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Holder's Inequalities for a New Class of p-Valent Analytic Functions

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Abstract

In this paper, we introduced a new class of p-valent analytic functions using a linear multiplier Dziok-Srivastava operator $D_{p,\lambda,\ell}^{m,q,s}f(z)(m \in \mathbb{N}_0 = \{0,1,...\}$, $q \leq s+1; q,s \in \mathbb{N}_0$, $\lambda \geq 0$, $\ell \geq 0)$. Hölders inequalities results and modified Hadamard produt for functions belonging to this class are obtained.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent functions in the open unit disc $U = \{z : |z| < 1\}$. We denote by $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ the subclasses functions of $\mathcal{A}(p)$ consisting of all functions which are, respectively, p-valently starlike and p-valently convex of order α $(0 \leq \alpha < p)$. Thus,

$$S_p^*(\alpha) = \left\{ f \in A(p) : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ (0 \leq \alpha < p; z \in U \right\} \tag{1.2}$$

and

$$\mathcal{K}_p(\alpha) = \left\{ f \in A(p) : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ (0 \leq \alpha < p;\ z \in U) \right\}. \tag{1.3}$$

The classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were introduced by Patel and Thakare [16] and Owa [14]. From (1.2) and (1.3) it follows that

$$f(z) \in \mathcal{K}_p(\alpha) \Leftrightarrow \frac{z}{p}f'(z) \in \mathcal{S}_p^*(\alpha). \tag{1.4}$$

We note that:

$$\mathcal{S}_p^*(0) = \mathcal{S}_p^*,\ \mathcal{K}_p(0) = \mathcal{K}_p$$

(see Goodman [10]).
Let $f \in \mathcal{A}(p)$ be given by (1.1) and $g \in \mathcal{A}(p)$ is given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k . \tag{1.5}$$

The Hadamard product (or convolution) of f and g is defind by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.6}$$

Also denote by $\mathcal{T}(p)$ the subclass of $\mathcal{A}(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k (\ a_k \geq 0; z \in U). \tag{1.7}$$

Recently, Nishiwaki and Owa [13] have studied some results of Hölder-type inequalities for a subclass of p-valent functions. Now, we recall the generalization of the convolution due to Choi et al. [6].

For functions $f_j(z) \in \mathcal{T}(p)$ are given by

$$f_j(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,j} z^k \left(a_{k,j} \geq 0; j = 1, 2, ..., m\right), \tag{1.8}$$

we define

$$G_m(z) = z^p - \sum_{k=p+1}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) z^k, \tag{1.9}$$

and

$$H_m(z) = z^p - \sum_{k=p+1}^{\infty} \left(\prod_{j=1}^{m} (a_{k,j})^{q_j}\right) z^k \quad (q_j > 0), \tag{1.10}$$

where $G_m(z)$ denotes the modified Hadamard product of $f_j(z)(j = 1, 2, ..., m)$ which are given by (1.8). Therefore, $H_m(z)$ are the generalization modified Hadamard product.

Remark 1.

(i) For $m = 2$, then $G_2(z) = (f_1 * f_2)(z)$.

(ii) For $q_j = 1$, we have $G_m(z) = H_m(z)$.

Further for functions $f_j(z)(j = 1, 2, ..., m)$ which are given by (1.8), the familiar Hölder inequality assumes the following form (see [12,20]):

$$\sum_{k=p+1}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) \leq \prod_{j=1}^{m} \left(\sum_{k=p+1}^{\infty} (a_{k,j})^{q_j}\right)^{\frac{1}{q_j}} \quad (q_j \geq 1; j = 1, 2, ..., m; \sum_{j=1}^{m} \frac{1}{q_j} \geq 1). \tag{1.11}$$

For positive real values of $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [19])

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ...(\alpha_q)_k}{(\beta_1)_k ...(\beta_s)_k} . \frac{z^k}{k!} \tag{1.12}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)} = \begin{cases} 1 & (m = 0), \\ a(a + 1)....(a + m - 1) & (m \in \mathbb{N}). \end{cases} \tag{1.13}$$

Corresponding to the function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ defined by

$$h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^p \, {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \tag{1.14}$$

we consider a linear operator $H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ which is defined by following Hadamard product (or convolution):

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z). \tag{1.15}$$

We observe that for function $f(z)$ of the form (1.1) we have

$$H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=z^p+\sum_{k=p+1}^{\infty}\Gamma_{k-p}(\alpha_1)a_kz^k, \tag{1.16}$$

where

$$\Gamma_{k-p}(\alpha_1)=\frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}....(\beta_s)_{k-p}}\cdot\frac{1}{(1)_{k-p}}\,(k\geq p+1)\,. \tag{1.17}$$

For convenience, we write

$$H^p_{q,s}[\alpha_1]=H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s). \tag{1.18}$$

The linear operator $H^p_{q,s}[\alpha_1]$ was introduced and studied by Dziok and Srivastava [7], and it includes (as its special cases) various other linear operators for example Carlson and Shaffer [4] and Ruscheweyh [17].

We define the linear multiplier Dziok-Srivastava operator $D^{m,q,s}_{p,\lambda,\ell}f(z)$ is given by

$$D^{0,q,s}_{p,\lambda,\ell}f(z)=f(z)\,(f(z)\in A(p)\,,$$

$$D^{1,q,s}_{p,\lambda,\ell}f(z)=(1-\lambda)H^p_{q,s}(\alpha_1)f(z)+\frac{\lambda}{(\ell+p)z^{\ell-1}}[z^\ell H^p_{q,s}(\alpha_1)f(z)]'(\lambda\geq 0;\ell\geq 0),$$

$$D^{2,q,s}_{p,\lambda,\ell}f(z)=D^{q,s}_{p,\lambda,\ell}f(z)(D^{1,q,s}_{p,\lambda,\ell}f(z))$$

and (in general)

$$D^{m,q,s}_{p,\lambda,\ell}f(z)=D^{q,s}_{p,\lambda,\ell}f(z)(D^{m-1,q,s}_{p,\lambda,\ell}f(z))(m\in\mathbb{N}_0).$$

If $f(z)$ is given by (1.1), then from (1.16) we see that

$$D^{m,q,s}_{p,\lambda,\ell}f(z)=z^p+\sum_{k=p+1}^{\infty}\Phi^m_{p,k}(\alpha_1,\lambda,\ell)a_kz^k\quad, \tag{1.19}$$

where

$$\Phi^m_{p,k}(\alpha_1,\lambda,\ell)=\left[\frac{\ell+p+\lambda(k-p)}{\ell+p}\Gamma_{k-p}(\alpha_1)\right]^m. \tag{1.20}$$

The operator $D^{m,q,s}_{p,\lambda,\ell}f(z)$, can be written in terms of convolution as follows:

$$\begin{aligned}D^{m,q,s}_{p,\lambda,\ell}f(z) &= [(h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*g_{p,\lambda,\ell}(z))*...\\ &\quad *(h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*g_{p,\lambda,\ell}(z))]*f(z),1.21 \quad (1)\end{aligned}$$

where

$$g_{p,\lambda,\ell}(z)=\frac{(\ell+p)\,(1-z)\,z^{p}+\lambda z^{p+1}}{(\ell+p)(1-z)^{2}}=\frac{z^{p}-(1-\frac{\lambda}{\ell+p})z^{p+1}}{(1-z)^{2}}.$$

By specializing the parameters q, s, α_1, β_1, λ and ℓ, we obtain the following operators studied by various authors:
(i) For $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, we have $D_{p,\lambda,\ell}^{m,2,1}f(z)=I_p^m(\lambda,\ell)\,f(z)$ (see Catas [5]);
(ii) For $\ell=0$, $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$, we have $D_{p,\lambda,\ell}^{m,2,1}f(z)=D_{\lambda,p}^{m}f(z)$ (see El-Ashwah and Aouf [9]);
(iii) $D_{p,0,0}^{1,q,s}f(z)=H_{q,s}^{p}[\alpha_1]$ (see Dziok and Sirvastava [7]);
(iv) For $p=1$, we have $D_{1,\lambda,\ell}^{m,q,s}f(z)=D_{\lambda,\ell}^{m,q,s}f(z)$ (see El-Ashwah et al. [8]);
(v) For $p=1$, $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$, $\beta_1=2-\alpha$ ($\alpha\neq 2,3,...$) and $\ell=0$, we have $D_{1,\lambda,0}^{m,2,1}f(z)=D_{\lambda}^{m,\alpha}f(z)$ (see Al-Oboudi and Al-Amoudi [2] and Aouf and Mostafa [3]);
(vi) For $p=1$, $q=2$, $s=1$, $\alpha_1=a$ ($a>0$), $\alpha_2=1$, $\beta_1=c$ ($c>0$) and $\ell=0$, we have $D_{1,\lambda,0}^{m,2,1}f(z)=I_{a,c,\lambda}^{m}f(z)$ (see Prajapat and Raina [15]);
(vii) For $p=1$, $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$, we have $D_{1,\lambda,\ell}^{m,2,1}f(z)=I^{m}(\lambda,\ell)f(z)$ (see Catas [5]);
(viii) For $p=1$, $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$ and $\ell=0$, we have $D_{1,\lambda,0}^{m,2,1}f(z)=D_{\lambda}^{m}f(z)$ (see Al-Oboudi [1]) and $D_{1,0,0}^{m,2,1}f(z)=D^{m}f(z)$ (see Salagean [18]);
(ix) $D_{1,0,0}^{1,q,s}f(z)=H_{q,s}[\alpha_1]$ (see Dziok and Sirvastava [7]).

For $0\le\delta\le1$, $0<\beta\le1$, $-1\le A<B\le1$, $0\le\gamma\le1$ and $0\le\alpha<p$ we let the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)$ denote the subclass of $\mathcal{T}(p)$ consisting of functions of the form (1.7) and satisfing the condition:

$$\left|\frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-p}{(B-A)\,\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-\alpha\right)-B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-p\right)}\right|<\beta\quad(z\in U)\,, \tag{1.22}$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)}=\frac{(1-\delta+\frac{\delta}{p})z(D_{p,\lambda,\ell}^{m,q,s}f(z))'+\frac{\delta}{p}z^{2}(D_{p,\lambda,\ell}^{m,q,s}f(z))''}{(1-\delta)D_{p,\lambda,\ell}^{m,q,s}f(z)+\frac{\delta}{p}z(D_{p,\lambda,\ell}^{m,q,s}f(z))'}. \tag{1.23}$$

We note that:
(i) For $p=m=1$, $\lambda=0$ and $\ell=0$, we have $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)=\mathcal{HF}_{\gamma}^{\delta}(\alpha,\beta,A,B)$ (see Murugusundaramoothy et al. [11]);
(ii) For $m=\delta=0$, $\beta=\gamma=B=1$ and $A=-1$, we have $DF_{\lambda,\ell,\gamma}^{0,p,0}(\alpha,\beta,A,B)=S_p^{*}(\alpha)$, for $m=0$ and $\delta=\beta=\gamma=1$, we have $DF_{\lambda,\ell,\gamma}^{0,p,1}(\alpha,\beta,A,B)=K_p(\alpha)$ (see also Nishiwaki and Owa [13], with $n=1$).

Also we note that:
(i) For $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$, we have

$$DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)=\left|\frac{\frac{zF_\delta'(z)}{F_\delta(z)}-p}{(B-A)\gamma\left(\frac{zF_\delta'(z)}{F_\delta(z)}-\alpha\right)-B\left(\frac{zF_\delta'(z)}{F_\delta(z)}-p\right)}\right|<\beta\quad(z\in U),$$

where

$$\frac{zF_\delta'(z)}{F_\delta(z)}=\frac{(1-\delta+\frac{\delta}{p})z(I_p^m(\lambda,\ell)f(z))'+\frac{\delta}{p}z^2(I_p^m(\lambda,\ell)f(z))''}{(1-\delta)I_p^m(\lambda,\ell)f(z)+\frac{\delta}{p}z(I_p^m(\lambda,\ell)f(z))'}\quad(0\le\delta\le1);$$

(ii) For $\ell=0,\ q=2,\ s=1$ and $\alpha_1=\alpha_2=\beta_1=1$, we have

$$DF_{\lambda,0,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)=DF_{\lambda,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)=\left|\frac{\frac{zF_\delta'(z)}{F_\delta(z)}-p}{(B-A)\gamma\left(\frac{zF_\delta'(z)}{F_\delta(z)}-\alpha\right)-B\left(\frac{zF_\delta'(z)}{F_\delta(z)}-p\right)}\right|<\beta,$$

where

$$\frac{zF_\delta'(z)}{F_\delta(z)}=\frac{(1-\delta+\frac{\delta}{p})z(D_{\lambda,p}^mf(z))'+\frac{\delta}{p}z^2(D_{\lambda,p}^mf(z))''}{(1-\delta)D^mf(z)+\frac{\delta}{p}z(D_{\lambda,p}^mf(z))'}\quad(0\le\delta\le1);$$

(iii) For $\ell=\lambda=0$, we have

$$DF_{0,0,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)=DF_{\gamma}^{m,p,\delta}(\alpha,\beta,A,B)=\left|\frac{\frac{zF_\delta'(z)}{F_\delta(z)}-p}{(B-A)\gamma\left(\frac{zF_\delta'(z)}{F_\delta(z)}-\alpha\right)-B\left(\frac{zF_\delta'(z)}{F_\delta(z)}-p\right)}\right|<\beta,$$

where

$$\frac{zF_\delta'(z)}{F_\delta(z)}=\frac{(1-\delta+\frac{\delta}{p})z(H_{q,s}^p[\alpha_1]f(z))'+\frac{\delta}{p}z^2(H_{q,s}^p[\alpha_1]f(z))''}{(1-\delta)H_{q,s}^p[\alpha_1]f(z)+\frac{\delta}{p}z(H_{q,s}^p[\alpha_1]f(z))'}\quad(0\le\delta\le1).$$

In this paper, we discuss some interesting Hölders inequalities results and modified Hadamard product for functions $f(z)\in DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)$.

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0\le\delta\le1$, $-1\le A<B\le1$, $0<B\le1$, $q\le s+1, q,s\in\mathbb{N}_0$, $\lambda\ge0$, $\ell\ge0$, $m\in\mathbb{N}_0$, $0\le\alpha<p$, $0<\beta\le1$ and $z\in U$.

In the following theorem we obtain necessary and sufficient conditions for functions $f(z) \in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\alpha,\beta,A,B)$.

Theorem 1. Let the function $f(z)$ be defined by (1.7). Then $f(z)$ is in the class $DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\alpha,\beta,A,B)$ if and only if

$$\sum_{k=p+1}^{\infty} R_k a_k \leq (B-A)\,\beta\gamma\,(p-\alpha)\,, \tag{2.1}$$

where

$$R_k = (1+\frac{k}{p}\delta-\delta)[(1-\beta B)\,(k-p)+(B-A)\,\beta\gamma\,(k-\alpha)]\Phi^m_{p,k}(\alpha_1,\lambda,\ell) \tag{2.2}$$

and $\Phi^m_{p,k}(\alpha_1,\lambda,\ell)$ is given by (1.20).

Proof. Assume that the inequality (2.1) holds true, we find from (1.7) and (1.22) that

$$\begin{aligned}
&|zF'_\delta(z)-pF_\delta(z)| \\
&\qquad -\beta\,|(B-A)\,\gamma\,[zF'_\delta(z)-\alpha F_\delta(z)]-B\,[zF'_\delta(z)-pF_\delta(z)]| \\
= &\left|\sum_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)\,(k-p)\,\Phi^m_{p,k}(\alpha_1,\lambda,\ell)a_k z^k\right| \\
&-\beta\,|(B-A)\,\gamma\,(p-\alpha)\,z^p \\
&+\sum_{k=p+1}^{\infty}\left[(1+\frac{k}{p}\delta-\delta)\,(B-A)\,\gamma\,(k-\alpha)-B\,(k-p)\right]\Phi^m_{p,k}(\alpha_1,\lambda,\ell)a_k z^k
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)\,(k-1)\,\Phi_{p+1,k}(\alpha_1,\lambda,\ell)a_k r^k-(B-A)\,\beta\gamma\,(p-\alpha)\,r \\
&+\beta\sum_{k=p+1}^{\infty}\left[(1+\frac{k}{p}\delta-\delta)\,(B-A)\,\gamma\,(k-\alpha)-B\,(k-p)\right]\Phi^m_{p,k}(\alpha_1,\lambda,\ell)a_k r^k \\
&\leq \sum_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)[(1-\beta B)\,(k-p)+(B-A)\,\beta\gamma\,(k-\alpha)]\Phi^m_{p,k}(\alpha_1,\lambda,\ell)a_k \\
&\qquad\qquad -(B-A)\,\beta\gamma\,(p-\alpha)\leq 0\ \ (z\in U)\,.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\alpha,\beta,A,B)$.

Conversely, Let

$$\left|\frac{\frac{zF_\delta'(z)}{F_\delta(z)}-p}{(B-A)\,\gamma\left(\frac{zF_\delta'(z)}{F_\delta(z)}-\alpha\right)-B\left(\frac{zF_\delta'(z)}{F_\delta(z)}-p\right)}\right|$$

$$=\left|\frac{\sum\limits_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)(k-1)\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_kz^k}{(B-A)\gamma(p-\alpha)z^p-\sum\limits_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)[(B-A)\gamma(k-\alpha)+B(k-p)]\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_kz^k}\right|<\beta\ \ (z\in U)\,.$$

Now since $Re\{z\}\leq|z|$ for all z, we have

$$Re\left\{\frac{\sum\limits_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)(k-p)\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_kz^k}{(B-A)\gamma(p-\alpha)-\sum\limits_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)[(B-A)\gamma(k-\alpha)-B(k-p)]\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_kz^k}\right\}<\beta. \tag{2.3}$$

Choose values of z on the real axis so that $f'(z)$ is real. Then upon clearing the denominator in (2.3) and letting $z\to1^-$through real values, we have

$$\sum_{k=p+1}^{\infty}(1+\frac{k}{p}\delta-\delta)[(1-\beta B)\,(k-p)+(B-A)\,\beta\gamma\,(k-\alpha)]\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_k$$

$$-\,(B-A)\,\beta\gamma\,(p-\alpha)\leq0.$$

This completes the proof of Theorem 1.

3. Hölder's inequality

Theorem 2. Let the functions $f_j(z)(j=1,2,...m)$ defined by (1.8) are in the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\eta_j,\beta,A,B)$. Then $H_m(z)$ are in the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\zeta,\beta,A,B)$ with

$$\zeta\leq p-\frac{(k-p)\mu_j+(k-p)(1-\beta B)_{j=1}^{m}[(B-A)\beta\gamma]^{s_j-1}(p-\eta_j)^{s_j}}{{}_{j=1}^{m}(1+k\frac{\delta}{p}-\delta)^{s_j-1}[\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)]^{s_j-1}[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]^{s_j}-\mu_j},\,k\geq p+1,$$

where

$$\left(r=\sum_{j=1}^{m}s_j\geq1;s_j\geq\frac{1}{q_j};\sum_{j=1}^{m}\frac{1}{q_j}\geq1;q_j>1;j=1,2...m\right).$$

Proof. Let $f_j(z)\in DF_{\lambda,\ell,\gamma}^{n,q,s}(\eta_j,\beta,A,B)$, we have

$$\sum_{k=p+1}^{\infty}\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]}{(B-A)\beta\gamma(p-\eta_j)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_{k,j}\leq1, \tag{2.1}$$

which implies

$$\left(\sum_{k=p+1}^{\infty}\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]}{(B-A)\beta\gamma(p-\eta_j)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_{k,j}\right)^{\frac{1}{q_j}}\leq 1, \qquad (2.2)$$

with $q_j > 1$ and $\sum_{j=1}^{m}\frac{1}{q_j}\geq 1$.
From (2.2), we have

$$\mathop{}_{j=1}^{m}\left(\sum_{k=p+1}^{\infty}\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]}{(B-A)\beta\gamma(p-\eta_j)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)a_{k,j}\right)^{\frac{1}{q_j}}\leq 1.$$

Applying Hölder's inequality (1.11), we find that

$$\sum_{k=p+1}^{\infty}\left[\mathop{}_{j=1}^{m}\left(\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]}{(B-A)\beta\gamma(p-\eta_j)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\right)^{\frac{1}{q_j}}a_{k,j}^{\frac{1}{q_j}}\right]\leq 1.$$

Thus, we have to determine the largest ζ such that

$$\sum_{k=p+1}^{\infty}\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]}{(B-A)\beta\gamma(p-\zeta)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\left(\mathop{}_{j=1}^{m}a_{k,j}^{s_j}\right)\leq 1$$

that is

$$\begin{aligned}&\sum_{k=p+1}^{\infty}\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]}{(B-A)\beta\gamma(p-\zeta)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\left(\mathop{}_{j=1}^{m}a_{k,j}^{s_j}\right)\\ \leq\ &\sum_{k=p+1}^{\infty}\left[\mathop{}_{j=1}^{m}\left(\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]}{(B-A)\beta\gamma(p-\eta_j)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\right)^{\frac{1}{q_j}}a_{k,j}^{\frac{1}{q_j}}\right].\end{aligned}$$

Therefore, we need to find the largest ζ such that

$$\begin{aligned}&\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]}{(B-A)\beta\gamma(p-\zeta)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\left(\mathop{}_{j=1}^{m}a_{k,j}^{s_j-\frac{1}{q_j}}\right)\\ \leq\ &\mathop{}_{j=1}^{m}\left(\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]}{(B-A)\beta\gamma(p-\eta_j)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\right)^{\frac{1}{q_j}},\end{aligned}$$

for $(k\geq p+1)$. Since

$$\mathop{}_{j=1}^{m}\left(\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]}{(B-A)\beta\gamma(p-\zeta)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\right)^{s_j-\frac{1}{q_j}}a_{k,j}^{s_j-\frac{1}{q_j}}\leq 1$$

$$(s_j-\frac{1}{q_j}\geq 0),$$

we see that,

$$ {}_{j=1}^{m} a_{k,j}^{s_j-\frac{1}{q_j}} \leq \frac{1}{{}_{j=1}^{m}\left(\frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma\left(k-\eta_j\right)]}{(B-A)\beta\gamma\left(p-\eta_j\right)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)\right)^{s_j-\frac{1}{q_j}}}. $$

This implies that

$$ \frac{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]}{(B-A)\beta\gamma(p-\zeta)}\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell) $$
$$ \leq \frac{{}_{j=1}^{m}(1+\frac{k}{p}\delta-\delta)^{s_j}[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)]^{s_j}}{{}_{j=1}^{m}[(B-A)\beta\gamma(p-\eta_j)]^{s_j}}, $$

which is equivalent to

$$ \zeta \leq p-\frac{(k-p)\mu_j+(k-p)(1-\beta B){}_{j=1}^{m}[(B-A)\beta\gamma]^{s_j-1}(p-\eta_j)^{s_j}}{{}_{j=1}^{m}(1+k\frac{\delta}{p}-\delta)^{s_j-1}[\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)]^{s_j-1}[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]^{s_j}-\mu_j}, $$

where $\mu_j ={}_{j=1}^{m} [(B-A)\,\beta\gamma]^{s_j}\,(p-\eta_j)^{s_j}$. Let

$$ \Psi(k) \leq p-\frac{(k-p)\mu_j+(k-p)(1-\beta B){}_{j=1}^{m}[(B-A)\beta\gamma]^{s_j-1}(p-\eta_j)^{s_j}}{{}_{j=1}^{m}(1+\frac{k}{p}\delta-\delta)^{s_j-1}[\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)]^{s_j-1}[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\eta_j)]^{s_j}-\mu_j}, $$

which is an increasing function in k. This completes the proof of Theorem 2.

Remark 2.
Putting $m=\delta=0, \gamma=\beta=B=1, A=-1$ in Theorem 2, we obtain the corresponding result obtained by Nishiwaki and Owa [13, Theorem 2.1, with $n=1$].
Putting $s_j=1$ in Theorem 2, we obtain of the following corollary:

Corollary 1. Let the functions $f_j(z)(j=1,2,...m)$ defined by (1.8) are in the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\eta_j,\beta,A,B)$. Then $H_m(z)$ are in the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\zeta,\beta,A,B)$ with

$$ \zeta \leq p-\frac{\prod\limits_{j=1}^{m}(p-\eta_j)[(B-A)\beta\gamma]^{r-1}[(1-\beta B)+(B-A)\beta\gamma]}{[(1+\frac{\delta}{p})\Phi_{p,p+1}^{m}(\alpha_1,\lambda,\ell)]^{r-1}_{j=1}{}^{m}[(1-\beta B)+(B-A)\beta\gamma(p+1-\eta_j)]-{}_{j=1}^{m}(p-\eta_j)[(B-A)\beta\gamma]^{r}}, $$

where $(r=\sum_{j=1}^{m}s_j \geq 1; s_j \geq \frac{1}{q_j}; \sum_{j=1}^{m}\frac{1}{q_j} \geq 1; q_j>1; j=1,2...m)$.
Putting $\eta_j=\zeta$ in Theorem 2, we obtain of the following corollary:

Corollary 2. Let the functions $f_j(z)(j=1,2,...m)$ defined by (1.8) are in the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\eta_j,\beta,A,B)$. Then $H_m(z)$ are in the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\zeta,\beta,A,B)$ with

$$ \zeta \leq p-\frac{(p-\eta)^r[(B-A)\beta\gamma]^{r-1}[(1-\beta B)+(B-A)\beta\gamma]}{[(1+\frac{\delta}{p})\Phi_{p,p+1}^{m}(\alpha_1,\lambda,\ell)]^{r-1}[(1-\beta B)+(B-A)\beta\gamma(p+1-\eta)]^r-(p-\eta)^r[(B-A)\beta\gamma]^r}, $$

where

$$(r=\sum_{j=1}^{m}s_j\geq p+1;s_j\geq\frac{1}{q_j};\sum_{j=1}^{m}\frac{1}{q_j}\geq 1;q_j>1;j=1,2...m)\ .$$

Example 1. Let the functions $f_j(z)(j=1,2,...,m)$ defined as follows:

$$f_j(z)=z^p-\tfrac{(B-A)\beta\gamma(p-\alpha)}{R_{p+1}}\epsilon z^{p+1}-\tfrac{(B-A)\beta\gamma(p-\alpha)}{R_{p+j+1}}\epsilon_j z^{p+j+1}\qquad(\epsilon+\epsilon_j\leq 1),\quad(2.13)$$

where R_{p+j+1} is given by (2.2) Then $H_m(z)\in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\zeta,\beta,A,B)$ with

$$\zeta=p-\tfrac{[(B-A)\beta\gamma]^r(p-\alpha)^r+(1-\beta B)[(B-A)\beta\gamma]^{r-1}(p-\alpha)^r}{[(1+\frac{\delta}{p})\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)]^{r-1}[(1-\beta B)+(B-A)\beta\gamma(p-\alpha+1)]^r-[(B-A)\beta\gamma]^r(p-\alpha)^r}.$$

Because, for functions from (2.13), for $j=1,2,...m$, we have

$$\begin{aligned}\sum_{k=p+1}^{\infty}\tfrac{R_k}{(B-A)\beta\gamma(1-\alpha)}a_k &= \tfrac{R_{p+1}}{(B-A)\beta\gamma(p-\alpha)}\epsilon a_{p+1}+\tfrac{R_{p+j+1}}{(B-A)\beta\gamma(p-\alpha)}\epsilon_j a_{p+j+1}\\ &= \epsilon+\epsilon_j\leq 1,\end{aligned}$$

from Theorem 1, then $f_j(z)\in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\eta_j,\beta,A,B)$. From (2.13), we have

$$H_m(z)=z^p-\left(\frac{(B-A)\,\beta\gamma\,(p-\alpha)}{R_{p+1}}\epsilon\right)^r.$$

Therefore $H_m(z)\in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\zeta,\beta,A,B)$.
Putting $s_j=1$ and $m=2$ in Theorem 2, we obtain of the following corollary:

Corollary 3. Let the function $f_1(z)$ defined by (1.8), be in the class $DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\eta_1,\beta,A,B)$. Suppose also that a function $f_2(z)$ defined by (1.8), be in the class $DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\eta_2,\beta,A,B)$. Then $(f_1*f_2)\in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\zeta,\beta,A,B)$, where

$$\zeta\leq p-\tfrac{(B-A)\beta\gamma(P-\eta_1)(P-\eta_2)[(B-A)\beta\gamma-B\beta+1]}{(1+\frac{\delta}{p})\Phi^2_{p,p+1}(\alpha_1,\lambda,\ell)\theta_1(\eta_1,\beta,\gamma,A,B,p+1)\theta_2(\eta_2,\beta,\gamma,A,B,p+1)-[(B-A)^2\beta^2\gamma^2](p-\eta_1)(p-\eta_2)},\quad(2.3)$$

where

$$\theta_1\left(\eta_1,\beta,\gamma,A,B,p+1\right)=\left(B-A\right)\beta\gamma\left(p-\eta_1+1\right)+\left(1-\beta B\right)\qquad(2.4)$$

and

$$\theta_2\left(\eta_2,\beta,\gamma,A,B,p+1\right)=\left(B-A\right)\beta\gamma\left(p-\eta_2+1\right)+\left(1-\beta B\right).\qquad(2.5)$$

This result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_1(z)=z^p-\tfrac{(B-A)\beta\gamma(p-\eta_1)}{(1+\frac{\delta}{p})[(B-A)\beta\gamma(p+1-\eta_1)+(1-\beta B)]\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)}z^{p+1}$$

and

$$f_2(z) = z^p - \frac{(B-A)\beta\gamma(1-\eta_2)}{(1+\frac{\delta}{p})[(B-A)\beta\gamma(p+1-\eta_2)+(1-\beta B)]\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)} z^{p+1}.$$

Putting $\eta_1 = \eta_2 = \eta$ in Corollary 3, we obtain of the following corollary:

Corollary 4. Let the functions $f_j(z)(j = 1, 2)$ defined by (1.8) are in the class $DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\eta, \beta, A, B)$. Then $(f_1 * f_2) \in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\varepsilon, \beta, A, B)$, where

$$\varepsilon = p - \frac{(B-A)\beta\gamma(p-\eta)^2[(B-A)\beta\gamma+1-\beta B]}{(1+\frac{\delta}{p})\Phi^2_{p,p+1}(\alpha_1,\lambda,\ell)\theta_1^2(\eta,\beta,\gamma,A,B,p+1)-(B-A)^2\beta^2\gamma^2(p-\eta)^2}. \tag{2.6}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z^p - \frac{(B-A)\beta\gamma(1-\eta)}{(1+\frac{\delta}{p})[(B-A)\beta\gamma(p-\eta+1)+(1-\beta B)]\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)} z^{p+1}(j = 1, 2). \tag{2.7}$$

Theorem 3. Let the functions $f_j(z)(j = 1, 2)$ defined by (1.8) are in the class $DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\zeta, \beta, A, B)$. Then the function $h(z)$ defined by

$$h(z) = z^p - \sum_{k=p+1}^{\infty} \left(a^2_{k,1} + a^2_{k,2}\right) z^k, \tag{2.8}$$

is in the class $DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\mu, \beta, A, B)$, where

$$\mu = 1 - \frac{2(B-A)\beta\gamma(1-\zeta)^2[1-\beta B+(B-A)\beta\gamma]}{(1+\frac{\delta}{p})[1-\beta B+(B-A)\beta\gamma(p-\zeta+1)]^2\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)-2(B-A)^2\beta^2\gamma^2(1-\zeta)^2}. \tag{2.9}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ defined by (2.7).

Proof. By virtue of Theorem 1, it is sufficient prove that

$$\sum_{k=p+1}^{\infty} \frac{(1+\frac{k}{p}\delta-\delta)[(B-A)\beta\gamma(k-\mu)+(1-\beta B)(k-p)]\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\mu)} \left(a^2_{k,1} + a^2_{k,2}\right) \le 1. \tag{2.10}$$

Since $f_j(z)(j = 1, 2) \in DF^{m,p,\delta}_{\lambda,\ell,\gamma}(\zeta, \beta, A, B)$, we have

$$\left\{\sum_{k=p+1}^{\infty} \frac{(1+\frac{k}{p}\delta-\delta)[(B-A)\beta\gamma(k-\zeta)+(1-\beta B)(k-p)]\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\zeta)}\right\}^2 a^2_{k,1}$$

$$\le \left\{\sum_{k=p+1}^{\infty} \frac{(1+\frac{k}{p}\delta-\delta)[(B-A)\beta\gamma(k-\zeta)+(1-\beta B)(k-p)]\Phi^m_{p,p+1}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\zeta)} a_{k,1}\right\}^2 \le 1 \tag{2.11}$$

and

$$\left\{\sum_{k=p+1}^{\infty}\frac{(1+k\frac{\delta}{p}-\delta)[(B-A)\beta\gamma(k-\zeta)+(1-\beta B)(k-p)]\Phi_{p+1,k}^{m}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\zeta)}\right\}^2 a_{k,2}^2$$

$$\leq\left\{\sum_{k=p+1}^{\infty}\frac{(1+k\frac{\delta}{p}-\delta)[(B-A)\beta\gamma(k-\zeta)+(1-\beta B)(k-p)]\Phi_{p+1,k}^{m}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\zeta)}a_{k,2}\right\}^2\leq 1. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\sum_{k=p+1}^{\infty}\frac{1}{2}\left\{\frac{(1+\frac{k}{p}\delta-\delta)[(B-A)\beta\gamma(k-\zeta)+(1-\beta B)(k-p)]\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\zeta)}\right\}^2(a_{k,1}^2+a_{k,2}^2)\leq 1.$$

Now to find the largest μ, such that

$$\frac{(1+\frac{k}{p}\delta-\delta)[(B-A)\beta\gamma(k-\mu)+(1-\beta B)(k-p)]\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\mu)}$$
$$\leq\ \frac{1}{2}\left\{\frac{(1+\frac{k}{p}\delta-\delta)[(B-A)\beta\gamma(k-\zeta)+(1-\beta B)(k-p)]\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(p-\zeta)}\right\}^2\quad(k\geq p+1)$$

that is

$$\mu\leq p-\frac{2(k-p)(B-A)\beta\gamma(1-\beta B)(p-\zeta)^2+2k[(B-A)\beta\gamma(p-\zeta)]^2-2p[(B-A)\beta\gamma(p-\zeta)]^2}{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]^2\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)-2(B-A)^2\beta^2\gamma^2(p-\zeta)^2}.$$

Now, defining the function $\Psi(k)$ by

$$\Psi(k)=p-\frac{2(k-p)(B-A)\beta\gamma(1-\beta B)(p-\zeta)^2+2k[(B-A)\beta\gamma(p-\zeta)]^2-2p[(B-A)\beta\gamma(p-\zeta)]^2}{(1+\frac{k}{p}\delta-\delta)[(1-\beta B)(k-p)+(B-A)\beta\gamma(k-\zeta)]^2\Phi_{p,k}^{m}(\alpha_1,\lambda,\ell)-2(B-A)^2\beta^2\gamma^2(p-\zeta)^2}.$$

We see that $\Psi(k)$ is an increasing function of k $(k\geq p+1)$. Therefore, we conclued that

$$\mu\leq\Psi(p+1)=p-\frac{2(B-A)\beta\gamma(1-\beta B)(p-\zeta)^2+2(p+1)[(B-A)\beta\gamma(p-\zeta)]^2-2p[(B-A)\beta\gamma(p-\zeta)]^2}{(1+\frac{\delta}{p})[(1-\beta B)+(B-A)\beta\gamma(p-\zeta+1)]^2\Phi_{p,p+1}^{m}(\alpha_1,\lambda,\ell)-2(B-A)^2\beta^2\gamma^2(p-\zeta)^2}.$$

This completes the proof of Theorem 3.
Remark 3. By specializing the parameters m, q, s, α_1, α_2, β_1, λ and ℓ, we obtain results corresponding to the classes $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)$, $DF_{\lambda,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)$ and $DF_{\gamma}^{m,p,\delta}(\alpha,\beta,A,B)$, mentioned in the introduction.

Open problem

The authors suggest to discuss the integral mean, neighbourhood and partial sum for the class $DF_{\lambda,\ell,\gamma}^{m,p,\delta}(\alpha,\beta,A,B)$.

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