(1+\beta)+(1-\delta)]\,[j(1+\beta)+(1-\mu)]-(1-\delta)(1-\mu)\}. \tag{2.35}$$

Since Theorem 2 leads to $f_1 * f_2 * ... * f_h \in \mathbb{Q}_j(m, n, \lambda, \delta, \beta)$, where δ is defined by (2.3) and $g_1 * g_2 * ... * g_s \in \mathbb{Q}_j(m, n, \lambda, \mu, \beta)$, with

$$\mu = 1 - \frac{j(1+\beta)\prod_{s=1}^{t}(1-\alpha_s)}{(j+1)^{n(t-1)}[1+\lambda((j+1)^m-1)]^{t-1}\prod_{s=1}^{t}[(j+1)(1+\beta)-(\alpha_s+\beta)] - \prod_{s=1}^{t}(1-\alpha_s)}. \tag{2.36}$$

Then, we have $(f_1 * f_2 * ... * f_h * g_1 * g_2 * ...... * g_t)(z) \in \mathbb{Q}_j(m, n, \lambda, \psi, \beta)$, where ψ is given by (2.33), this completes the proof of Theorem 6.

Letting $\alpha_\ell = \alpha(\ell = 1, 2, ..., h)$ and $\alpha_s = \alpha(s = 1, 2, ..., t)$ in Theorem 6, we obtain the following corollary.

Corollary 10. Let the functions $f_\ell(z)\,(\ell = 1, ..., h)$ defined by (1.4) be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)\,(\ell = 1, ..., h)$ and let the functions $g_s(z)$ defined by (2.32) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then we have $f_1 * f_2 * ... * f_h * g_1 * g_2 * ... * g_t \in \mathbb{Q}_j(m,n,\lambda,\psi,\beta)$, where

$$\psi = 1 - \frac{j(1+\beta)(1-\alpha)^{h+t}}{(j+1)^{n(h+t-1)}[1+\lambda((j+1)^m-1)]^{h+t-1}[j(1+\beta)+(1-\alpha)]^{h+t}-(1-\alpha)^{h+t}}. \tag{2.37}$$

The result is sharp for the functions $f_\ell(z)$ given by (2.5) and the functions $g_s(z)$ given by

$$g_s(z) = z - \frac{1-\alpha_s}{(j+1)^n[1+\lambda((j+1)^m-1)][j(1+\beta)+(1-\alpha_s)]} z^{j+1} \quad (s = 1,2,..,t).$$

Letting $h = t = 2$ in Colloary 10, we obtain the following corollary.

Corollary 11. Let the functions $f_\ell(z)\,(\ell = 1,2)$ defined by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ and let the functions $g_s(z)\,(s = 1,2)$ defined by (2.32) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then we have $(f_1 * f_2 * g_1 * g_2)(z) \in \mathbb{Q}_j(m,n,\lambda,\psi,\beta)$, where

$$\psi = 1 - \frac{j(1+\beta)(1-\alpha)^4}{(j+1)^{3n}[1+\lambda((j+1)^m-1)]^3\,[j(1+\beta)+(1-\alpha)]^4-(1-\alpha)^4}.$$

The result is sharp.

3 Open Problem

The authors suggest to study the properties of the same class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ by replacing of f by $(f * g)$.

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