Int. J. Open Problems Complex Analysis, Vol. 7, No. 2, June 2015
ISSN 2074-2827;
www.i-csrs.org

Spectral theorems associated with the Dunkl type operator on the real line

Hatem Mejjaoli

The National Higher Engineering School of Tunis, Department of Mathematics
5 Avenue Taha Hussein, BP, 56, Bab Manara, Tunis, Tunisia
e-mail: hatem.mejjaoli@yahoo.fr

Abstract

We consider a singular differential-difference operator Λ on the real line which generalizes the Dunkl operator associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. We establish some spectral theorems for the generalized Fourier transform on $\mathbb{R}$ tied to Λ.

Keywords: *Dunkl type operator, generalized Fourier transform, real Paley-Wiener theorems, Roe's theorem.*
2000 Mathematical Subject Classification: 35L05, 22E30.

1 Introduction

In this paper, we consider the first-order singular differential-difference operator on $\mathbb{R}$

$$\Lambda f = \frac{df}{dx} + \frac{A'(x)}{A(x)} \left(\frac{f(x) - f(-x)}{2} \right),$$

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. We suppose in addition that

i) For all $x \geq 0, A(x)$ is increasing and $\lim\limits_{x\to\infty} A(x) = \infty$.

ii) For all $x > 0$, $\dfrac{A'(x)}{A(x)}$ is decreasing and $\lim\limits_{x\to\infty} \dfrac{A'(x)}{A(x)} = 2\rho \geq 0$.

iii) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty), x_0 > 0$, we have
$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x} D(x),$$
where D is a C^∞-function, bounded together with its derivatives.

The generalized Fourier transform is defined for a suitable function f on $\mathbb{R}$ by
$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\lambda(-x)A(x)dx$$
where Φ_λ is the solution of the differential-difference equation
$$\begin{cases} \Lambda u &= i\lambda u, \\ u(0) &= 1. \end{cases}$$

For
$$A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$$
we regain the differential-difference operator
$$\Lambda f(x) = \frac{d}{dx} f(x) + \Big((2\alpha+1)\coth(x) + (2\beta+1)\tanh(x)\Big)\Big(f(x) - f(-x)\Big),$$
which is referred to as the Jacobi-Dunkl operator (see [4, 8]). In this case, the generalized Fourier transform $\mathcal{F}_\Lambda$ coincides with the Jacobi-Dunkl transform.

For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, we regain the differential-difference operator
$$D_\alpha f = \frac{df}{dx} + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x},$$
which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl in connection with a generalization of the classical theory of spherical harmonics (see [9] and the references therein). In this case, the generalized Fourier transform $\mathcal{F}_\Lambda$ coincides with the Dunkl transform of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$.

Motivated by the treatment in the Euclidean setting of the Paley-Wiener and Roe's theorems, we will derive in this paper new real Paley-Wiener theorems for the generalized Fourier transform, on the Lebesgue space $L^2_A(\mathbb{R})$ and on the generalized tempered distribution space $\mathcal{S}'_2(\mathbb{R})$. Study the generalized tempered distributions with the spectral gaps. Finally we prove the Roe's theorem in the context of the Dunkl type operator. We note that the real Paley-Wiener theorems has been studied by many authors for various Fourier transforms, for examples (cf. [1, 7, 12, 21]) and others.

The contents of the paper is as follows. In §2 we recall some basic results about the harmonic analysis associated with the Dunkl type operator on the real line which we need in the sequel. The §3 is devoted to study the L^2 Paley-Wiener theorems for the generalized Fourier transform on the generalized Schwartz space of functions. In §4 we prove new versions of real Paley-Wiener theorems associated with the generalized Fourier transform. In §5 we prove the Roe's theorem for the Dunkl type operator on $\mathbb{R}$. Finally, in the last section we study the generalized tempered distributions with spectral gaps.

2 Preliminaries

This section gives an introduction to the harmonic analysis associated with the Dunkl type operator. The main references are [15, 20].

Notation. We denote by

$\mathcal{E}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$.

$\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

$\mathcal{S}_e(\mathbb{R})$ (resp. $\mathcal{S}_o(\mathbb{R})$) the subspace of $\mathcal{S}(\mathbb{R})$ consisting of even (resp. odd) functions.

$D(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ which are of compact support.

2.1 The eigenfunctions of the Dunkl type operator on the real line

To study the eigenfunctions of Λ, we consider first those of the second-order singular differential operator on $\mathbb{R}$ defined by

$$L = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

Our basic reference about L will be the paper [19] from which we recall the following result.

Lemma 1 *(i) For each $\lambda \in \mathbb{C}$ the differential equation*

$$Lu = -(\lambda^2 + \varrho^2)u, \quad u(0) = 1, \tag{1}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted φ_λ.

(ii) For every $x \in \mathbb{R}$, the function $\lambda \mapsto \varphi_\lambda$ is analytic.

(iii) For every $x \in \mathbb{R}$,

$$e^{-\varrho|x|} \le \varphi_0(x) \le 1. \tag{2}$$

(iv) There is a positive constant C such that for all $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)| \le C|x|^n e^{((|Im\lambda|-\varrho)\,|x|}. \tag{3}$$

Remark 1 *If* $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, *then the differential operator* L *reduced to the so-called Jacobi operator. The eigenfunction* φ_λ *is the Jacobi function of index* (α, β) *given by*

$$\varphi_\lambda(x) = F(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda);\ \alpha + 1;\ -(\sinh(x))^2), \tag{4}$$

where F *is the hypergeometric function* ${}_2F_1$ *of Gauss.*

Proposition 1 *For each* $\lambda \in \mathbb{C}$ *the differential-difference equation*

$$\Lambda u = i\lambda u, \quad u(0) = 1, \tag{5}$$

admits a unique C^∞ *solution on* $\mathbb{R}$, *denoted* Φ_λ *given by*

$$\Phi_\lambda(x) = \begin{cases} \varphi_{\sqrt{\lambda^2-\varrho^2}}(x) - \frac{1}{i\lambda}\frac{d}{dx}\varphi_{\sqrt{\lambda^2-\varrho^2}}(x) & \text{if } \lambda \neq 0 \\ \\ 1 & \text{if } \lambda = 0. \end{cases} \tag{6}$$

The following estimate for the eigenfunction $\Phi_\lambda(x)$ shall be useful.

Proposition 2 *Let* $\varrho > 0$. *There exist positive constant* C *such that for all* $x \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}$, *with* $|\lambda| \geq \varrho$ *and* $n \in \mathbb{N}_0$, *we have*

$$|\frac{d^n}{d\lambda^n}\Phi_\lambda(x)| \leq C(1 + |\lambda|)(1 + |x|)^n e^{-\varrho|x|}. \tag{7}$$

2.2 The generalized Fourier transform

Notations We denote by

$L^p_A(\mathbb{R})$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\begin{aligned} \|f\|_{L^p_A(\mathbb{R})} &= \left(\int_{\mathbb{R}} |f(x)|^p A(x)dx\right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty \\ \|f\|_{L^\infty_A(\mathbb{R})} &= ess\sup_{x\in\mathbb{R}} |f(x)| < \infty. \end{aligned}$$

$\mathcal{S}^2(\mathbb{R})$, the space of C^∞-functions on $\mathbb{R}$ such that for all $m, n \in \mathbb{N}$

$$q_{n,m}(f) := \sup_{x\in\mathbb{R}}(\cosh x)^\varrho(1 + x^2)^m|\frac{d^n}{dx^n}f(x)| < \infty.$$

The topology of $\mathcal{S}^2(\mathbb{R})$ is defined by the semi-norms $q_{n,m}$, $m, n \in \mathbb{N}$.

$\mathcal{S}^2_e(\mathbb{R})$ (resp. $\mathcal{S}^2_o(\mathbb{R})$) the subspace of $\mathcal{S}^2(\mathbb{R})$ consisting of even (resp. odd) functions.

Definition 1 *The generalized Fourier transform of a function $f \in L^1_A(\mathbb{R})$ is defined by*

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\lambda(-x)A(x)dx, \quad \text{for all } \lambda \in \mathbb{R}. \tag{8}$$

Remarks 1 *(i) From (2) we see that $\mathcal{S}^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.*

(ii) The generalized Schwartz space $\mathcal{S}^2(\mathbb{R})$ is invariant under the differential-difference operator Λ.

(iii) Due to our assumptions on the function A there is a positive constant C such that

$$\forall\, x \in \mathbb{R}, \quad A(x) \leq \begin{cases} C\, e^{2\varrho|x|} & if \quad \varrho > 0 \\ C|x|^{2\alpha+1} & if \quad \varrho = 0 \end{cases} \tag{9}$$

(iv) The generalized Fourier transform $\mathcal{F}_\Lambda$ is well defined on $\mathcal{S}^2(\mathbb{R})$.

(v) For all $f \in D(\mathbb{R})$, we have

$$\forall\, \lambda \in \mathbb{C}, \quad \mathcal{F}_\Lambda(\tau_x f)(\lambda) = \Phi_\lambda(-x)\mathcal{F}_\Lambda(f)(\lambda). \tag{10}$$

Proposition 3 *For all $f \in \mathcal{S}^2(\mathbb{R})$ the decomposition*

$$\mathcal{F}_\Lambda f(\lambda) = 2\,\mathcal{F}_L(f_e)(\sqrt{\lambda^2-\varrho^2}) - 2i\lambda\,\mathcal{F}_L J(f_o)(\sqrt{\lambda^2-\varrho^2}), \tag{11}$$

where J is the integral operator defined by

$$Jf(x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \mathbb{R}, \tag{12}$$

and $\mathcal{F}_L$ stands for the Fourier transform related to the differential operator L, defined on $\mathcal{S}^2_e(\mathbb{R})$ by

$$\mathcal{F}_L(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x)A(x)dx, \quad \lambda \in \mathbb{R},$$

φ_λ being the eigenfunction of L as defined by (1).

We shall need the following properties.

Proposition 4 *(Transmutation formula)*

(i) Let $f \in \mathcal{S}^2(\mathbb{R})$ and g a nice function. Then

$$\int_{\mathbb{R}} \Lambda f(x)g(-x)A(x)dx = \int_{\mathbb{R}} f(x)\Lambda g(-x)A(x)dx. \tag{13}$$

(ii) For $f \in \mathcal{S}^2(\mathbb{R})$

$$\mathcal{F}_\Lambda\left(\Lambda f\right)(y) = iy\mathcal{F}_\Lambda f(y), \quad y \in \mathbb{R}. \tag{14}$$

(iii) For $f \in \mathcal{S}^2(\mathbb{R})$

$$\mathcal{F}_\Lambda\,(\triangle_A\, f)(y) = -y^2 \mathcal{F}_\Lambda\,(f)(y), \quad \text{for all } y \in \mathbb{R}, \tag{15}$$

where $\triangle_A$ is the generalized Laplace operator on $\mathbb{R}$ given by

$$\triangle_A\, f(x) := \Lambda^{\,2} f(x). \tag{16}$$

Notation. We denote by

$L^p_c(\mathbb{R}_+)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\begin{aligned} \|f\|_{L^p_c(\mathbb{R}_+)} &= \left(\int_{\mathbb{R}_+} |f(x)|^p \frac{dx}{|c(x)|^2}\right)^{1/p} < \infty, \quad \text{if } 1 \le p < \infty \\ \|f\|_{L^\infty_c(\mathbb{R}_+)} &= ess \sup_{x \in \mathbb{R}_+} |f(x)| < \infty, \end{aligned}$$

where $c(s)$ is a continuous function on $(0, \infty)$ such that

$$c^{-1}(s) \sim k_1\; s^{\alpha + \frac{1}{2}}, \quad \text{as } s \to \infty, \tag{17}$$

$$c^{-1}(s) \sim \begin{cases} k_2\; s, & \text{as} \quad s \to 0 \quad if\ \varrho > 0 \\ k_3\; s^{2\alpha+1}, & \text{as} \quad s \to 0 \quad if\ \varrho = 0 \end{cases} \tag{18}$$

for some k_1, k_2, and $k_3 \in \mathbb{C}$.

$L^p_\nu(\mathbb{R})$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\begin{aligned} \|f\|_{L^p_\nu(\mathbb{R})} &= \left(\int_{\mathbb{R}} |f(x)|^p d\nu(x)\right)^{1/p} < \infty, \quad \text{if } 1 \le p < \infty \\ \|f\|_{L^\infty_\nu(\mathbb{R})} &= ess \sup_{x \in \mathbb{R}} |f(x)| < \infty, \end{aligned}$$

where $d\nu$ is the measure given by

$$d\nu(\lambda) = \frac{|\lambda|}{4\sqrt{\lambda^2 - \varrho^2}\,|c(\sqrt{\lambda^2 - \varrho^2})|^2} 1_{\mathbb{R}\backslash(-\varrho,\varrho)} d\lambda, \tag{19}$$

with $1_{\mathbb{R}\backslash(-\varrho,\varrho)}$ is the characteristic function of $\mathbb{R}\backslash(-\varrho, \varrho)$.

Theorem 1 *For all $f \in D(\mathbb{R})$, we have*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda) \Phi_\lambda(x) d\nu(\lambda), \tag{20}$$

where $d\nu$ is given by (19).

Theorem 2 *Let $f \in L^1_A(\mathbb{R})$ such that $\mathcal{F}_\Lambda(f)$ belongs to $L^1_\nu(\mathbb{R})$ then we have the following inversion formula*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Phi_\lambda(x)d\nu(\lambda) \quad a.e.\ x \in \mathbb{R}.$$

Theorem 3 *(Plancherel formula)*

(i) For all $f \in \mathcal{S}^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\nu(\lambda), \tag{21}$$

where $d\nu$ is the measure given by (19).

(ii) The generalized Fourier transform $\mathcal{F}_\Lambda$ extends uniquely to a unitary isomorphism from $L^2_A(\mathbb{R})$ onto $L^2_\nu(\mathbb{R})$.

3 Paley-Wiener theorems of functions for the generalized Fourier transform

We begin by the Paley-Wiener theorem for the generalized Fourier transform on the generalized Schwartz space of functions.

Proposition 5 *The generalized Fourier transform $\mathcal{F}_\Lambda$ is a bijection from $\mathcal{S}^2(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$.*

Proof. By [19] we know that the transform $\mathcal{F}_L$ is bijective from $\mathcal{S}^2_e(\mathbb{R})$ onto $\mathcal{S}_e(\mathbb{R})$. Hence it suffices in view of (11) to show that the map $f \to \lambda\mathcal{F}_L(J(f))$ is bijective from $\mathcal{S}^2_o(\mathbb{R})$ onto $\mathcal{S}_o(\mathbb{R})$. But this is immediate. First, because the operator J is one-to-one from $\mathcal{S}^2_o(\mathbb{R})$ onto $\mathcal{S}_e(\mathbb{R})$. Next, since the map $f \to \lambda f$ is one-to-one from $\mathcal{S}_e(\mathbb{R})$ onto $\mathcal{S}_o(\mathbb{R})$.

Notations. We denote by

$\mathcal{S}'^2(\mathbb{R})$ the space of generalized temperate distributions on $\mathbb{R}$, it is the dual space of $\mathcal{S}^2(\mathbb{R})$.

$\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support.

Definition 2 *i) The generalized Fourier transform of a distribution τ in $\mathcal{S}'^{\,2}(\mathbb{R})$ is defined by*

$$\langle \mathcal{F}_\Lambda(\tau), \phi\rangle = \langle \tau, \mathcal{F}_\Lambda^{-1}(\phi)\rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}). \tag{22}$$

ii) The inverse of the generalized Fourier transform of a distribution τ in $\mathcal{E}'(\mathbb{R})$ is defined by

$$\forall \quad x \in \mathbb{R},\ \mathcal{F}_\Lambda^{-1}(\tau)(x) = \langle 1_{\mathbb{R}\setminus(-\varrho,\varrho)}\tau_\lambda, \Phi_\lambda(x)\rangle. \tag{23}$$

From above it is easy to obtain the following.

Corollary 1 *The generalized Fourier transform $\mathcal{F}_\Lambda$ is a topological isomorphism from $\mathcal{S}'^2(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$. Moreover, for all $\tau \in \mathcal{S}'^2(\mathbb{R})$, we have*

$$\mathcal{F}_\Lambda(\Lambda\,\tau) \;=\; iy\mathcal{F}_\Lambda(\tau). \tag{24}$$

and

$$\mathcal{F}_\Lambda(\triangle_A\,\tau) \;=\; -y^2\mathcal{F}_\Lambda(\tau). \tag{25}$$

We consider f in $L^2_A(\mathbb{R})$. We define the distribution $\mathcal{T}_f$ in $\mathcal{S}'^2(\mathbb{R})$ by

$$\langle \mathcal{T}_f, \varphi\rangle = \int_{\mathbb{R}} f(x)\overline{\varphi(x)}A(x)dx,\ \varphi \in \mathcal{S}^2(\mathbb{R}).$$

Notations. We denote by

$L^2_{A,c}(\mathbb{R})$ the space of functions in $L^2_A(\mathbb{R})$ with compact support.

$\mathcal{H}_{L^2}(\mathbb{C})$ the space of entire functions f on $\mathbb{C}$ of exponential type such that $f_{|\mathbb{R}}$ belongs to $L^2_\nu(\mathbb{R})$.

Theorem 4 *The generalized Fourier transform $\mathcal{F}_\Lambda$ is bijective from $L^2_{A,c}(\mathbb{R})$ onto $\mathcal{H}_{L^2}(\mathbb{C})$.*

Proof. i) We consider the function f on $\mathbb{C}$ given by

$$\forall\, z \in \mathbb{C},\ f(z) = \int_{\mathbb{R}} g(x)\Phi_z(-x)A(x)dx, \tag{26}$$

with $g \in L^2_{A,c}(\mathbb{R})$. By derivation under the integral sign and by using the inequality (7), we deduce that the function f is entire on $\mathbb{C}$ and of exponential type. On the other hand the relation (26) can also be written in the form

$$\forall\, y \in \mathbb{R},\ f(y) = \mathcal{F}_\Lambda(g)(y).$$

Thus from Theorem 3 the function $f_{|\mathbb{R}}$ belongs to $L^2_\nu(\mathbb{R})$. Hence $f \in \mathcal{H}_{L^2}(\mathbb{C})$.

ii) Reciprocally let ψ be in $\mathcal{H}_{L^2}(\mathbb{C})$. From [20] there exists $S \in \mathcal{E}'(\mathbb{R})$ with support in $[-a, a]$, such that

$$\forall\, \lambda \in \mathbb{R},\ \psi(\lambda) = \langle S_x, \Phi_\lambda(-x)\rangle. \tag{27}$$

On the other hand as $\psi_{|\mathbb{R}}$ belongs to $L^2_\nu(\mathbb{R})$, then from Theorem 3 there exists $h \in L^2_A(\mathbb{R})$ such that

$$\psi_{|\mathbb{R}} = \mathcal{F}_\Lambda(h). \tag{28}$$

Thus from (27), for all $\varphi \in D(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \psi(y)\overline{\mathcal{F}_\Lambda(\varphi)(y)}d\nu(y) \;=\; \langle S_x, \int_{\mathbb{R}} \Phi_y(x)\overline{\mathcal{F}_\Lambda(\varphi)(y)}d\nu(y)\rangle.$$

Thus using Theorem 2 we deduce that

$$\int_{\mathbb{R}} \psi(y)\overline{\mathcal{F}_\Lambda(\varphi)(y)}d\nu(y) = \langle S, \varphi\rangle. \tag{29}$$

On the other hand (28) implies

$$\int_{\mathbb{R}} \psi(y)\overline{\mathcal{F}_\Lambda(\varphi)(y)}d\nu(y) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(h)(y)\overline{\mathcal{F}_\Lambda(\varphi)(y)}d\nu(y).$$

But from Theorem 3 we deduce that

$$\int_{\mathbb{R}} \mathcal{F}_\Lambda(h)(y)\overline{\mathcal{F}_\Lambda(\varphi)(y)}d\nu(y) \quad = \quad \int_{\mathbb{R}} h(y)\overline{\varphi(y)}A(y)dy = \langle \mathcal{T}_h, \varphi\rangle. \tag{30}$$

Thus the relations (29),(30) imply

$$S = \mathcal{T}_h.$$

This relation shows that the support h is compact. Then $h \in L^2_{A,c}(\mathbb{R})$.

In the following $\mathcal{T}_f$ will be denoted by f.

Definition 3 *i) We define the support of $g \in L^2_\nu(\mathbb{R})$ and we denote it by supp g, the smallest closed set, outside which the function g vanishes almost everywhere.*

ii) We denote by

$$R_g := \sup_{\lambda \in suppg} |\lambda|,$$

the radius of the support of g.

Remark 2 *It is clear that R_g is finite if and only if, g has compact support.*

Notations. We denote by

$L^2_{\nu,c}(\mathbb{R})$ the space of functions in $L^2_\nu(\mathbb{R})$ with compact support.

$L^2_{\nu,c,R}(\mathbb{R}) := \Big\{g \in L^2_{\nu,c}(\mathbb{R}) : R_g = R\Big\}$, for $R \geq 0$.

$D_R(\mathbb{R}) := \Big\{g \in D(\mathbb{R}) : R_g = R\Big\}$, for $R \geq 0$.

Definition 4 *We define the Paley-Wiener spaces $PW^2(\mathbb{R})$ and $PW^2_R(\mathbb{R})$ as follows*

i) $PW^2(\mathbb{R})$ is the space of functions $f \in \mathcal{E}(\mathbb{R})$ satisfying

a) $\triangle^n_A f \in L^2_A(\mathbb{R})$ for all $n \in \mathbb{N}$.

b) $R^{\triangle_A}_f := \lim_{n\to\infty} ||\triangle^n_A f||^{\frac{1}{2n}}_{L^2_{A(\mathbb{R})}} < \infty$.

ii) $PW^2_R(\mathbb{R}) := \Big\{f \in PW^2(\mathbb{R}) : R^{\triangle_A}_f = R\Big\}$.

The real L^2-Paley-Wiener theorem for the generalized Fourier transform can be formulated as follows

Theorem 5 *The generalized Fourier transform $\mathcal{F}_\Lambda$ is a bijection*
i) From $PW_R^2(\mathbb{R})$ onto $L^2_{\nu,c,R}(\mathbb{R})$.
ii) From $PW^2(\mathbb{R})$ onto $L^2_{\nu,c}(\mathbb{R})$.

Proof. Let $g \in PW^2(\mathbb{R})$. Then from (25) the function

$$\mathcal{F}_\Lambda(\triangle_A^n g)(\xi) = (-1)^n \xi^{2n} \mathcal{F}_\Lambda(g)(\xi) \in L^2_\nu(\mathbb{R}), \quad \forall\, n \in \mathbb{N}.$$

On the other hand from Theorem 3 we deduce that

$$\begin{aligned}\lim_{n\to\infty}\left\{\int_{\mathbb{R}} \xi^{4n}|\mathcal{F}_\Lambda(g)(\xi)|^2 d\nu(\xi)\right\}^{\frac{1}{4n}} &= \lim_{n\to\infty}\left\{\int_{\mathbb{R}} |\triangle_A^n g(x)|^2 A(x)dx\right\}^{\frac{1}{4n}}\\ &= R_g^{\triangle_A} < \infty.\end{aligned}$$

Moreover, by a simple calculations, it is easy to see that $\mathcal{F}_\Lambda(g)$ has compact support with

$$R_{\mathcal{F}_\Lambda(g)} = R_g^{\triangle_A}.$$

Conversely let $f \in L^2_{\nu,c,R}(\mathbb{R})$. Then $\xi^n f(\xi) \in L^1_\nu(\mathbb{R})$ for any $n \in \mathbb{N}$, and $\mathcal{F}_\Lambda^{-1}(f)$ belongs to $\mathcal{E}(\mathbb{R})$. On the other hand from Theorem 3 we have

$$\begin{aligned}\lim_{n\to\infty}\left\{\int_{\mathbb{R}} |\triangle_A^n(\mathcal{F}_\Lambda^{-1} f)(x)|^2 A(x)dx\right\}^{\frac{1}{4n}} &= \lim_{n\to\infty}\left\{\int_{\mathbb{R}} \xi^{4n}|f(\xi)|^2 d\nu(\xi)\right\}^{\frac{1}{4n}}\\ &= R.\end{aligned}$$

Thus $\mathcal{F}_\Lambda^{-1}(f) \in PW^2(\mathbb{R})$.
ii) We deduce the result from the i).

Definition 5 *Let u be a distribution on $\mathbb{R}$ and P a polynomial. Then we let*

$$R(P,u) = \sup\Big\{|P(y)| : \quad y \in suppu\Big\} \in [0,\infty],$$

where by convention $R(P,u) = 0$ if $u = 0$.

We will now study the real L^2-Paley-Wiener theorem for the generalized Fourier transform, for which we need the following key propositions.

Proposition 6 *Let P be a polynomial and $f \in \mathcal{S}^2(\mathbb{R})$. Then in the extended positive real numbers*

$$\limsup_{n\to\infty} ||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}^{\frac{1}{n}} \le R(P, \mathcal{F}_\Lambda(f)). \tag{31}$$

Proof. Suppose firstly that $R(P,\mathcal{F}_\Lambda(f))=0$. Then $\mathcal{F}_\Lambda(f)=0$, and hence from Proposition 5, $f=0$. Thus (31) is immediately.

Moreover, the inequality (31), is clear when $R(P,\mathcal{F}_\Lambda(f))=\infty$. So we can assume that

$$0<R(P,\mathcal{F}_\Lambda(f))<\infty.$$

Hölder's inequality gives

$$||f||^2_{L^2_A(\mathbb{R})}=\int_{\mathbb{R}}(1+x^2)^{-1}(1+x^2)|f(x)|^2A(x)dx\le C\sup_{x\in\mathbb{R}}e^{2\varrho|x|}(1+x^2)^{2m}|f(x)|^2, \tag{32}$$

for $m\ge 1$. Thus

$$||f||_{L^2_A(\mathbb{R})}\le C\sup_{x\in\mathbb{R}}e^{\varrho|x|}(1+x^2)^m|f(x)|.$$

Consequently for all $n\in\mathbb{N}$, we deduce that

$$\begin{aligned}||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})} &\le C\sup\nolimits_{x\in\mathbb{R}}e^{\varrho|x|}(1+x^2)^m|P^n(-i\Lambda)f(x)|\\ &\le C\sup\nolimits_{x\in\mathbb{R}}e^{\varrho|x|}(1+x^2)^m\Big|\Big[\mathcal{F}_\Lambda^{-1}\Big(P^n(\xi)\mathcal{F}_\Lambda(f)(x)\Big)\Big]\Big|.\end{aligned}$$

Using the continuity of $\mathcal{F}_\Lambda^{-1}$ we can show that

$$||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}\le C\sup_{\xi\in\mathbb{R}}\sum_{1\le l,j\le M}(1+\xi^2)^j\Big|\frac{d^l}{d\xi^l}\Big[P^n(\xi)\mathcal{F}_\Lambda(f)(\xi)\Big]\Big|, \tag{33}$$

with positive constants C and integer M, independent of n. Using Leibniz's rule we deduce that

$$||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}\ \le\ Cn^M\sup_{y\in supp\mathcal{F}_\Lambda(f)}|P(y)|^{n-M},$$

with C is a constant independent of n. Hence, from the previous inequalities we obtain

$$\limsup_{n\to\infty}||P^n(-i\Lambda)f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})}\le\sup_{y\in supp\mathcal{F}_\Lambda(f)}|P(y)|=R(P,\mathcal{F}_\Lambda(f)).$$

Proposition 7 *Let P be a polynomial. Suppose that $P^n(-i\Lambda)f\in L^2_A(\mathbb{R})$ for all $n\in\mathbb{N}_0$. Then in the extended positive real numbers*

$$\liminf_{n\to\infty}||P^n(-i\Lambda)f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})}\ge R(P,\mathcal{F}_\Lambda(f)). \tag{34}$$

Proof. Fix $\xi_0\in supp\,\mathcal{F}_\Lambda(f)$. We can assume that $|P(\xi_0)|\neq 0$. We will show that

$$\liminf_{n\to\infty}||P^n(-i\Lambda)f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})}\ge|P(\xi_0)|-\varepsilon,$$

for any fixed $\varepsilon > 0$ such that $0 < 2\varepsilon < |P(\xi_0)|$.
To this end, choose and fix $\chi \in D(\mathbb{R})$ such that $\langle \mathcal{F}_\Lambda(f), \chi\rangle \neq 0$, and

$$supp\,\chi \subset \Big\{\xi \in \mathbb{R} :\ |P(\xi_0)| - \varepsilon < |P(\xi)| < |P(\xi_0)| + \varepsilon\Big\}.$$

For $n \in \mathbb{N}$, let $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$. On the follow we want to estimate $||\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^2_A(\mathbb{R})}$. Indeed as above we have

$$\begin{aligned} ||\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^2_A(\mathbb{R})} &\leq C \sup_{x\in\mathbb{R}} e^{\varrho|x|}(1+x^2)^m |\mathcal{F}_\Lambda^{-1}(\chi_n)(x)| \\ &\leq C \sup_{x\in\mathbb{R}} e^{\varrho|x|}(1+x^2)^m \Big|\Big[\mathcal{F}_\Lambda^{-1}\Big(P^{-n}(\xi)\chi\Big)(x)\Big]\Big|, \end{aligned}$$

with $m \geq 1$. Using the continuity of $\mathcal{F}_\Lambda^{-1}$ we can show that

$$||\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^2_A(\mathbb{R})} \leq C \sup_{\xi\in\mathbb{R}} \sum_{1\leq l,j\leq M} (1+\xi^2)^j \Big|\frac{d^l}{d\xi^l}\Big[P^{-n}(\xi)\chi(\xi)\Big]\Big|, \tag{35}$$

with positive constants C and integer M, independent of n. Using Leibniz's rule we deduce that

$$||\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^2_A(\mathbb{R})} \leq Cn^M(|P(\xi_0)| - \varepsilon)^{-n}.$$

Then, since

$$\begin{aligned} \langle \mathcal{F}_\Lambda(f), \chi\rangle &= \langle \mathcal{F}_\Lambda(f), P^n(\xi)\chi_n\rangle \\ &= \langle P^n(\xi)\mathcal{F}_\Lambda(f), \chi_n\rangle \\ &= \langle \mathcal{F}_\Lambda(P^n(-i\Lambda)f), \chi_n\rangle \\ &= \langle (P^n(-i\Lambda)f), \mathcal{F}_\Lambda^{-1}(\chi_n)\rangle. \end{aligned}$$

Hence, from the Hölder inequality we obtain

$$\begin{aligned} |\langle \mathcal{F}_\Lambda(f), \chi\rangle| &\leq C||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})} ||\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^2_A(\mathbb{R})} \\ &\leq Cn^M(|P(\xi_0)| - \varepsilon)^{-n} ||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}. \end{aligned}$$

Since $|\langle \mathcal{F}_\Lambda(f), \chi\rangle| > 0$, we deduce that

$$\liminf_{n\to\infty} ||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}^{\frac{1}{n}} \geq |P(\xi_0)| - \varepsilon.$$

Thus

$$\liminf_{n\to\infty} ||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}^{\frac{1}{n}} \geq \sup_{y\in supp\,\mathcal{F}_\Lambda(f)} |P(y)| = R(P, \mathcal{F}_\Lambda(f)).$$

Combining Proposition 6 and Proposition 7 together, we get

Theorem 6 *Let P be a non-constant polynomial. For any function $f \in \mathcal{S}^2(\mathbb{R})$ the following relation holds*

$$\lim_{n\to\infty} ||P^n(-i\Lambda)f||_{L^2_A(\mathbb{R})}^{\frac{1}{n}} = R(P, \mathcal{F}_\Lambda(f)). \tag{36}$$

Definition 6 *Let P be a non-constant polynomial, we define the polynomial domain U_p by*

$$U_p := \Big\{x \in \mathbb{R} : |P(x)| \leq 1\Big\}.$$

We have the following result.

Corollary 2 *Let $f \in \mathcal{S}^2(\mathbb{R})$. The generalized Fourier transform $\mathcal{F}_\Lambda(f)$ vanishes outside a domain U_P, if and only if,*

$$\limsup_{n\to\infty} ||P^n(-i\Lambda)f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \leq 1. \tag{37}$$

Remark 3 *If we take $P(y) = -y^2$, then $P(-i\Lambda) = \triangle_A$, and Theorem 6 and Corollary 2 characterize functions such that the support of their generalized Fourier transform is $[-1, 1]$.*

4 Characterization of the functions whose generalized Fourier transform has support in antipodal points

Theorem 7 *Let $u \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^2(\mathbb{R})$. Then the support of $\mathcal{F}_\Lambda(u)$ is contained in the compact $V_r := \Big\{\xi \in \mathbb{R} : |P(\xi)| \leq r\Big\}$ for a polynomial P and a constant $r \geq 0$, if, and only if, for each $R > r$, there exist $N_R \in \mathbb{N}_0$ and a positive constant $C(R)$ such that*

$$|P^n(-i\Lambda)(u)(x)| \leq C(R)R^n(1 + |x|)^{N_R} e^{-\varrho|x|}, \tag{38}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof. Assume that support of $\mathcal{F}_\Lambda(u)$ is contained in the compact V_r. Let $R > r$ and let $\varepsilon \in (0, R - r)$. We choose $\chi \in D(\mathbb{R})$ such that $\chi \equiv 1$ on an open neighborhood of support of $\mathcal{F}_\Lambda(u)$, and $\chi \equiv 0$ outside $V_{R-\frac{\varepsilon}{3}}$. As $\mathcal{F}_\Lambda(u)$ is of order N, there exists a positive constant C such that for all $x \in \mathbb{R}$

$$\begin{aligned}
|P^n(-i\Lambda)(u)(x)| &= \Big|\mathcal{F}_\Lambda^{-1}\Big(P^n(\xi)\mathcal{F}_\Lambda(u)\Big)(x)\Big| \\
&= \Big|\mathcal{F}_\Lambda^{-1}\Big(\chi(\xi)P^n(\xi)\mathcal{F}_\Lambda(u)\Big)(x)\Big| \\
&= |\langle\chi(\xi)P^n(\xi)\mathcal{F}_\Lambda(u)(\xi), \Phi_\xi(x)\rangle| \\
&= |\langle\mathcal{F}_\Lambda(u)(\xi), \chi(\xi)P^n(\xi)\Phi_\xi(x)\rangle| \\
&\leq C \sup_{|\xi|\geq\varrho} \sum_{0\leq j\leq N} \Big|D^j\Big(\chi(\xi)P^n(\xi)\Phi_\xi(x)\Big)\Big|.
\end{aligned}$$

Thus from the Leibniz formula (7) we obtain that

$$\begin{array}{rcl}\forall\, n \in \mathbb{N}_0, \quad |P^n(-i\Lambda)(u)(x)| & \leq & C_1(R) n^N (R - \frac{\varepsilon}{3})^n (1+|x|)^{N+2} e^{-\varrho|x|}\\ & \leq & C_2(R) R^n (1+|x|)^{N+2} e^{-\varrho|x|}.\end{array}$$

Conversely we assume that we have (38).

Suppose $\xi_0 \in \mathbb{R}$ is fixed and such that $|P(\xi_0)| \geq R + \varepsilon$, for some $\varepsilon > 0$. Choose and fix $\chi \in D(\mathbb{R})$ such that $\mathrm{supp}\chi \subset \Big\{\xi \in \mathbb{R} : |P(\xi)| \geq R + \frac{\varepsilon}{3}\Big\}$.
For $n \in \mathbb{N}$, we introduce the function χ_n defined by $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$.
We have

$$\begin{array}{rcl}\langle \mathcal{F}_\Lambda(u), \chi\rangle & = & \langle \mathcal{F}_\Lambda(u), P^n(\xi)\chi_n\rangle = \langle P^n(\xi)\mathcal{F}_\Lambda(u), \chi_n\rangle\\ & = & \langle \mathcal{F}_\Lambda(P^n(-i\Lambda)u), \chi_n\rangle\\ & = & \langle \Big(e^{\varrho|x|}(1+|x|)^{-N} P^n(-i\Lambda)u\Big), e^{-\varrho|x|}(1+|x|)^N \mathcal{F}_\Lambda^{-1}(\chi_n)\rangle.\end{array}$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{F}_\Lambda(u), \chi\rangle| \leq ||e^{\varrho|x|}(1+|x|)^{-N} P^n(-i\Lambda)u||_{L^\infty_A(\mathbb{R})} ||e^{-\varrho|x|}(1+|x|)^N \mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^1_A(\mathbb{R})}.$$

We proceed as in Proposition 7, we prove that

$$||e^{-\varrho|x|}(1+|x|)^N \mathcal{F}_\Lambda^{-1}(\chi_n)||_{L^1_A(\mathbb{R})} \leq C n^M (R + \frac{\varepsilon}{3})^{-n}.$$

Thus

$$|\langle \mathcal{F}_\Lambda(u), \chi\rangle| \leq C(R) n^M \Big(\frac{R}{R + \frac{\varepsilon}{3}}\Big)^n.$$

Hence we deduce $\langle \mathcal{F}_\Lambda(u), \chi\rangle = 0$, which implies that $\xi_0 \notin \mathrm{supp}\, \mathcal{F}_\Lambda(u)$.
Thus support of $\mathcal{F}_\Lambda(u)$ is contained in the compact V_r.

Notations. Let $r > 0$, we denote by

$$B_r := \Big\{\xi \in \mathbb{R} : |P(\xi)| < r\Big\}, \quad S_r := \Big\{\xi \in \mathbb{R} : |P(\xi)| = r\Big\}.$$

Theorem 8 *Let $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^2(\mathbb{R})$, and consider the infinite series $\{u_{-n}\}_{n\in\mathbb{N}}$ of generalized tempered distributions defined as $u_{-n+1} = P(-i\Lambda)u_n$, for a polynomial P and for all $n \in \mathbb{N}$. Let $r > 0$. Assume, for all $R \in (0, r)$ there exist constants $N_R \in \mathbb{N}_0$ and $C(R) > 0$, such that*

$$\forall\, x \in \mathbb{R}, \quad |u_{-n}(x)| \leq C(R) R^{-n} (1+|x|)^{N_R} e^{-\varrho|x|}, \tag{39}$$

for all $n \in \mathbb{N}$. Then $\mathrm{supp}\mathcal{F}_\Lambda(u) \cap B_r = \emptyset$.
On the other hand, if $\mathrm{supp}\mathcal{F}_\Lambda(u) \cap B_r = \emptyset$ and $\mathrm{supp}\mathcal{F}_\Lambda(u)$ is compact, then (39) holds, for all $R \in (0, r)$.

Proof. Assume that $\mathrm{supp}\mathcal{F}_\Lambda(u) \cap B_r = \emptyset$ and $\mathrm{supp}\mathcal{F}_\Lambda(u)$ is compact. Let $R \in (0, r)$ and let $\varepsilon \in (0, r - R)$. Choose $\chi \in D(\mathbb{R})$ such that $\chi \equiv 1$ on an open neighborhood of support of $\mathcal{F}_\Lambda(u)$, and $\chi \equiv 0$ on $V_{R+\frac{\varepsilon}{3}}$. As $u = P^n(-i\Lambda)u_{-n}$, we have

$$\begin{aligned}
|u_{-n}(x)| &= \Big|\mathcal{F}_\Lambda^{-1}\Big(P^{-n}(\xi)\mathcal{F}_\Lambda(u)\Big)(x)\Big| \\
&= \Big|\mathcal{F}_\Lambda^{-1}\Big(\chi(\xi)P^{-n}(\xi)\mathcal{F}_\Lambda(u)\Big)(x)\Big| \\
&= |\langle \chi(\xi)P^{-n}(\xi)\mathcal{F}_\Lambda(u)(\xi), \Phi_\xi(x)\rangle| \\
&= |\langle \mathcal{F}_\Lambda(u)(\xi), \chi(\xi)P^{-n}(\xi)\Phi_\xi(x)\rangle| \\
&\leq C \sup\nolimits_{|\xi|\geq\varrho} \sum_{0\leq j\leq N} \Big|D^j\Big(\chi(\xi)P^{-n}(\xi)\Phi_\xi(x)\Big)\Big|.
\end{aligned}$$

Thus from the Leibniz formula (7) we obtain that

$$\begin{aligned}
\forall\, n \in \mathbb{N}_0, \quad |u_{-n}(x)| &\leq C_1(R)n^N(R + \tfrac{\varepsilon}{3})^{-n}(1 + |x|)^{N+2}e^{-\varrho|x|} \\
&\leq C_2(R)R^{-n}(1 + |x|)^{N+2}e^{-\varrho|x|}.
\end{aligned}$$

Assume that we have (39). For a fixed $R \in (0, r)$ let $\varepsilon > 0$. Choose and fix $\chi \in D(\mathbb{R})$ such that

$$\mathrm{supp}\,\chi \subset \Big\{\xi \in \mathbb{R} : |P(\xi)| \leq R - \frac{\varepsilon}{3}\Big\},$$

and put $\chi_n = P^n(\xi)\chi$. We have

$$\begin{aligned}
\langle \mathcal{F}_\Lambda(u), \chi\rangle &= \langle \mathcal{F}_\Lambda(u), P^{-n}(\xi)\chi_n\rangle = \langle P^{-n}(\xi)\mathcal{F}_\Lambda(u), \chi_n\rangle \\
&= \langle \mathcal{F}_\Lambda(u_{-n}), \chi_n\rangle \\
&= \langle \Big(e^{\varrho|x|}(1 + |x|)^{-N}u_{-n}\Big), e^{-\varrho|x|}(1 + |x|)^N\mathcal{F}_\Lambda^{-1}(\chi_n)\rangle.
\end{aligned}$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{F}_\Lambda(u), \chi\rangle| \leq ||e^{\varrho|x|}(1 + |x|)^{-N}u_{-n}||_{L_A^\infty(\mathbb{R})}||e^{-\varrho|x|}(1 + |x|)^N\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L_A^1(\mathbb{R})}.$$

We proceed as in Proposition 7, we prove that

$$||e^{-\varrho|x|}(1 + |x|)^N\mathcal{F}_\Lambda^{-1}(\chi_n)||_{L_A^1(\mathbb{R})} \leq Cn^M(R - \frac{\varepsilon}{3})^n.$$

Thus

$$\forall\, n \in \mathbb{N}, \quad |\langle \mathcal{F}_\Lambda(u), \chi\rangle| \leq C(R)n^M\Big(\frac{R - \frac{\varepsilon}{3}}{R}\Big)^n.$$

Thus we deduce $\langle \mathcal{F}_\Lambda(u), \chi\rangle = 0$, which implies that $\mathrm{supp}\mathcal{F}_\Lambda(u) \cap B_r = \emptyset$.

Combining Theorem 7 and Theorem 8 together, we get

Corollary 3 *Let $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^{\,2}(\mathbb{R})$, and consider the infinite series $\{u_n\}_{n\in\mathbb{Z}}$ of generalized tempered distributions defined as $u_{n+1} = P(-i\Lambda)u_n$, for a polynomial P and for all $n \in \mathbb{Z}$. Let $R > 0$. Then* $\operatorname{supp}\mathcal{F}_\Lambda(u)$ *is contained in S_R, if and only if for all $\varepsilon > 0$, there exist constants $N_\varepsilon \in \mathbb{N}_0$ and $C_\varepsilon > 0$, such that*

$$\forall\, x \in \mathbb{R}, \quad |u_n(x)| \le C_\varepsilon R^n (1+\varepsilon)^{|n|}(1+|x|)^{N_\varepsilon} e^{-\varrho|x|} \tag{40}$$

for all $n \in \mathbb{Z}$.

Remark 4 *(i) The previous corollary, gives a characterization of the functions whose generalized Fourier transform has support at the endpoints.*

(ii) We note that the results of this section generalize and improve the version presented in [13, 14].

5 Roe's theorem associated with Dunkl type operators

In [16] Roe proved that if a doubly-infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies $\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \le M$ for all $j = 0, \pm1, \pm2, ...$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x+b)$ where a and b are real constants.

The purpose of this section is to generalize this theorem for the Dunkl type operator Λ.

Theorem 9 *Suppose $P(\xi) = \displaystyle\sum_n a_n\xi^n$ is real-valued and let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-i\Lambda)f_j.$$

(i) Let $a \ge 0$, $R > 0$, and assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies

$$|f_j(x)| \le M_j R^j (1+|x|)^a e^{-\varrho|x|}, \tag{41}$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{42}$$

Then $f = f_+ + f_-$ where $P(-i\Lambda)f_+ = Rf_+$ and $P(-i\Lambda)f_- = -Rf_-$. If R (or $-R$) is not in the range of P then $f_+ = 0$ (or $f_- = 0$).

(ii) If we replace (42) with

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0, \tag{43}$$

for all $j > 0$, then the span of $(f_j)_j$ is finite dimensional.
Moreover, $f_0 = f_+ + f_-$, where, for some integer N, $(P(-i\Lambda) - R)^N f_+ = 0$ and $(P(-i\Lambda) + R)^N f_- = 0$. Thus f_+ (or f_-) is a generalized eigenfunction of $P(-i\Lambda)$ with eigenvalue R (or $-R$).

In order to prove Theorem 9 we need the following lemmas:

Lemma 2 *Let $(f_j)_{j\in\mathbb{Z}}$ is be a sequence of functions on $\mathbb{R}$ satisfying*

$$f_{j+1} = P(-i\Lambda)f_j, \tag{44}$$

$$|f_j(x)| \leq M_j R^j (1 + |x|)^a e^{-\varrho|x|}, \tag{45}$$

and

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0, \tag{46}$$

for all $\varepsilon > 0$, then

$$\mathrm{supp}(\mathcal{F}_\Lambda(f_0)) \subset S_R := \Big\{\xi : |P(\xi)| = R\Big\}.$$

Proof. First we show that $\mathcal{F}_\Lambda(f_0)$ is supported in $\Big\{\xi : |P(\xi)| \leq R\Big\}$.
To do this we need to show that

$$\langle \mathcal{F}_\Lambda(f_0), \phi\rangle = 0, \text{ if } \phi \in D(\mathbb{R})$$

and

$$\mathrm{supp}(\phi) \cap \Big\{\xi : |P(\xi)| \leq R\Big\} = \emptyset.$$

Since $\mathrm{supp}(\phi)$ is compact, there is some $r < \frac{1}{R}$ so that $\frac{1}{|P(\xi)|} \leq r$, for all $\xi \in \mathrm{supp}(\phi)$. Then

$$\begin{aligned} \langle \mathcal{F}_\Lambda(f_0), \phi\rangle &= \langle P^j \mathcal{F}_\Lambda(f_0), \tfrac{\phi}{P^j}\rangle \\ &= \langle \mathcal{F}_\Lambda\Big(P^j(-i\Lambda)f_0\Big), \tfrac{\phi}{P^j}\rangle \\ &= \langle P^j(-i\Lambda)f_0, \mathcal{F}_\Lambda^{-1}(\tfrac{\phi}{P^j})\rangle. \end{aligned}$$

Choose an integer m with $2m \geq 2a + 2$. A calculation, using the hypothesis of the lemma and Cauchy-Schwartz inequality, implies

$$\begin{aligned} |\langle \mathcal{F}_\Lambda(f_0), \phi\rangle| &\leq \int_{\mathbb{R}} |P^j(-i\Lambda)f_0(x)||\mathcal{F}_\Lambda^{-1}(\frac{\phi}{P^j})(x)|A(x)dx \\ &\leq CM_j \sup_{x\in\mathbb{R}} |e^{\varrho|x|}(1 + x^2)^m \mathcal{F}_\Lambda^{-1}(\tfrac{\phi}{P^j})(x)]|. \end{aligned}$$

Using the continuity of $\mathcal{F}_\Lambda^{-1}$ and the fact that ϕ is supported in $\Big\{\xi : |P(\xi)| \geq R + \varepsilon\Big\}$ for some fixed $\varepsilon > 0$, it is not hard to prove that the

right-hand side of this goes to zero as $j \to \infty$ and so $\langle \mathcal{F}_\Lambda(f_0), \phi \rangle = 0$.
To complete the proof we need to show that $\mathcal{F}_\Lambda(f_0)$ is also supported in

$$\Big\{\xi : |P(\xi)| \geq R\Big\},$$

which means $\langle \mathcal{F}_\Lambda(f_0), \phi \rangle = 0$ if ϕ is supported in

$$\Big\{\xi : |P(\xi)| \leq R\Big\}.$$

Here we use (44) to obtain

$$\langle \mathcal{F}_\Lambda(f_0), \phi \rangle = \langle \mathcal{F}_\Lambda(f_{-j}), P^j \phi \rangle$$

and the argument proceeds as before.

Lemma 3 *We assume that $-R$ is not a value of $P(\xi)$. There exists an integer N such that*

$$(P(\xi) - R)^{N+1} \mathcal{F}_\Lambda(f_0) = 0. \tag{47}$$

Proof. Using Lemma 2 and proceeding as in [11], we prove the result.

Lemma 4 *([6]). Let X be a finite dimensional complex vector space, and let $T : X \to X$ be a linear map with eigenvalues $\lambda_1, ..., \lambda_p$.*
Then $X = X_1 \oplus ... \oplus X_p$, where $X_j = ker((T - \lambda_j)^N)$ and $dimX = N$.

Proof of Theorem 9

We want to prove (i). Inverting the generalized transform in (47) yields that

$$(P(-i\Lambda) - R)^{N+1} f_0 = 0. \tag{48}$$

This equation implies

$$\begin{aligned} \mathrm{span}\Big\{f_0, f_1, f_2, ...\Big\} &= \mathrm{span}\Big\{f_0, P(-i\Lambda)f_0, P(-i\Lambda)^2 f_0, ...\Big\} \\ &= \mathrm{span}\Big\{f_0, P(-i\Lambda)f_0, ..., P^N(-i\Lambda)f_0\Big\}. \end{aligned}$$

We shall now show that we can take $N = 0$ in (48).
If not then $(P(-i\Lambda) - R)f_0 \neq 0$. Let p be the largest positive integer so that $(P(-i\Lambda) - R)^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := (P(-i\Lambda) - R)^{p-1} f_0 \in \mathrm{span}\Big\{f_0, f_1, ..., f_N\Big\}$$

will satisfy

$$(P(-i\Lambda) - R)^2 f = 0 \quad \text{and} \quad (P(-i\Lambda) - R)f \neq 0. \tag{49}$$

Write

$$f = a_0 f_0 + ... + a_N f_N,$$

for constants $a_0, ..., a_N$. Then

$$P^j(-i\Lambda)f = a_0 f_j + ... + a_N f_{N+j}.$$

If

$$C_j = |a_0|R^0 M_j + ... + |a_N|R^N M_{j+N},$$

then this and (41) imply

$$|P^j(-i\Lambda)f(x)| \leq C_j R^j (1+|x|)^a e^{-\varrho|x|}. \tag{50}$$

By (42) these satisfy the sublinear growth condition

$$\lim_{j\to\infty} \frac{C_j}{j} = 0. \tag{51}$$

An induction using (49) implies for $j \geq 2$ that

$$P^j(-i\Lambda)f = R^{j-1}jP(-i\Lambda)f - R^j(j-1)f = R^{j-1}j(P(-i\Lambda)-R)f + R^j f.$$

Thus

$$|(P(-i\Lambda)-R)f(x)| \leq \frac{1}{jR^{j-1}}|P^j(-i\Lambda)f(x)| + \frac{R|f(x)|}{j} \leq \frac{C_jR}{j}(1+|x|)^a e^{-\varrho|x|} + \frac{R|f(x)|}{j}.$$

Letting $j \to \infty$ and using (51) implies $(P(-i\Lambda)-R)f = 0$. But this contradicts (49). Consequently, $N = 0$ in (48). This completes the proof in the case that $-R$ is not in the range of P.

In the case that R is not in the range of P we apply the same argument to $-P(-i\Lambda)$ to conclude $P(-i\Lambda)f_0 = -Rf_0$.
In the general case, let $\mathfrak{L} = P^2(-i\Lambda)$. Then $\mathcal{F}_\Lambda(\mathfrak{L}f)(\xi) = P^2(\xi)\mathcal{F}_\Lambda(f)(\xi)$. $\mathfrak{L}f_{2p} = f_{2(p+1)}$ and $P^2(\xi) \neq -R$. Thus we can (as before) conclude, for the sequence $(f_{2p})_{p\in\mathbb{Z}}$ that

$$\mathfrak{L}f_0 = P^2(-i\Lambda)f_0 = R^2 f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + \frac{1}{R}P(-i\Lambda)f_0)$ and $f_- = \frac{1}{2}(f_0 - \frac{1}{R}P(-i\Lambda)f_0)$.
Then $f = f_+ + f_-$, $P(-i\Lambda)f_+ = Rf_+$ and $P(-i\Lambda)f_- = -Rf_-$. This completes the proof of (i).

Now we want to prove (ii). We first prove (ii) under the assumption that $P(\xi) \neq -R$. Using the growth condition (43) and Lemma 4, we may still conclude that $\mathrm{supp}(\mathcal{F}_\Lambda(f_0)) \subset S_R := \Big\{\xi : P(\xi) = R\Big\}$. But then, as before, we can conclude that (48) holds. But this is enough to complete the proof in this case. A similar argument shows that if $P(\xi) \neq R$, then $(P(-i\Lambda)+R)^N f_0 = 0$. In the general case we again let $\mathfrak{L} = P^2(-i\Lambda)$ and $P_0 = P^2$. Then $P_0(\xi) \neq -R$ and the span of $(f_{2j})_j$ is finite dimensional. The map $P(-i\Lambda)$ takes the span of $(f_{2j})_j$ onto the span of $(f_{2j+1})_j$. Thus X is finite dimensional. Any $f \in X$ will have $\mathrm{supp}(f)$ inside the set defined by $P(\xi) = \pm R$. From this it is not hard to show the only possible eigenvalues of $P(-i\Lambda)$ restricted to X are R and $-R$. The result now follows from the last lemma.

Remark 5 *(i) If we take $P(y) = -y^2$, then $P(-i\Lambda) = \triangle_A$ and Theorem 9 give $\triangle_A f_0 = -Rf_0$. This characterizes eigenfunctions f of generalized Laplace operator $\triangle_A$ with polynomial growth in terms of the size of the powers $\triangle_A^j f$, $-\infty < j < \infty$.*

(ii) The previous theorem generalizes and improves the version presented in [13, 14].

Theorem 10 *Suppose $P(\xi) = \sum_n a_n \xi^n$ is a non-constant polynomial with complex coefficients. Let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-i\Lambda) f_j.$$

1) Let $a \geq 0$ and let $R > 0$. Assume that for all $\varepsilon > 0$, there exist constants $N \in \mathbb{N}_0$ and $C > 0$, such that

$$\forall\, x \in \mathbb{R}, \quad |f_n(x)| \leq CR^n (1+\varepsilon)^{|n|}(1+|x|)^N e^{-\varrho|x|} \tag{52}$$

is satisfied for all $n \in \mathbb{Z}$. Then

$$f_0 = \sum_{\lambda \in S_R} \sum_{j=0}^{N} c(\lambda, j) \frac{d^j}{d\xi^j}_{|\xi=\lambda} \Phi_\xi, \tag{53}$$

for constants $c(\lambda, j) \in \mathbb{C}$ and $N \in \mathbb{N}$.

2) Let $a \geq 0$ and let $R > 0$ and assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies

$$|f_j(x)| \leq M_j R^j (1+|x|)^a e^{-\varrho|x|}, \tag{54}$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the subpotential growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j^m} = 0, \tag{55}$$

for some $m \geq 0$.

We have

(i) If $P'(\lambda_p) \neq 0$, for all $\lambda_p \in S_R$, then $N < m$ in (53).

In particular, if $m = 1$, then

$$f_0 = \sum_{\lambda_p \in S_R} f_{\lambda_p}, \quad \text{where } f_{\lambda_p} = c(\lambda_p)\Phi_{\lambda_p}$$

.

(ii) If S_R consists of one point λ_0 and $m = 1$ in (55), then

$$P(-i\Lambda)f_0 = P(\lambda_0)f_0.$$

Proof. 1) Assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies (52). Then Corollary 3 implies that the support of $\mathcal{F}_\Lambda(f_0)$ is contained in the finite set S_R. A standard result in distribution theory, see e.g., [[17], Theorem 6.25], infers that

$$\mathcal{F}_\Lambda(f_0) = \sum_{\lambda \in S_R} \sum_{0 \le j \le N} c(\lambda, j)\delta_\lambda^{(j)}$$

for constants $c(\lambda, j) \in \mathbb{C}$, and some integer N.
Here δ_ξ^j denotes the jth distributional derivative of the delta function δ_ξ at ξ. The result follows with $f_0 = \mathcal{F}_\Lambda^{-1}\Big(\sum_{\lambda \in S_R} \sum_{0 \le j \le N} c(\lambda, j)\delta_\lambda^{(j)}\Big)$.

We want to prove 2) (i). For $n \ge 0$, we have

$$\langle f_n, \chi \rangle = \langle \mathcal{F}_\Lambda(f_0), P^n(\lambda)\mathcal{F}_\Lambda(\chi) \rangle,$$

for any $\chi \in \mathcal{S}^2(\mathbb{R})$. Fix $\lambda_p \in S_R$ such that $P'(\lambda_p) \ne 0$ and let N_p be the order of $\mathcal{F}_\Lambda(f)$ at λ_p. Choose $\chi \in \mathcal{S}^2(\mathbb{R})$ such that $\mathcal{F}_\Lambda(\chi) = 1$ in a small neighborhood of λ_p, and $\mathcal{F}_\Lambda(\chi) = 0$ around the points $V_R \backslash \{\lambda_p\}$. Then, for $n > N_p$

$$\begin{aligned} \langle f_n, \chi \rangle = \langle \mathcal{F}_\Lambda(f_0), P^n(\lambda)\mathcal{F}_\Lambda(\chi) \rangle &= \langle \textstyle\sum_{0 \le j \le N_p} \Big(c(\lambda_p, j)\delta_{\lambda_p}^{(j)}\Big), P^n(\lambda)\mathcal{F}_\Lambda(\chi) \rangle \\ &= c(\lambda_p, N_p) n^{N_p} P^{n-N_p}(\lambda_p)(P'(\lambda_p))^{N_p} + \ldots \end{aligned}$$

plus lower order terms in n. Since $|\langle f_n, \chi \rangle| \le C M_n R^n$ for a constant $C > 0$, by (54), we have $c(\lambda_p, N_p) = 0$ for $N_p \ge m$ by(55).

If we assume that $m = 1$, then $N_p = 0$ and condition (55) implies that the condition (40) is satisfied. Thus from the above, Eq. (53) becomes

$$f_0 = \sum_{\lambda_p \in S_R} f_{\lambda_p}, \quad \text{where } f_{\lambda_p} = c(\lambda_p)\Phi_{\lambda_p}$$

for a constant $c(\lambda_p) \in \mathbb{C}$.

We want to prove 2) (ii). Indeed, as in the above and from the assumptions on $\{f_j\}_{-\infty}^{\infty}$ we prove that

$$(P(-i\Lambda) - P(\lambda_0))^{N+1} f_0 = 0. \tag{56}$$

This equation implies

$$\begin{aligned}\mathrm{span}\Big\{f_0, f_1, f_2, ...\Big\} &= \mathrm{span}\Big\{f_0, P(-i\Lambda)f_0, P(-i\Lambda)^2 f_0, ...\Big\} \\ &= \mathrm{span}\Big\{f_0, P(-i\Lambda)f_0, ..., P^N(-i\Lambda)f_0\Big\}.\end{aligned}$$

We shall now show that we can take $N = 0$ in (56).
If not then $(P(-i\Lambda) - P(\lambda_0))f_0 \neq 0$. Let p be the largest positive integer so that $(P(-i\Lambda) - P(\lambda_0))^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := (P(-i\Lambda) - P(\lambda_0))^{p-1} f_0 \in \mathrm{span}\Big\{f_0, f_1, ..., f_N\Big\}$$

will satisfy

$$(P(-i\Lambda) - P(\lambda_0))^2 f = 0 \quad \text{and} \quad (P(-i\Lambda) - P(\lambda_0))f \neq 0. \tag{57}$$

Write

$$f = a_0 f_0 + ... + a_N f_N,$$

for constants $a_0, ..., a_N$. Then

$$P^j(-i\Lambda)f = a_0 f_j + ... + a_N f_{N+j}.$$

If we put

$$C_j := |a_0|R^0 M_j + ... + |a_N|R^N M_{j+N},$$

then by (54) we obtain

$$|P^j(-i\Lambda)f(x)| \leq C_j R^j (1 + |x|)^a e^{-\varrho|x|}. \tag{58}$$

By (55) C_j satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{C_j}{j} = 0. \tag{59}$$

An induction using (57) implies for $j \geq 2$ that

$$\begin{aligned}P^j(-i\Lambda)f &= jP(\lambda_0)^{j-1}P(-i\Lambda)f - (j-1)P(\lambda_0)^j f \\ &= jP(\lambda_0)^{j-1}(P(-i\Lambda) - P(\lambda_0))f + P(\lambda_0)^j f.\end{aligned}$$

Thus

$$\begin{aligned}|(P(-i\Lambda) - P(\lambda_0))f(x)| &\leq \tfrac{1}{jR^{j-1}}|P^j(-i\Lambda)f(x)| + \tfrac{R|f(x)|}{j} \\ &\leq \tfrac{C_j R}{j}(1 + |x|)^a e^{-\varrho|x|} + \tfrac{R|f(x)|}{j}.\end{aligned}$$

Letting $j \to \infty$ and using (59) implies $(P(-i\Lambda) - P(\lambda_0))f = 0$.
But this contradicts (57). Consequently, $N = 0$ in (56). This completes the proof.

Remark 6 *The previous theorem is the analogue for the Theorems 1 and 6 of [2].*

6 Real Paley-Wiener theorems for the generalized Fourier transform on $\mathcal{S}'^{\,2}(\mathbb{R})$

Let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$. We put $\Gamma_u := \inf\Big\{r \in (0,\infty] : \quad \mathrm{supp}(\mathcal{F}_\Lambda(u)) \subset [-r,r]\Big\}$.

Theorem 11 *Let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$. Then the support of $\mathcal{F}_\Lambda(u)$ is included in $[-M,M]$, $M>0$, if and only if for all $R>M$ we have*

$$\lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0, \quad \text{in} \quad \mathcal{S}'^{\,2}(\mathbb{R}).$$

Proof. Let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$ and $M>0$ such that

$$\lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0, \quad \text{for all} \quad R>M.$$

Let $\varphi \in D(\mathbb{R})$ satisfy $\mathrm{supp}(\varphi) \subset [-M,M]^c$. We have to prove that

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = 0.$$

Let $r>M$ satisfy $\varphi(x)=0$ for all $x\in[-r,r]$ and $R\in(M,r)$. Then for all $n\in\mathbb{N}$ the function $x^{-2n}\varphi$ is in $D(\mathbb{R})$ and we can write

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = \langle (-x^2)^n R^{-2n}\mathcal{F}_\Lambda(u), (-x^2)^{-n}R^{2n}\varphi\rangle,$$

and by formula (25), we have

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = \langle \mathcal{F}_\Lambda(R^{-2n}\triangle_A^n(u)), (-x^2)^{-n}R^{2n}\varphi\rangle.$$

The hypothesis implies that $\mathcal{F}_\Lambda(R^{-2n}\triangle_A^n(u)) \to 0$ in $\mathcal{S}'(\mathbb{R})$. Moreover from the Leibniz formula we deduce that $(-x^2)^{-n}R^{2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = 0.$$

Conversely, let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$ and $M>0$ such that $\mathrm{supp}\mathcal{F}_\Lambda(u) \subset [-M,M]$. We are going to prove that for all $R>M$

$$\lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0, \quad \text{in} \quad \mathcal{S}'^{\,2}(\mathbb{R}).$$

Let $M<R$ and choose $\varrho \in (M,R)$ and $\psi \in D(R)$ satisfying $\psi \equiv 1$ on a neighborhood of $[-M,M]$ and $\psi(x)=0$ for all $x \notin [-\varrho,\varrho]$. Then for all $\varphi \in D(\mathbb{R})$ we have

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = \langle \mathcal{F}_\Lambda(u), \psi\varphi\rangle,$$

and then

$$\langle \mathcal{F}_\Lambda(R^{-2n}\triangle_A^n(u)), \varphi\rangle = \langle \mathcal{F}_\Lambda(u), (-x^2)^n R^{-2n}\psi\varphi\rangle.$$

Finally we deduce the result by using the fact that $(-x^2)^n R^{-2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$.

Corollary 4 *From the previous theorem we obtain*

$$\Gamma_u = \inf\Big\{R > 0 : \quad \lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0, \quad \text{in} \quad \mathcal{S}'^{\,2}(\mathbb{R})\Big\}.$$

Let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$. We put $\gamma_u := \sup\Big\{r \in [0,\infty) : \quad \text{supp}(\mathcal{F}_\Lambda(u)) \subset (-r,r)^c\Big\}$.

Theorem 12 *Let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$ such that $(-x^2)^{-n}\mathcal{F}_\Lambda(u) \in \mathcal{S}'(\mathbb{R})$ for all $n \in \mathbb{N}$.*

Let $u_n = \mathcal{F}_\Lambda^{-1}\Big((-x^2)^{-n}\mathcal{F}_\Lambda(u)\Big)$. Then the support of $\mathcal{F}_\Lambda(u)$ is included in $(-M,M)^c$, $M > 0$, if and only if for all $R < M$ we have

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in} \quad \mathcal{S}'^{\,2}(\mathbb{R}).$$

Proof. Let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$ and $M > 0$ such that

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{for all} \quad R < M.$$

Let $\varphi \in D(\mathbb{R})$ satisfy $\text{supp}(\varphi) \subset (-M,M)$. We want to prove that

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = 0.$$

Let $r \in (0,M)$ such that $\text{supp}\,\varphi \subset (-r,r)$ and $R \in (r,M)$. Then for all $n \in \mathbb{N}$ the function $x^{2n}\varphi$ is in $D(\mathbb{R})$ and we can write

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = \left\langle (-x^2)^{-n}R^{2n}\mathcal{F}_\Lambda(u), (-x^2)^n R^{-2n}\varphi\right\rangle = \left\langle \mathcal{F}_\Lambda(R^{2n}u_n), (-x^2)^n R^{-2n}\varphi\right\rangle.$$

The hypothesis implies that $\mathcal{F}_\Lambda(R^{2n}u_n) \to 0$ in $\mathcal{S}'(\mathbb{R})$.
Moreover from the Leibniz formula we deduce that $(-x^2)^n R^{-2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = 0.$$

Conversely, let $u \in \mathcal{S}'^{\,2}(\mathbb{R})$ and $M > 0$ such that $\text{supp}\mathcal{F}_\Lambda(u) \subset (-M,M)^c$. We are going to prove that for all $R < M$

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in} \quad \mathcal{S}'^{\,2}(\mathbb{R}).$$

Let $M > R$ and choose $\varrho \in (R,M)$ and $\psi \in D(R)$ satisfying $\psi(x) \equiv 1$ for $|x| \geq \frac{M+\varrho}{2}$ and $\psi(x) = 0$ for all $|x| \leq \varrho$. Then for all $\varphi \in D(\mathbb{R})$ we have

$$\langle \mathcal{F}_\Lambda(u), \varphi\rangle = \langle \mathcal{F}_\Lambda(u), \psi\varphi\rangle,$$

and then

$$\langle \mathcal{F}_\Lambda(R^n u_n), \varphi\rangle = \langle \mathcal{F}_\Lambda(u), (-x^2)^{-n}R^{2n}\psi\varphi\rangle.$$

Finally we deduce the result by using the fact that $(-x^2)^{-n}R^{2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$.

Corollary 5 *From the previous theorem we obtain*

$$\gamma_u = \sup\Big\{R > 0, \lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in} \quad \mathcal{S}'^{\,2}(\mathbb{R})\Big\}.$$

7 Open Problem

In [18] Strichartz proved that
Theorem. Let f be a function on $\mathbb{R}^d$ such that

$$\forall j \in \mathbb{N},\ ||(-\triangle + ||x||^2)^j f||_{L^\infty(\mathbb{R}^d)} \leq Md^j.$$

Then $f(x) = Ce^{-\frac{||x||^2}{2}}$.

The purpose of the future work is to generalize this theorem.
In place of oscillator operator $-\triangle + ||x||^2$ of $\mathbb{R}^d$, we shall extended this to generalized oscillator operator $L_A := -\Lambda^2 + |x|^2$ on $\mathbb{R}$.

Acknowledgements. The author is deeply indebted to the referees for providing constructive comments and help in improving the contents of this paper. Thanks also to Professors K. Trimèche and N-B. Andersen for their helps and encouragements.

References

[1] **N.B. Andersen,** *On the range of the Chébli-Trimèche transform,* Monatsh. Math. **144** (2005), 193-201.

[2] **N.B. Andersen,** *Roe's theorem revisited,* Integral Transf. and Special Functions V. **26**, Issue 3, (2015), 165-172.

[3] **H.H. Bang,** *A property of infinitely differentiable functions,* Proc. Amer. Math. Soc. **108**, N^o.1, (1990), 73-76.

[4] **N. Ben Salem and A. Ould Ahmed Salem,** *Convolution structure associated with the Jacobi-Dunkl operator on* $\mathbb{R}$*,* Ramanujan J. 12 (2006), no. 3, 359-378.

[5] **J.J. Betancor, J.D. Betancor and J.M.R. Mendez,** *Paley-Wiener type theorems for Chébli-Trimèche transforms,* Publ. Math. Debrecen **60**, 3-4 (2002), 347-358.

[6] **G. Birkoff and S. MacLane,** *A Survey of Modern Algebra, MacMillan, New York, 1965.*

[7] **C. Chettaoui, Y. Othmani and K. Trimèche,** *On the range of the Dunkl transform on* $\mathbb{R}$*,* Anal. and Appl. **2**, N^{o}3, (2004), 177-192.

[8] **F. Chouchene, M. Mili and K. Trimèche,** *Positivity of the intertwining operator and harmonic analysis associated with the Jacobi-Dunkl operator on* $\mathbb{R}$, Anal. and Appl. Vol. **1** (2003), 387-412.

[9] **C.F. Dunkl,** *Hankel transforms associated to finite reflection groups*, Contemp. Math. 138(1992), 123-138.

[10] **J.-P. Gabardo,** *Tempered distributions with spectral gaps, Math. Proc. Camb. Phil. Soc. 106, (1989), 143-162.*

[11] **R. Howard and M. Reese,** *Characterization of eigenfunctions by boundedness conditions,* Canad. Math. Bull. **35** (1992), 204-213.

[12] **H. Mejjaoli and K. Trimèche,** *Spectrum of functions for the Dunkl transform on* $\mathbb{R}^d$*. Fract. Calc. Appl. Anal. 10 (2007), no. 1, 19–38.*

[13] **H. Mejjaoli,** *Roe's theorem associated with a Dunkl type differential-difference operator on the real Line* $\mathbb{R}$*,* Tamsui Oxford Journal of Mathematical Sciences 01/(2010); 26 237-254.

[14] **H. Mejjaoli,** *Spectrum of functions and distributions for the Jacobi-Dunkl Transform on* $\mathbb{R}$*,* Mediterr. J. Math. 10 (2013), 753-778.

[15] **H. Mejjaoli,** *Sobolev spaces associated with a Dunkl type differential-difference operator on the real line and applications,* Journal of Pseudo-Differential Operators and Applications Volume 6, Issue 1, (2015), 33-68.

[16] **J. Roe,** *A characterization of the sine function,* Math. Proc. Comb. Phil. Soc. **87** (1980), 69-73.

[17] **W. Rudin,** *Functional Analysis,* McGraw-Hill Book Co., 1973.

[18] **R.S. Strichartz,** *Characterization of eigenfunctions of the Laplacian by boundedness conditions,* Trans. Amer. Math. Soc. **338** (1993), 971-979.

[19] **K. Trimèche,** *Inversion of the J.L. Lions transmutation operators using generalized wavelets*, Applied and Computational Harmonic Analysis, 4(1997), 97-112.

[20] **K. Trimèche,** *The Transmutation Operators Relating to a Dunkl Type Operator on* $\mathbb{R}$ *and Their Positivity,* Mediterr. J. Math. DOI 10.1007/s00009-014-0407-0.

[21] **V.K. Tuan,** *Paley-Wiener theorems for a class of integral transforms,* J. Math. Anal. Appl. **266** (2002), 200-226.