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Characterization and other properties of analytic functions containing generalized operator

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Abstract

In this note, the authors introduce two new subclasses $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ and $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ of analytic functions in the open uint disk $\mathbb{U}$. The subclasses are obtained by making use of combinations of generalized differential operator with iterations of the Owa-Srivastava operator for normalized analytic functions. Characterization and other properties such as inclusion theorems, distortion bounds, extreme points, convex linear combination, integral means inequalities and Fekete-Szegö problems for the above mentioned classes are obtained.

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1 Introduction and Definitions

Let $\mathcal{H}(\mathbb{U})$ be the class of functions analytic in $\mathbb{U}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in \mathbb{U}). \tag{1}$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}), \tag{2}$$

which satisfy the following usual normalized condition:

$$f(0) = f'(0) - 1 = 0.$$

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order ζ if it satisfies the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \zeta \quad (z \in \mathbb{U}), \tag{3}$$

for some $0 \leq \zeta < 1$. We denote by $\mathcal{S}^*(\zeta)$, the class of such functions. Further, a function $f \in \mathcal{A}$ is said to be convex of order ζ if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \zeta \quad (z \in \mathbb{U}), \tag{4}$$

for some $0 \leq \zeta < 1$ and it is denoted by $\mathcal{K}(\zeta)$. In view of Alexander's transform $f(z) \in \mathcal{K}(\zeta) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\zeta)$ (for detail, see [13]). In particular, the class $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are familiar classes of starlike and convex functions in $\mathbb{U}$ respectively.

For the function $f \in \mathcal{A}$ given by (2), the operator $\Omega_z^\lambda : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\Omega_z^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n \quad (-\infty < \lambda < 1; z \in \mathbb{U}). \tag{5}$$

Note that

$$\Omega_z^0 f(z) = f(z), \quad \Omega_z^1 f(z) = zf'(z) \quad \text{ and } \quad \Omega_z^{-1} f(z) = \frac{2}{z}\int_0^z f(\tau)d\tau.$$

The operator Ω_z^λ is popularly known as the Owa-Srivastava operator in literature (see, e.g., [9, 10]).
We observe that $\Omega_z^\lambda f$ provides an analytic expression for the fractional derivative of order λ $(0 \leq \lambda < 1)$ and fractional integral of order λ $(-\infty < \lambda < 0)$ for functions of a complex variable, defined by Owa and Srivastava [11] and Srivastava and Owa [16].

Definition 1.1 *For $f \in \mathcal{A}$, the operator $\Omega_z^{(\lambda,m)} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by*

$$\Omega_z^{(\lambda,0)} f(z) = f(z)$$
$$\Omega_z^{(\lambda,1)} f(z) = \Omega_z^\lambda f(z)$$

and for $m = 2, 3, 4, \cdots; z \in \mathbb{U}$

$$\begin{aligned} \Omega_z^{(\lambda,m)} f(z) &= \Omega_z^\lambda \left(\Omega_z^{m-1} f(z)\right) \\ &= z + \sum_{n=2}^{\infty} \left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m a_n z^n \end{aligned} \tag{6}$$

Definition 1.2 *(see [12]) For $f \in \mathcal{A}$ given by (2), the generalized differential operator $\mathcal{D}^k_{\alpha,\beta,\eta,\delta} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by*

$$\begin{aligned}
\mathcal{D}^0_{\alpha,\beta,\eta,\delta} f(z) &= f(z) \\
\mathcal{D}^1_{\alpha,\beta,\eta,\delta} f(z) &= [1-(\eta-\delta)(\beta-\alpha)]f(z) + (\eta-\delta)(\beta-\alpha)zf'(z) \\
&= z + \sum_{n=2}^{\infty} [(\eta-\delta)(\beta-\alpha)(n-1)+1]a_n z^n, \\
\cdots & \\
\cdots & \\
\cdots & \\
\mathcal{D}^k_{\alpha,\beta,\eta,\delta} f(z) &= \mathcal{D}^1_{\alpha,\beta,\eta,\delta}\left(\mathcal{D}^{k-1}_{\alpha,\beta,\eta,\delta} f(z)\right) \\
&= z + \sum_{n=2}^{\infty} [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k a_n z^n \qquad (7)
\end{aligned}$$

$$(\alpha \geq 0,\ \beta > 0,\ \eta > 0,\ \delta \geq 0,\ \eta > \delta,\ \beta > \alpha,\ k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Thus, for the function $f \in \mathcal{A}$ given by (2), we introduce the operator $\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta} : \mathcal{A} \longrightarrow \mathcal{A}$ as the combination of the operators $\Omega_z^{(\lambda,m)}$ and $\mathcal{D}^k_{\alpha,\beta,\eta,\delta}$.
That is for $z \in \mathbb{U}$,

$$\begin{aligned}
\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta} f(z) &= \Omega_z^{(\lambda,m)} \mathcal{D}^k_{\alpha,\beta,\eta,\delta} f(z) \\
&= z + \sum_{n=2}^{\infty} \left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k a_n z^n \qquad (8)
\end{aligned}$$

The operator $\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}$ generalizes several previously studied familiar operators. The following are some of the interesting particular cases:

- For $k = 0$ and $m = 1$, the operator $\mathcal{Q}^{0,\lambda,1}_{\alpha,\beta,\eta,\delta} = \Omega^\lambda_z$ is the familiar Owa-Srivastava operator [11];
- for $m = \alpha = \delta = 0, \eta = \beta = 1$, the operator $\mathcal{Q}^{k,\lambda,0}_{0,1,1,0} = \mathrm{S}^k$ is the popular Sãlãgean operator [14];
- for $m = 0,\ \lambda = 0$ and $\alpha = 0$, the operator $\mathcal{Q}^{k,0,0}_{0,\beta,\eta,\delta} = D^k_{\delta,\beta,\eta}$ has been studied by Darus and Ibrahim [2];
- for $m = \alpha = \delta = 0, \eta = 1$, the operator $\mathcal{Q}^{k,\lambda,0}_{0,\beta,1,0} = D^k_\beta$ has been studied by Al-Oboudi [1].

Using the generalized operator $\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}$, we introduce the new classes of analytic functions as follows:

Definition 1.3 *Let $f(z) \in \mathcal{A}$. Then $f(z)$ is in the class $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ if and only if*

$$\Re\left[\frac{z\left(\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)\right)'}{\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)}\right] > \mu \quad (0 \leq \mu < 1; z \in \mathbb{U}). \tag{9}$$

Definition 1.4 *Let $f(z) \in \mathcal{A}$. Then $f(z)$ is in the class $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ if and only if*

$$\Re\left[\frac{\left\{z\left(\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)\right)'\right\}'}{\left\{\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)\right\}'}\right] > \mu \quad (0 \leq \mu < 1; z \in \mathbb{U}). \tag{10}$$

Note that for $m = k = 0$, the classes defined by (9) and (10) reduces to the class of starlike function and convex function order μ respectively.

The object of the present paper is to study characterization properties, distortion bounds, extreme points, linear combination and Fekete-Szgö inequalities for the classes $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ and $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$.

2 Characterization properties

In this section we study the characterization properties for function $f(z) \in \mathcal{A}$ given by (2) belong to the classes $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ and $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ by obtaining the coefficient bounds.

Theorem 2.1 *Let the function $f(z)$ of the form (2) be in the class $\mathcal{A}$. If*

$$\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \leq (1-\mu) \tag{11}$$

for some μ $(0 \leq \mu < 1)$, then $f(z) \in \mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$. The result (11) is sharp.

Proof. Suppose that (11) holds true for μ $(0 \leq \mu < 1)$. Since

$$\begin{aligned}
1-\mu \;&\geq\; \sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \\
&\geq\; \sum_{n=2}^{\infty}\mu\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \\
&-\; \sum_{n=2}^{\infty}n\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n|,
\end{aligned}$$

then this implies that

$$\frac{1+\sum_{n=2}^{\infty} n\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k |a_n|}{1+\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k |a_n|} > \mu.$$

So we conclude that

$$\Re\left[\frac{z\left(\mathcal{Q}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m} f(z)\right)'}{\mathcal{Q}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m} f(z)}\right] > \mu \quad (0 \le \mu < 1; z \in \mathbb{U}).$$

Thus, $f(z) \in \mathcal{S}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}(\mu)$.
The assertion (11) is sharp for the extremal function given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\mu}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n.$$

The proof of Theorem 2.1 is completed.

Corollary 2.2 *Let the hypothesis of Theorem 2.1 be satisfied. Then*

$$|a_n| \le \frac{1-\mu}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} \quad (\forall \;\; n \ge 2).$$

Corollary 2.3 *Let the hypothesis of Theorem 2.1 be satisfied. Then for* $\mu = m = k = 0$, *we have*

$$|a_n| \le \frac{1}{n} \quad (\forall \;\; n \ge 2).$$

In the similar manner we can derive the following result for the class $\mathcal{C}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}(\mu)$. We choose to omit the detail.

Theorem 2.4 *Let the function* $f(z) \in \mathcal{A}$ *be given by (2). If*

$$\sum_{n=2}^{\infty} n(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k |a_n| \le (1-\mu) \quad (12)$$

for some μ $(0 \le \mu < 1)$, *then* $f(z) \in \mathcal{C}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}(\mu)$. *The result (12) is sharp.*

Corollary 2.5 *Let the hypothesis of Theorem 2.4 be satisfied. Then*

$$|a_n| \le \frac{1-\mu}{n(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} \quad (\forall \;\; n \ge 2).$$

Corollary 2.6 *Let the hypothesis of Theorem 2.4 be satisfied. Then for* $\mu = m = k = 0$, *we have*

$$|a_n| \le \frac{1}{n^2} \quad (\forall \;\; n \ge 2).$$

3 Inclusion theorems

In this section we discuss the inclusion theorems for the class $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ and $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$. The proof is straight forward and follows from Theorem 2.1 and Theorem 2.4. We only mention the statement.

Theorem 3.1 *Let* $0 \le \mu_1 \le \mu_2 < 1$. *Then*

$$\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu_1) \supseteq \mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu_2).$$

Theorem 3.2 *Let* $0 \le \mu_1 \le \mu_2 < 1$. *Then*

$$\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu_1) \supseteq \mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu_2).$$

4 Distortion bounds

A distortion property for the function $f \in \mathcal{A}$ in the class $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ and $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ are considered in the following theorems.

Theorem 4.1 *Let the hypothesis of Theorem 2.1 be satisfied. Then for* $z \in \mathbb{U}$ *and* $0 \le \mu < 1$, *we have*

$$|z| - \frac{1-\mu}{2-\mu}|z|^2 \le |\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta} f(z)| \le |z| + \frac{1-\mu}{2-\mu}|z|^2. \tag{13}$$

Proof. By virtue of (11) of Theorem 2.1, we have

$$\begin{aligned} &(2-\mu)\textstyle\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \\ \le\ &\textstyle\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \\ &\le 1-\mu. \end{aligned}$$

Therefore, we have

$$\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \le \frac{1-\mu}{2-\mu}. \tag{14}$$

It follows from (8) that

$$\begin{aligned} &\left|\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta} f(z)\right| \\ \le\ & |z| + \textstyle\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n||z|^n \\ \le\ & |z| + \textstyle\sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n||z|^2 \\ &\le |z| + \tfrac{1-\mu}{2-\mu}|z|^2. \end{aligned} \tag{15}$$

Further,

$$\begin{aligned}
\left|\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)\right| &\geq |z| - \sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n||z|^n \\
&\geq |z| - \sum_{n=2}^{\infty}\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n||z|^2 \\
&\geq |z| - \frac{(1-\mu)}{(2-\mu)}|z|^2, \qquad (16)
\end{aligned}$$

by application of (14). Therefore, the assertion (13) follows from (15) and (16). The proof of Theorem 4.1 is thus completed.

The proof of the following theorem can be derived in the same line to that of Theorem 4.1. We use the hypothesis of Theorem 2.4 instead of Theorem 2.1. We mention here only the statement and choose to omit the detail

Theorem 4.2 *Let the hypothesis of Theorem 2.4 be satisfied. Then for* $z \in \mathbb{U}$ *and* $0 \leq \mu < 1$, *we have*

$$|z| - \frac{1-\mu}{2(2-\mu)}|z|^2 \leq |\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)| \leq |z| + \frac{1-\mu}{2(2-\mu)}|z|^2.$$

Theorem 4.3 *Let the hypothesis of Theorem 2.1 be satisfied. Then*

$$|z| - \frac{(1-\mu)(2-\lambda)^m}{2^m(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k} \leq |f(z)| \leq |z| + \frac{(1-\mu)(2-\lambda)^m}{2^m(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k}|z|^2. \qquad (17)$$

Proof. By application of Theorem 2.1, we have

$$\begin{aligned}
&(2-\mu)\tfrac{2^m}{(2-\lambda)^m}[(\eta-\delta)(\beta-\alpha)+1]^k \textstyle\sum_{n=2}^{\infty}|a_n| \\
\leq\ & \textstyle\sum_{n=2}^{\infty}(n-\mu)\left[\tfrac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n| \\
\leq\ & 1-\mu,
\end{aligned}$$

which implies

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(1-\mu)(2-\lambda)^m}{2^m(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k}.$$

Therefore, we have

$$\begin{aligned}
|f(z)| &= |z + \sum_{n=2}^{\infty}a_nz^n| \\
&\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\
&\leq |z| + \frac{(1-\mu)(2-\lambda)^m}{2^m(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k}|z|^2. \qquad (18)
\end{aligned}$$

Again,

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^2$$
$$\geq |z| - \frac{(1-\mu)(2-\lambda)^m}{2^m(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k} \qquad (19)$$

The assertion (17) follows from (18) and (19). This complete the proof of Theorem 4.3.

Using same technique we can get the following result.

Theorem 4.4 *Let the hypothesis of Theorem 2.4 be satisfied. Then*

$$(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k \geq 1 \qquad \textit{and} \qquad 0 \leq \mu < 1$$

implies

$$|z| - \frac{(1-\mu)(2-\lambda)^m}{2^{m+1}(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k}|z|^2 \leq |f(z)|$$
$$\leq |z| + \frac{(1-\mu)(2-\lambda)^m}{2^{m+1}(2-\mu)[(\eta-\delta)(\beta-\alpha)+1]^k}|z|^2.$$

5 Extreme points

The determination of the extreme point of a family $\mathcal{F}$ of univalent functions enable us to solve many extremal problems for $\mathcal{F}$.

Theorem 5.1 *Let* $f_1(z) = z$ *and*

$$f_n(z) = z + \frac{(1-\mu)\epsilon_n}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n \quad (\forall \ \ n \geq 2; |\epsilon_n| = 1).$$

Then the hypothesis of Theorem 2.1 is satisfied if and only if $f(z)$ *can be expressed in the form as* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \ \ \lambda_n \geq 0, n \geq 1$ with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \left[z + \frac{(1-\mu)\epsilon_n}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n \right] \end{aligned}$$

$$= \sum_{n=1}^{\infty}\lambda_n z + \sum_{n=2}^{\infty}\lambda_n \frac{(1-\mu)\epsilon_n}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n$$
$$= z + \sum_{n=2}^{\infty}\lambda_n \frac{(1-\mu)\epsilon_n}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n.$$

Now

$$\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k$$
$$\left|\frac{(1-\mu)\epsilon_n\lambda_n}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\mu)(\beta-\alpha)(n-1)+1]^k}\right|$$
$$= \sum_{n=2}^{\infty}(1-\mu)\lambda_n = (1-\mu)\sum_{n=2}^{\infty}\lambda_n = (1-\mu)\left(\sum_{n=1}^{\infty}\lambda_n - \lambda_1\right)$$
$$= (1-\mu)(1-\lambda_1) \leq (1-\mu) \quad (\lambda_1 \geq 0),$$

which is the condition (11) for $f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$.
Conversely, let the hypothesis of Theorem 2.1 be satisfied. Hence by Corollary 2.2 we have

$$|a_n| \leq \frac{(1-\mu)}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} \quad (\forall n \geq 2).$$

Put

$$\lambda_n = \frac{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k}{(1-\mu)\epsilon_n} a_n \quad (|\epsilon_n| = 1)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n.$$

Then

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z).$$

Corollary 5.2 *Let the hypothesis of Theorem 2.1 be satisfied. Then the extreme points of* $\mathcal{S}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}(\mu)$ *are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z + \frac{(1-\mu)\epsilon_n}{(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n \quad (n = 2, 3, \cdots, |\epsilon_n| = 1).$$

Proceeding in the same technique as Theorem 5.1, we get the following result for the class $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$.

Theorem 5.3 *Let $f_1(z) = z$ and*

$$f_n(z) = z + \frac{(1-\mu)\epsilon_n}{n(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n.$$

Then the hypothesis of Theorem 2.4 is satisfied if and only if it can be expressed in the form as $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Corollary 5.4 *Let the hypothesis of Theorem 2.4 is satisfied. The extreme points of $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ are the function $f_1(z) = z$ and*

$$f_n(z) = z + \frac{(1-\mu)\epsilon_n}{n(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k} z^n$$
$$(n = 2, 3, \cdots, |\epsilon_n| = 1).$$

6 Convex linear combination

Theorem 6.1 *Let the functions $f(z)$ given by (2) and $g \in \mathcal{A}$ defined by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

be satisfied the condition of Theorem 2.1. Then the function $h(z)$ defined by

$$h(z) = (1-l)f(z) + lg(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{20}$$

where

$$c_n = (1-l)a_n + lb_n \quad (0 \leq l \leq 1)$$

is in the class $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$.

Proof. Suppose that each of the functions f and g satisfy the condition of Theorem 2.1. To show $h(z)$ is in the class $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$, by virtue of Theorem 2.1 it is sufficient to show

$$\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k |c_n| \leq (1-\mu).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|c_n|\\
&=\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|(1-l)a_n+lb_n|\\
&\leq(1-l)\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|a_n|\\
&\quad+l\sum_{n=2}^{\infty}(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k|b_n|\\
&\leq(1-l)(1-\mu)+l(1-\mu)\\
&=(1-\mu),
\end{aligned}$$

The proof of Theorem 6.1 is thus completed.

Remark 6.2 *Using same technique as in Theorem 6.1 and making use of Theorem 2.4 one can shown that* $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ *is a convex set.*

7 Integral Means Inequalities

For any two functions f and g analytic in $\mathbb{U}$, f is said to be subordinate to g in $\mathbb{U}$, denoted by $f \prec g$ if there exists an analytic function ω defined in $\mathbb{U}$ satisfying $\omega(0)=0$ and $|\omega(z)|<1$ such that $f(z)=g(\omega(z))$ $(z\in\mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$ (see [8]).

In order to prove our result, we need the following lemma.

Lemma 7.1 *(see [7]) If f and g are any two functions, analytic in $\mathbb{U}$ with $f\prec g$, then for $\xi>0$ and $z=re^{i\theta}$ $(0<r<1)$,*

$$\int_0^{2\pi}|f(z)|^{\xi}d\theta\leq\int_0^{2\pi}|g(z)|^{\xi}d\theta. \tag{21}$$

Theorem 7.2 *Let the condition of Theorem 2.1 be satisfied and f_n is given by*

$$f_n(z)=z+\frac{(1-\mu)\epsilon_n}{\psi(\lambda,\mu,\alpha,\beta,\eta,\delta,k,n)}z^n,\quad (n=2,3,\cdots,|\epsilon_n|=1),$$

where

$$\psi(\lambda,\mu,\alpha,\beta,\eta,\delta,k,n)=(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m[(\eta-\delta)(\beta-\alpha)(n-1)+1]^k.$$

If there exists an analytic function $\omega(z)$ given by

$$[\omega(z)]^{n-1} = \frac{\psi(\lambda, \mu, \alpha, \beta, \eta, \delta, k, n)}{(1-\mu)\epsilon_n} \sum_{n=2}^{\infty} a_n z^{n-1}, \tag{22}$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^{\xi} d\theta \leq \int_0^{2\pi} |f_n(re^{i\theta})|^{\xi} d\theta \quad (\xi > 0). \tag{23}$$

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, (23) is equivalent to prove that

$$\int_0^{2\pi} \left| 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right|^{\xi} d\theta \leq \int_0^{2\pi} \left| 1 + \frac{(1-\mu)\epsilon_n}{\psi(\lambda, \mu, \alpha, \beta, \eta, \delta, k, n)} z^{n-1} \right|^{\xi} d\theta.$$

By Lemma 7.1, it is sufficient to show that

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 + \frac{(1-\mu)\epsilon_n}{\psi(\lambda, \mu, \alpha, \beta, \eta, \delta, k, n)} z^{n-1}.$$

By setting

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} = 1 + \frac{(1-\mu)\epsilon_n}{\psi(\lambda, \mu, \alpha, \beta, \eta, \delta, k, n)} [\omega(z)]^{n-1},$$

we get

$$[\omega(z)]^{n-1} = \frac{\psi(\lambda, \mu, \alpha, \beta, \eta, \delta, k, n)}{(1-\mu)\epsilon_n} \sum_{n=2}^{\infty} a_n z^{n-1}.$$

Clearly, $\omega(0) = 0$. By application of Theorem 2.1, we have

$$\begin{aligned} |[\omega(z)]|^{n-1} &= \left| \tfrac{\psi(\lambda,\mu,\alpha,\beta,\eta,\delta,k,n)}{(1-\mu)\epsilon_n} \textstyle\sum_{n=2}^{\infty} a_n z^{n-1} \right| \\ &\leq \tfrac{\psi(\lambda,\mu,\alpha,\beta,\eta,\delta,k,n)}{(1-\mu)|\epsilon_n|} \textstyle\sum_{n=2}^{\infty} |a_n||z|^{n-1} \\ &\leq |z| < 1. \end{aligned}$$

This complete the proof of Theorem 7.2.

In the same manner we can prove the integral means inequalities for the functional class $\mathcal{C}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}(\mu)$.

Theorem 7.3 *Let the hypothesis of Theorem 2.4 be satisfied. Define the function g_n by*

$$g_n(z) = z + \frac{(1-\mu)\epsilon_n}{\phi(\lambda, \mu, \alpha, \beta, \eta, \delta, k, n)} z^n, \quad (n = 2, 3, \cdots, |\epsilon_n| = 1),$$

where

$$\phi(\lambda,\mu,\alpha,\beta,\eta,\delta,k,n) = n(n-\mu)\left[\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\right]^m [(\eta-\delta)(\beta-\alpha)(n-1)+1]^k.$$

If there exists an analytic function $\omega_1(z)$ *given by*

$$[\omega_1(z)]^{n-1} = \frac{\phi(\lambda,\mu,\alpha,\beta,\eta,\delta,k,n)}{(1-\mu)\epsilon_n}\sum_{n=2}^{\infty} a_n z^{n-1}, \tag{24}$$

then for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^{\xi} d\theta \leq \int_0^{2\pi} |g_n(re^{i\theta})|^{\xi} d\theta \quad (\xi > 0). \tag{25}$$

8 Fekete-Szegö Problems

It is well-known ([3]) that for $f \in \mathcal{S}$ and given by (2), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. Fekete and Szegö [4] obtained sharp upper bounds for $|a_3 - \nu a_2^2|$ for $f \in \mathcal{S}$ when μ is real. Thus the determination of sharp upper bounds for the nonlinear functional $|a_3 - \nu a_2^2|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{A}$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. For a brief history of the Fekete-Szegö problem see([15]).

In this section we determine the sharp upper bound for $|a_2|$ for the classes $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$ and $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$. Moreover, we calculate the Fekete-Szegö $|a_3 - \nu a_2^2|$ functional for the above mentioned classes. For this we need the following preliminary.

Let $\mathcal{P}$ be the family of all functions p analytic in $\mathbb{U}$ for which $\Re(p(z)) > 0$, $p(0) = 1$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U}). \tag{26}$$

Lemma 8.1 *(see[5, 6]) Let the function* $p \in \mathcal{P}$ *be given by the series (26). Then for any complex number* μ,

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\} \tag{27}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}). \tag{28}$$

Theorem 8.2 *Let the function* f *given by (2) be in the class* $\mathcal{S}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$. *Then*

$$|a_2| \leq \frac{(1-\mu)(2-\lambda)^m}{2^{m-1}[(\eta-\delta)(\beta-\alpha)+1]^k} \tag{29}$$

and for any complex number ν,

$$|a_3 - \nu a_2^2| \leq \frac{[(2-\lambda)(3-\lambda)]^m}{6^m[2(\eta-\delta)(\beta-\alpha)+1]^k}(1-\mu)$$

$$\max\left\{1, \left|1+2(1-\mu)\left(1-\nu\frac{6^m(2-\lambda)^m[2(\eta-\delta)(\beta-\alpha)+1]^k}{2^{2m-1}(3-\lambda)^m[(\eta-\delta)(\beta-\alpha)+1]^{2k}}\right)\right|\right\}. \quad (30)$$

The result is sharp.

Proof. Since $f \in \mathcal{S}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}(\mu)$, hence by Definition 1.3 we have

$$\Re\left[\frac{z(\mathcal{Q}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}f(z))'}{\mathcal{Q}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}f(z)}\right] > \mu \quad (0 \leq \mu < 1; z \in \mathbb{U}). \quad (31)$$

The expression (31) is equivalent to

$$z(\mathcal{Q}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}f(z))' = [(1-\mu)p(z)+\mu]\mathcal{Q}_{\alpha,\beta,\eta,\delta}^{k,\lambda,m}f(z), \quad (32)$$

for some $p \in \mathcal{P}$. Equating coefficients of z^2 and z^3 on both sides of (32) we obtain

$$a_2 = \frac{(2-\lambda)^m}{2^m[(\eta-\delta)(\beta-\alpha)+1]^k}(1-\mu)p_1 \quad (33)$$

and

$$a_3 = \frac{[(2-\lambda)(3-\lambda)]^m}{6^m[2(\eta-\delta)(\beta-\alpha)+1]^k}\frac{(1-\mu)}{2}[p_2+(1-\mu)p_1^2], \quad (34)$$

Using well-known inequality $|p_n| \leq 2$ for all $n \in \mathbb{N}$, it follows from (33) that

$$|a_2| \leq \frac{(1-\mu)(2-\lambda)^m}{2^{m-1}[(\eta-\delta)(\beta-\alpha)+1]^k}.$$

This proves the assertion (29).
Further for $\nu \in \mathbb{C}$, we have

$$a_3 - \nu a_2^2 = \frac{[(2-\lambda)(3-\lambda)]^m}{6^m[2(\eta-\delta)(\beta-\alpha)+1]^k}\frac{(1-\mu)}{2}[p_2 - sp_1^2], \quad (35)$$

where

$$s = (1-\mu)\left[\nu\frac{6^m(2-\lambda)^m[2(\eta-\delta)(\beta-\alpha)+1]^k}{2^{2m-1}(3-\lambda)^m[(\eta-\delta)(\beta-\alpha)+1]^{2k}} - 1\right].$$

An application of Lemma 8.1 to (35) gives

$$|a_3 - \nu a_2^2| \leq \frac{[(2-\lambda)(3-\lambda)]^m}{6^m[2(\eta-\delta)(\beta-\alpha)+1]^k}(1-\mu)\max\{1, |2s-1|\}$$

$$= \frac{[(2-\lambda)(3-\lambda)]^m}{6^m[2(\eta-\delta)(\beta-\alpha)+1]^k}(1-\mu)$$

$$\max\left\{1, \left|1+2(1-\mu)\left(1-\nu\frac{6^m(2-\lambda)^m[2(\eta-\delta)(\beta-\alpha)+1]^k}{2^{2m-1}(3-\lambda)^m[(\eta-\delta)(\beta-\alpha)+1]^{2k}}\right)\right|\right\}.$$

This is precisely the assertion (30).
Equality is attained for the functions given by

$$\Re\left[\frac{z(\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z))'}{\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)}\right]=\frac{1+(1-2\mu)z^2}{1-z^2},$$

and

$$\Re\left[\frac{z(\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z))'}{\mathcal{Q}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}f(z)}\right]=\frac{1+(1-2\mu)z}{1-z}$$

respectively.

Corollary 8.3 *Let the assumption of Theorem 8.2 holds. Then for $\mu=0$, we have*

$$|a_2|\le\frac{(2-\lambda)^m}{2^{m-1}[(\eta-\delta)(\beta-\alpha)+1]^k},$$

and for any complex number mu,

$$|a_3-\nu a_2^2|\le\frac{[(2-\lambda)(3-\lambda)]^m}{6^m[2(\eta-\delta)(\beta-\alpha)+1]^k}$$
$$\max\left\{1,\left|1+2\left(1-\nu\frac{6^m(2-\lambda)^m[2(\eta-\delta)(\beta-\alpha)+1]^k}{2^{2m-1}(3-\lambda)^m[(\eta-\delta)(\beta-\alpha)+1]^{2k}}\right)\right|\right\}.$$

In the similar manner we can prove the following result for the class $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$.

Theorem 8.4 *Let the function f given by (2) be in the class $\mathcal{C}^{k,\lambda,m}_{\alpha,\beta,\eta,\delta}(\mu)$. Then*

$$|a_2|\le\frac{(1-\mu)(2-\lambda)^m}{2^m[(\eta-\delta)(\beta-\alpha)+1]^k}$$

and for any complex number ν,

$$|a_3-\nu a_2^2|\le\frac{2[(2-\lambda)(3-\lambda)]^m}{6^{m+1}[2(\eta-\delta)(\beta-\alpha)+1]^k}(1-\mu)$$
$$\max\left\{1,\left|1+2(1-\mu)\left(2-\nu\frac{6^{m+1}(2-\lambda)^m[2(\eta-\delta)(\beta-\alpha)+1]^k}{2^{2m+2}(3-\lambda)^m[(\eta-\delta)(\beta-\alpha)+1]^{2k}}\right)\right|\right\}.$$

The result is sharp.

Corollary 8.5 *Let the assumption of Theorem 8.4 holds. Then for $\mu = 0$, we have*

$$|a_2| \leq \frac{(2-\lambda)^m}{2^m[(\eta-\delta)(\beta-\alpha)+1]^k},$$

and for any complex number ν,

$$|a_3 - \nu a_2^2| \leq \frac{2[(2-\lambda)(3-\lambda)]^m}{6^{m+1}[2(\eta-\delta)(\beta-\alpha)+1]^k}$$
$$\max\left\{1, \left|1 + 2\left(2 - \nu\frac{6^{m+1}(2-\lambda)^m[2(\eta-\delta)(\beta-\alpha)+1]^k}{2^{2m+2}(3-\lambda)^m[(\eta-\delta)(\beta-\alpha)+1]^{2k}}\right)\right|\right\}.$$

9 Open Problem

Using another differential operator, define the other classes of analytic functions using the Definitions 1.3 and 1.4. For the defined classes find various properties such as inclusion theorems, distortion theorems, closure theorems, raddi of starlikeness, convexity and close-to-convex of functions belonging to these classes.

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