Int. J. Open Problems Complex Analysis, Vol. 7, No. 2, July 2015
ISSN 2074-2827; Copyright ©ICSRS Publication, 2015
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Properties of Certain Class of Uniformly Starlike and Convex Functions Defined by Convolution

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Received 5 January 2015; Accepted 25 February 2015

Abstract: *The aim of this paper is to obtain the modified Hadamard products and properties associated with generalized fractional calculus operators for functions belonging to the class* $TS_\gamma(f, g; \alpha, \beta)$ *of* β*-uniformly univalent functions defined by convolution.*

Keywords and phrases: *Univalent functions, uniformly starlike, uniformly convex, modified Hadamard products, fractional calculus operator.*
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $f(z) \in S$ be given by (1.1) and $g(z) \in S$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{1.2}$$

then the Hadamard product (or convolution) $(f * g)(z)$ of $f\ (z)$ and $g(z)$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Following Goodman ([6] and [7]), Rønning ([13] and [14]) introduced and studied the following subclasses:

(i) A function $f(z)$ of the form (1.1) is said to be in the class $S_p(\alpha, \beta)$ of uniformly $\beta-$starlike functions of order α if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{U}), \tag{1.4}$$

where $-1 \leq \alpha < 1$ and $\beta \geq 0$.

(ii) A function $f(z)$ of the form (1.1) is said to be in the class $UCV\ (\alpha, \beta)$ of uniformly $\beta-$convex functions of order α if it satisfies the condition:

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}), \tag{1.5}$$

where $-1 \leq \alpha < 1$ and $\beta \geq 0$.

It follows from (1.4) and (1.5) that

$$f(z) \in UCV(\alpha, \beta) \iff zf'(z) \in S_p(\alpha, \beta). \tag{1.6}$$

In [2] Aouf et al. defined the class $S_\gamma(f, g; \alpha, \beta)$ as follows:

. For $-1 \leq \alpha < 1$, $0 \leq \gamma \leq 1$ and $\beta \geq 0$, let $S_\gamma(f, g; \alpha, \beta)$ be the subclass of S consisting of functions $f(z)$ of the form (1.1) and functions $g(z)$ of the form (1.2) with $b_k > 0 (k \geq 2)$ and satisfying the analytic criterion:

$$Re\left\{\frac{z(f * g)'(z) + \gamma z^2 (f * g)''(z)}{(1-\gamma)\,(f * g)(z) + \gamma z (f * g)'(z)} - \alpha\right\}$$

$$> \beta\left|\frac{z(f * g)'(z) + \gamma z^2 (f * g)''(z)}{(1-\gamma)\,(f * g)(z) + \gamma z (f * g)'(z)} - 1\right| (z \in \mathbb{U}). \tag{1.7}$$

Let T denote the subclass of S consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0)\ . \tag{1.8}$$

Further, we define the class $TS_\gamma(f, g; \alpha, \beta)$ by

$$TS_\gamma(f, g; \alpha, \beta) = S_\gamma(f, g; \alpha, \beta) \cap T. \tag{1.9}$$

We note that:

(i) $TS_0(f, \frac{z}{(1-z)}; \alpha, 1) = TS^0(\alpha, 1)$ and $TS_0(f, \frac{z}{(1-z)^2}; \alpha, 1) =$

$TS_1(f, \frac{z}{(1-z)}; \alpha, 1) = UCV(\alpha)$ $(-1 \le \alpha < 1)$ (see Bharati et al. [4]);
(ii) $TS_1(f, \frac{z}{(1-z)}; 0, \beta) = UCT(\beta)$ $(\beta \ge 0)$ (see Subramanian et al. [21]);
(iii) $TS_0(f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta) = UCV\,(\alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0, c \ne 0, -1, -2, ...)$ (see Murugusundaramoorthy and Magesh [9, 10]);
(iv) $TS_0(f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta) = S\,(n, \alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\})$(see Rosy and Murugusundaramoorthy [16] and mostafa [8]);
(v) $TS_0(f, z + \sum_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k; \alpha, \beta) = D\,(\alpha, \beta, \lambda)\,(-1 \le \alpha < 1, \beta \ge 0, \lambda > -1)$ (see Shams et al. [18]);
(vi) $TS_0(f, z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n z^k; \alpha, \beta) = TS_\lambda\,(n, \alpha, \beta)\,(-1 \le \alpha < 1, \beta \ge 0,\ \lambda \ge 0,\ n \in \mathbb{N}_0)$ (see Aouf and Mostafa [3]);
(vii) $TS_\gamma(f, z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta) = TS\,(\gamma, \alpha, \beta)\,(-1 \le \alpha < 1,\ \beta \ge 0,\ 0 \le \gamma \le 1,\ c \ne 0, -1, -2, ...)$ (see Murugusundaramoorthy et al. [11]);
(viii) $TS_\gamma(f, z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k; \alpha, \beta) = UH(q, s, \gamma, \beta, \alpha)$(see Ahuja et al. [1]), where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} ... (\alpha_q)_{k-1}}{(\beta_1)_{k-1} ... (\beta_s)_{k-1}} \frac{1}{(k-1)!} \tag{1.10}$$

$(\alpha_i > 0,\ i = 1, ..., q;\ \beta_j > 0,\ j = 1, ..., s;\ q \le s+1;\ q,\ s \in \mathbb{N}_0)$.
(ix)$TS_\gamma(f, z + \sum_{k=2}^{\infty} k^n z^k; \alpha, \beta) = TS_\gamma\,(n, \alpha, \beta)\,(n \in \mathbb{N}_0)$(see Aouf et al. [2]).

Also we obtain the following new subclasses as follows:
(i) $TS_\gamma(f, z + \sum_{k=2}^{\infty} \Omega\ \sigma_k(\alpha_1) z^k; \alpha, \beta) = UW_{q,s}((\alpha_1, A_1), \gamma, \beta, \alpha)$

$$= \left\{ \begin{array}{c} f \in T:\ Re\left\{ \frac{z(W[\alpha_1, A_1]f(z))' + \gamma z^2 (W[\alpha_1, A_1]f(z))''}{(1-\gamma)(W[\alpha_1, A_1]f(z)) + \gamma z (W[\alpha_1, A_1]f(z))'} - \alpha \right\} \\ > \beta \left| \frac{z(W[\alpha_1, A_1]f(z))' + \gamma z^2 (W[\alpha_1, A_1]f(z))''}{(1-\gamma)(W[\alpha_1, A_1]f(z)) + \gamma z (W[\alpha_1, A_1]f(z))'} - 1 \right| (z \in \mathbb{U}) \end{array} \right\}, \tag{1.11}$$

where $W[\alpha_1, A_1]$ is the Wright generalized hypergeometric function (see Wright [22]) and Ω is given by

$$\Omega = \left(\prod_{t=1}^{q} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=1}^{s} \Gamma(\beta_t)\right), \tag{1.12}$$

and $\sigma_k\ (\alpha_1)$ is defined by

$$\sigma_k\ (\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n-1)) ... \Gamma(\alpha_q + A_q(n-1))}{(n-1)! \Gamma(\beta_1 + B_1(n-1)) ... \Gamma(\beta_s + B_s(n-1))}, \tag{1.13}$$

$(\alpha_1, A_1...\alpha_q, A_q$ and $\beta_1, B_1...\beta_s, B_s\ , q, s \in \mathbb{N})$ are positive real parameters.

(ii) $TS_\gamma(f, z +_{k=2}^{\infty} \left(\frac{1+l+\lambda(k-1)}{1+l}\right)^m z^k; \alpha, \beta) = TS_{l,\lambda}(m, \gamma, \alpha, \beta)$

$$= \left\{ \begin{array}{c} f \in T: \ Re\left\{ \dfrac{z(I^m(\lambda,l)f(z))' + \gamma z^2(I^m(\lambda,l)f(z))''}{(1-\gamma)(I^m(\lambda,l)f(z)) + \gamma z(I^m(\lambda,l)f(z))'} - \alpha \right\} \\ \\ > \beta \left| \dfrac{z(I^m(\lambda,l)f(z))' + \gamma z^2(I^m(\lambda,l)f(z))''}{(1-\gamma)(I^m(\lambda,l)f(z)) + \gamma z(I^m(\lambda,l)f(z))'} - 1 \right| \end{array} \right\} \quad (1.14)$$

$-1 \leq \alpha < 1,\ 0 \leq \gamma < 1,\ \beta \geq 0,\ \lambda > 0,\ m,\ l \in \mathbb{N}_0, z \in \mathbb{U},$

where $I^m(\lambda, l)$ is the Cãtãs operator (see Cãtãs et al. [5]).

To prove our main results, we need the following lemma.

[2]. A necessary and sufficient condition for the function $f(z)$ of the form (1.8) to be in the class $TS_\gamma(f,\ g\ ; \alpha, \beta)\ )(1 \leq \alpha \leq 1\ \ 0 \leq \gamma \leq 1$ and $\beta \geq 0)$ is that:

$$_{k=2}^{\infty}\ [k(1+\beta) - (\alpha+\beta)][1+\gamma(k-1)]b_k a_k \leq 1-\alpha. \quad (1.15)$$

2 Modified Hadamard products

Let the functions $f_j(z)$ be defined for $j = 1, 2, ...m$ by

$$f_j(z) = z -_{k=2}^{\infty} a_{k,j}\ z^k\ (a_{k,j} \geq 0). \quad (2.1)$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z -_{k=2}^{\infty} a_{k,1}\ a_{k,2}\ z^k = (f_2 * f_1)(z)\ \ . \quad (2.2)$$

. Let the functions $f_j(z)(j = 1, 2)$ defined by (2.1) be in the class $TS_\gamma(f, g\ ; \alpha, \beta)$.Then

$(f_1 * f_2)(z)\ \in TS_\gamma(f, g\ ; \varphi, \beta)$, where

$$\varphi = 1 - \frac{(1+\beta)(1-\alpha)^2}{(2-\alpha+\beta)^2(1+\gamma)b_2 - (1-\alpha)^2}. \quad (2.3)$$

The result is sharp for functions $f_j(z)\ (j = 1, 2)$ given by

$$f_j(z) = z - \frac{(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2} z^2 \quad (j = 1, 2). \quad (2.4)$$

Proof. Employing the technique used earlier by Schild and Silverman [17], we need to find the largest real parameter φ such that

$$\mathop{}_{k=2}^{\infty}\frac{[k(1+\beta)-(\varphi+\beta)][1+\gamma(k-1)]b_k}{(1-\varphi)}a_{k,1}\ a_{k,2}\ \leq 1. \tag{2.5}$$

Since $f_j(z)\in TS_\gamma(f,g\,;\alpha,\beta)$ $(j=1,2)$, we readily see that

$$\mathop{}_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}a_{k,1}\quad \leq 1 \tag{2.6}$$

and

$$\mathop{}_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}a_{k,2}\leq 1. \tag{2.7}$$

By the Cauchy-Schwarz inequality we have

$$\mathop{}_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}\leq 1. \tag{2.8}$$

Thus it sufficient to show that

$$\begin{aligned} &\frac{[k(1+\beta)-(\varphi+\beta)][1+\gamma(k-1)]b_k}{(1-\varphi)}a_{k,1}a_{k,2}\ 2.9 \qquad (1)\\ \leq\ &\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}\sqrt{a_{k,1}a_{k,2}} \end{aligned}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{(1-\varphi)[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)[k(1+\beta)-(\varphi+\beta)]}. \tag{2.10}$$

Hence, in the light of the inequality (2.8), it is sufficient to prove that

$$\frac{(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]\ [1+\gamma(k-1)]b_k}\leq\frac{(1-\varphi)\ [k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)\ [k(1+\beta)-(\varphi+\beta)]}\ . \tag{2.11}$$

It follows from (2.11) that

$$\varphi\leq\ 1-\frac{(k-1)(1+\beta)(1-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2[1+\gamma(k-1)]b_k-(1-\alpha)^2}\quad . \tag{2.12}$$

Now defining the function $G(k)$ by

$$G(k) = 1 - \frac{(k-1)(1+\beta)(1-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2[1+\gamma(k-1)]b_k-(1-\alpha)^2}. \tag{2.13}$$

We see that $G(k)$ is increasing function of $k(k \geq 2)$. Therefore, we conclude that,

$$\varphi \leq G(2) = 1 - \frac{(1+\beta)(1-\alpha)^2}{(2-\alpha+\beta)^2(1+\gamma)b_2-(1-\alpha)^2} \tag{2.14}$$

and hence the proof of Theorem **??** is completed.

. Let the function $f_1(z)$ defined by (2.1) be in the class $TS_\gamma(f, g\ ; \alpha, \beta)$.

Suppose also that the function $f_2(z)$ defined by (2.1) be in the class $TS_\gamma(f, g\ ; \delta, \beta)$. Then $(f_1 * f_2)(z) \in TS_\gamma(f, g\ ; \rho, \beta)$, where

$$\rho = 1 - \frac{(1+\beta)(1-\alpha)(1-\delta)}{(2-\alpha+\beta)\ (2-\delta+\beta)(1+\gamma)b_2-(1-\alpha)(1-\delta)}. \tag{2.15}$$

The result is sharp for functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z - \frac{(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2}\ z^2 \tag{2.16}$$

and

$$f_2(z) = z - \frac{(1-\delta)}{(2-\delta+\beta)(1+\gamma)b_2}\ z^2.$$

Proof. We need to find the largest real parameter ρ such that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\rho+\beta)][1+\gamma(k-1)]b_k}{(1-\rho)} a_{k,1}a_{k,2} \leq 1. \tag{2.18}$$

Since $f_1(z) \in TS_\gamma(f, g\ ; \alpha, \beta)$, we readily see that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)} a_{k,1} \leq 1, \tag{2.19}$$

and $f_2(z) \in TS_\gamma(f, g\ ; \delta, \beta)$, we readily see that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\delta+\beta)][1+\gamma(k-1)]b_k}{(1-\delta)} a_{k,2} \leq 1. \tag{2.20}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]^{\frac{1}{2}}[k(1+\beta)-(\delta+\beta)]^{\frac{1}{2}}[1+\gamma(k-1)]\ b_k}{\sqrt{1-\alpha}\ \sqrt{1-\delta}} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{2.21}$$

Thus it sufficient to show that

$$\frac{[k(1+\beta)-(\rho+\beta)][1+\gamma(k-1)]b_k}{(1-\rho)}a_{k,1}a_{k,2}2.22 \tag{2}$$
$$\leq \frac{[k(1+\beta)-(\alpha+\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\delta+\beta)]^{\frac{1}{2}}[1+\gamma(k-1)]b_k}{\sqrt{1-\alpha}\sqrt{1-\delta}}\sqrt{a_{k,1}a_{k,2}}$$

or, equivalently, that

$$\sqrt{a_{k,1}\ a_{k,2}} \ \leq \ \frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\delta+\beta)]^{\frac{1}{2}}}{\sqrt{1-\alpha}\sqrt{1-\delta}[k(1+\beta)-(\rho+\beta)]}. \tag{2.23}$$

Hence, in the light of the inequality (2.21), it is sufficient to prove that

$$\frac{\sqrt{1-\alpha}\ \sqrt{1-\delta}}{[k(1+\beta)-(\alpha+\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\delta+\beta)]^{\frac{1}{2}}[1+\gamma(k-1)]b_k}2.24 \tag{3}$$
$$\leq \quad \frac{(1-\rho)\ \ [k(1+\beta)-(\alpha+\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\delta+\beta)]^{\frac{1}{2}}}{\sqrt{1-\alpha}\ \sqrt{1-\delta}\ [k(1+\beta)-(\rho+\beta)]}.$$

It follows from (2.24) that

$$\rho \leq 1-\frac{(k-1)(1+\beta)(1-\alpha)(1-\delta)}{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\delta+\beta)][1+\gamma(k-1)]b_k-(1-\alpha)(1-\delta)}. \tag{2.25}$$

Now defining the function $M(k)$ by

$$M(k) = 1- \frac{(k-1)(1+\beta)(1-\alpha)(1-\delta)}{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\delta+\beta)][1+\gamma(k-1)]b_k-(1-\alpha)(1-\delta)}. \tag{2.26}$$

We see that $M(k)$ is increasing function of $k (k \geq 2)$. Therefore, we conclude that

$$\rho \ \leq M(2) \ = 1-\frac{(1+\beta)(1-\alpha)(1-\delta)}{(2-\alpha+\beta)(2-\delta+\beta)(1+\gamma)b_2-(1-\alpha)(1-\delta)} \tag{2.27}$$

and hence the proof of Theorem **??** is completed.

. Let the functions $f_j(z) (j = 1,\ 2, 3)$ defined by (2.1) be in the class $TS_\gamma(f, g\ ; \alpha, \beta)$.Then $(f_1 * f_2 * f_3)(z) \in TS_\gamma(f, g\ ; \tau, \beta)$, where

$$\tau = 1-\frac{(1+\beta)(1-\alpha)^3}{(2-\alpha+\beta)^3(1+\gamma)^2b_2^2-(1-\alpha)^3}. \tag{2.28}$$

The result is best possible for functions $f_j(z)(j = 1, 2, 3)$ given by (2.4) .

Proof. Form Theorem **??**, we have $(f_1*f_2)(z) \in TS_\gamma(f, g\,; \varphi, \beta)$, where φ is given by (2.3). Now, using Theorem **??**, we get $(f_1*f_2*f_3)(z) \in TS_\gamma(f, g\,; \tau, \beta)$, where

$$\tau \leq 1 - \frac{(1+\beta)(1-\alpha)\ (1-\varphi)(k-1)}{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\varphi+\beta)][1+\gamma(k-1)]b_k-(1-\alpha)(1-\varphi)}.$$

Now defining the function $S(k)$ by

$$S(k) = 1 - \frac{(1+\beta)(1-\alpha)(1-\varphi)(k-1)}{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\varphi+\beta)][1+\gamma(k-1)]b_k-(1-\alpha)(1-\varphi)}. \tag{2.29}$$

We see that $S(k)$ is increasing function of $k(k \geq 2)$. Therefore, we conclude that

$$\tau \leq S(2) \ = 1 - \frac{(1+\beta)(1-\alpha)(1-\varphi)}{(2-\alpha+\beta)(2-\varphi+\beta)(1+\gamma)b_2-(1-\alpha)(1-\varphi)}, \tag{2.30}$$

substituting from (2.3), we have

$$\tau = 1 - \frac{(1+\beta)(1-\alpha)^3}{(2-\alpha+\beta)^3(1+\gamma)^2 b_2^2-(1-\alpha)^3}$$

and hence the proof of Theorem **??** is completed.

. Let the functions $f_j(z)(j = 1, 2)$ defined by (2.1) be in the class $TS_\gamma(f, g\,; \alpha, \beta)$.Then the function

$$h\ (z) = z\ -_{k=2}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{2.31}$$

belongs to the class $TS_\gamma(f, g\ ; \zeta, \beta)$, where

$$\zeta = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(2-\alpha+\beta)^2(1+\gamma)b_2-2(1-\alpha)^2}. \tag{2.32}$$

The result is sharp for functions $f_j(z)(j = 1, 2)$ defined by (2.4).

Proof. By using Lemma **??**, we obtain

$$_{k=2}^{\infty}\left\{\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}\right\}^2 a_{k,1}^2.$$

$$\leq \left\{_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)} a_{k,1}\right\}^2 \leq 1, \tag{2.33}$$

and

$$\sum_{k=2}^{\infty}\left\{\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}\right\}^2 a_{k,2}^2$$

$$\leq \left\{\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}a_{k,2}\right\}^2 \leq 1. \qquad (2.34)$$

It follows from (2.33) and (2.34) that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left\{\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}\right\}^2 (a_{k,1}^2+a_{k,2}^2) \leq 1. \qquad (2.35)$$

Therefore, we need to find the largest ζ such that

$$\frac{[k(1+\beta)-(\zeta+\beta)][1+\gamma(k-1)]b_k}{(1-\zeta)}2.36 \qquad (4)$$

$$\leq \frac{1}{2}\left\{\frac{[k(1+\beta)-(\alpha+\beta)][1+\gamma(k-1)]b_k}{(1-\alpha)}\right\}^2,$$

that is

$$\zeta \leq 1-\frac{2(1+\beta)(k-1)(1-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2[1+\gamma(k-1)]b_k-2(1-\alpha)^2}. \qquad (2.37)$$

Now defining the function $H(k)$ by

$$H(k)=1-\frac{2(1+\beta)(k-1)(1-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2[1+\gamma(k-1)]b_k-2(1-\alpha)^2}. \qquad (2.38)$$

We see that $H(k)$ is increasing function of $k(k\geq 2)$.Therefore, we conclude that

$$\zeta \leq H(2) = 1-\frac{2(1+\beta)(1-\alpha)^2}{(2-\alpha+\beta)^2(1+\gamma)b_2-2(1-\alpha)^2}. \qquad (2.39)$$

and hence the proof of Theorem **??** is completed .

. Putting $\gamma = 0$ and $b_k = [1+\lambda(k-1)]^n$ $(\lambda \geq 0, k \geq 2, n \in \mathbb{N}_0)$ in Theorems **??**, **??** and **??**, respectively, we obtain the results obtained by Aouf and Mostafa [3, Theorems 8, 9 and 10, respectively].

3 Properties associated with generalized fractional calculus operators

In terms of the Gaussian hypergeometric function:

$$
\begin{aligned}
{}_2F_1(\delta,\mu;\nu;z) &= \sum_{k=0}^{\infty}\frac{(\delta)_k(\mu)_k}{(\nu)_k}\frac{z^k}{k!}3.1 \qquad (5)\\
(z &\in \mathbb{U};\delta,\mu,\nu\in\mathbb{C};\upsilon\neq 0,-1,-2,...),
\end{aligned}
$$

where(and in what follows) $(\theta)_k$ denotes that Pochhammer symbol defined in terms of Gamma functions, by

$$
(\theta)_k=\frac{\Gamma(\theta+k)}{\Gamma(\theta)}=\begin{cases}1 & (k=0)\\ \theta(\theta+1)...(\theta+k-1) & (k=\mathbb{N}).\end{cases}
$$

The generalized fractional calculus operators $I_{0,z}^{\mu,\nu,\eta}$ and $J_{0,z}^{\mu,\nu,\eta}$ are defined below (cf., eg., [12] and [20]).

[Generalized Fractional Integral Operators]. The generalized fractional integral of order μ is defined, for function $f(z)$, by

$$
\begin{aligned}
I_{0,z}^{\mu,\nu,\eta}f(z) &= \frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_0^z(z-\zeta)^{\mu-1}\,{}_2F_1(\mu+\nu;-\eta;\mu;1-\frac{\zeta}{z})f(\zeta)d\,\zeta \qquad (3.2)\\
(\mu &> 0;\ \ \epsilon>max\{0,\nu-\mu\}-1),
\end{aligned}
$$

where $f(z)$is an analytic function in a simply -connected region of the $z-$plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$to be real when $(z-\zeta)>0$, provided further that

$$
f(z)=O(|z|^{\epsilon})(z\to 0). \qquad (3.3)
$$

[Generalized Fractional Derivative Operators]. The generalized fractional derivative of order μ is defined, for function $f(z)$, by

$$
J_{0,z}^{\mu,\nu,\eta}f(z)=\begin{cases}\frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\left\{\int_0^z z^{\mu-\nu}(z-\zeta)^{-\mu}{}_2F_1(\nu-\mu;1-\eta;1-\mu;1-\frac{\zeta}{z})f(\zeta)d\,\zeta\right\} & (0\leq\mu<1);\\ \frac{d^n}{dz^n}J_{0,z}^{\mu-n,\nu,\eta}f(z) & (n\leq\mu<n+1;n\in\mathbb{N})\end{cases} \qquad (3.4)
$$

$$
\epsilon>max\{0,\gamma-\eta\}-1),
$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed, as in Definition **??** , and ϵ is given by the order estimate (3.3) .

It follows from Definition **??** and Definition **??** that

$$
I_{0,z}^{\mu,-\mu,\eta}f(z)=D_z^{-\mu}f(z) \qquad (\mu>0), \qquad (3.5)
$$

and

$$
J_{0,z}^{\mu,\mu,\eta}f(z)=D_z^{\mu}f(z) \qquad (\ 0\leq\ \mu<1), \qquad (3.6)
$$

where $D_z^{\mu}(\mu \in R)$ is the fractional operator considered by Owa [12] and (subsequently) by Srivastava and Owa [19]. Furthermore, in terms of Gamma function, Definition ?? and Definition ?? readily yield.

[20]. The generalized fractional integral and the generalized fractional derivative of a power function are given by

$$I_{0,z}^{\mu,\nu,\eta} z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho+\mu+\eta+1)} z^{\rho-\nu} 3.7 \tag{7}$$
$$(\mu > 0; \rho > max\{0, \nu-\eta\}-1),$$

and

$$J_{0,z}^{\mu,\nu,\eta} z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho-\mu+\eta+1)} z^{\rho-\nu} 3.8 \tag{8}$$
$$(0 \leq \mu < 1;\ \rho > max\{0, \nu-\eta\}-1).$$

. Let the function $f(z)$ defined by (1.8) be in the class $TS_{\gamma}(f, g\ ; \alpha, \beta)$. Then

$$\begin{aligned} & \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)} |z|^{1-\nu} \left\{1 - \frac{2(1-\alpha)(2-\nu+\eta)}{(2-\alpha+\beta)(1+\gamma)b_2(2-\nu)(2+\mu+\eta)} |z|\right\} \\ \leq & \left|I_{0,z}^{\mu,\nu,\eta} f(z)\right| 3.9 \\ \leq & \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)} |z|^{1-\nu} \left\{1 + \frac{2(1-\alpha)(2-\nu+\eta)}{(2-\alpha+\beta)(1+\gamma)b_2(2-\nu)(2+\mu+\eta)} |z|\right\} \\ (z \in & \mathbb{U}_0; \mu > 0; \max\{\nu, \nu-\eta, -\mu-\eta\} < 2; \gamma(\mu+\eta) \leq 3\mu) \end{aligned} \tag{9}$$

and

$$\begin{aligned} & \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)} |z|^{1-\nu} \left\{1 - \frac{2(1-\alpha)\Gamma(2-\nu+\mu)}{(2-\alpha+\beta)(1+\gamma)b_2(2-\nu)(2-\mu+\eta)} |z|\right\} \\ \leq & \left|J_{0,z}^{\mu,\nu,\eta} f(z)\right| \\ \leq & \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)} |z|^{1-\nu} \left\{1 + \frac{2(1-\alpha)\Gamma(2-\nu+\mu)}{(2-\alpha+\beta)(1+\gamma)b_2(2-\nu)(2-\mu+\eta)} |z|\right\} \\ (z \in & \mathbb{U}_0; 0 \leq \mu < 1; \max\{\nu, \nu-\eta, \mu-\eta\} < 2; \gamma(\mu-\eta) \geq 3\mu), \end{aligned} \tag{3.10}$$

where

$$\mathbb{U}_0 = \begin{cases} \mathbb{U} & (\nu \leq 1) \\ \mathbb{U}\backslash\{0\} & (\nu > 1)\ . \end{cases} \tag{3.11}$$

Each of these results is sharp for the function $f(z)$defined by

$$f(z) = z - \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2} z^2 \qquad (z \in \mathbb{U}). \tag{3.12}$$

Proof. First of all, since the function $f(z)$ defined by (1.8) is in the class $TS_\gamma(f, g\,; \alpha, \beta)$, we can apply Lemma **??** to deduce that

$$\sum_{k=2}^{\infty} a_k = \frac{1-\alpha}{(2-\alpha+\beta)(1+\gamma)b_2}. \tag{3.13}$$

Next, making use of the assertion (3.7) of Lemma **??**, we find from (1.8) that

$$F(z) = \frac{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}{\Gamma(2-\nu+\eta)} z^\nu I_{0,z}^{\mu,\nu,\eta} f(z) = z - \sum_{k=2}^{\infty} \Theta(k) a_k z^k, \tag{3.14}$$

where, for convenience,

$$\Theta(k) = \frac{(1)_k (2-\nu+\eta)_{k-1}}{(2-\nu)_{k-1}(2+\mu+\eta)_{k-1}} \qquad (k \in \mathbb{N} \setminus \{1\}). \tag{3.15}$$

The function $\Theta(k)$ defined by(3.15) can easily be seen to be non-increasing under the parametric constraints stated already with (3.9), we thus have

$$0 < \Theta(k) \leq \Theta(2) = \frac{2(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)} \qquad (k \in \mathbb{N} \setminus \{1\}). \tag{3.16}$$

Now the assertion (3.9) of Theorem **??** would follow readily from (3.13) and (3.16).

The assertion (3.10) of Theorem **??** can be proved similarly by noting from (3.8) that

$$G(z) = \frac{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}{\Gamma(2-\nu+\eta)} z^\nu J_{0,z}^{\mu,\nu,\eta} f(z) = z - \sum_{k=2}^{\infty} \Psi(k) a_k z^k, \tag{3.17}$$

where

$$\begin{aligned} 0 \ &< \ \Psi(k) = \frac{(1)_k (2-\nu+\eta)_{k-1}}{(2-\nu)_{k-1}(2-\mu+\eta)_{k-1}} 3.18 \qquad (11) \\ &\leq \ \Psi(2) = \frac{2(2-\nu+\eta)}{(2-\nu)(2-\mu+\eta)} \qquad (k \in \mathbb{N} \setminus \{1\}), \end{aligned}$$

where the parametric constraints stated already with (3.10).

Finally, by observing that the equalities in each of the constraints (3.9) and (3.10) are attained by the function $f(z)$ given by (3.12), we complete the proof of Theorem **??**.

In view of the relationships (3.5) and (3.6), by setting $\nu = -\mu$ and $\nu = \mu$ in our constraints

(3.9) and (3.10), respectively, we obtain

. Let the function $f(z)$ defined by (1.8) in the class $TS_{\gamma}(f,g\ ;\alpha,\beta)$. Then

$$\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2(2+\mu)}\,|z|\right\}\leq \left|D_z^{-\mu}f(z)\right| 3.19 \tag{12}$$
$$\leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2(2+\mu)}\,|z|\right\} \qquad (z\in\mathbb{U};\mu>0).$$

The result is sharp for the function $f(z)$ given by (3.12).

. Let the function $f(z)$ defined by (1.8) in the class $TS_{\gamma}(f,g\ ;\alpha,\beta)$. Then

$$\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2(2-\mu)}\,|z|\right\}\leq \left|D_z^{\mu}f(z)\right| 3.20 \tag{13}$$
$$\leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{2(1-\alpha)}{(2-\alpha+\beta)(1+\gamma)b_2(2-\mu)}\,|z|\right\} \qquad (z\in\mathbb{U};0\leq\mu<1).$$

The result is sharp for the function $f(z)$ given by (3.12).

. Putting $\gamma=0$ and $b_k=[1+\lambda(k-1)]^n$ $(\lambda\geq 0, k\geq 2, n\in\mathbb{N}_0)$in Theorem **??**, and Corollaries **??**, **??**, respectively, we obtain the results obtained by Aouf and Mostafa [3, Theorem 11, and Corollaries 5,6, respectively].

. Specializing the parameters γ, α, β, and function $g(z)$ in our results, we obtain new results associated to the subclasses mentioned in the introduction.

4 Open problem

The authors suggest to study the properties of the class $S_p^{\lambda}(f,g;\alpha,\beta)$ when the functions $f(z)$ and $g(z)$ are p-valent functions.

5 Acknowledgment.

The authors thank the referees for their valuable suggestions which led to the improvement of this paper.

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