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Certain subclass of higher order derivatives of $p-$valent functions

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Abstract

In this paper, we make use the differential operator to introduce and study the new subclass $\mathcal{TC}_m(p,q,n,\alpha)$ of $p-$valent analytic function with negative coefficient, and we derive distortion theorems, closure theorems, modified Hadamard products and radii of close-to-convexity, starlikeness and convexity.

Keywords: *Analytic function, p-valent functions, starlike function, convex function, Hadamard product, distortion theorems, closure theorems.*
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1 Introduction

Let $\mathcal{T}(p,n)$ denote the class of analytic and $p-$valent function in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0\,; k \geq n+p; p,n \in \mathbb{N} = \{1,2,...\}). \tag{1}$$

A function $f \in \mathcal{T}(p,n)$ is said to be in the class $\mathcal{T}_n^*(p,\alpha)$ of $p-$valently starlik functions of order α in $\mathbb{U}$ if the following inequality holds

$$Re\left(\frac{z f'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < p; p \in \mathbb{N}). \tag{2}$$

Also, a function $f \in \mathcal{T}(p,n)$ is said to be in the class $\mathcal{C}_n(p,\alpha)$ of $p-$valently convex functions of order α in $\mathbb{U}$ if the following inequality holds

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < p; p \in \mathbb{N}). \tag{3}$$

The classes $\mathcal{T}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ were introduced and studied by Owa [5]. It is known that (see [3, 5])

$$f \in \mathcal{C}_n(p,\alpha) \Leftrightarrow \frac{zf'}{p} \in \mathcal{T}_n^*(p,\alpha). \tag{4}$$

We note that

$$\mathcal{T}_1^*(p,\alpha) = \mathcal{T}^*(p,\alpha) \quad and \quad \mathcal{C}_1(p,\alpha) = \mathcal{C}(p,\alpha) \quad (0 \leq \alpha < p).$$

The classes $\mathcal{T}^*(p,\alpha)$ and $\mathcal{C}(p,\alpha)$ were introduced and studied by Owa [4].

Let $f_j \in \mathcal{T}(p,n)$ $(j = 1,2)$ be given by

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0; j = 1,2; k \geq n+p; n,p \in \mathbb{N}), \tag{5}$$

then the Hadamard product (or convolution) $(f_1 * f_2)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{6}$$

The main purpose of the present paper is to investigate various interesting properties and characteristics of functions belonging to the two subclasses $\mathcal{S}_n(p,q,\alpha)$ and $\mathcal{C}_n(p,q,\alpha)$ defined by

$$\mathcal{S}_n(p,q,\alpha) = \left\{ f \in \mathcal{T}(p,n) : Re\left(\frac{z f^{(1+q)}(z)}{f^{(q)}(z)}\right) > \alpha \quad (z \in \mathbb{U}) \right\} \tag{7}$$

and

$$\mathcal{C}_n(p,q,\alpha)=\left\{f\in\mathcal{T}(p,n):Re\left(1+\frac{z\ f^{\ (2+q)}(z)}{f^{(1+q)}(z)}\right)>\alpha\quad(z\in\mathbb{U})\right\},\qquad(8)$$

where, for each $f\in\mathcal{T}(p,n)$,

$$f^{(q)}(z)=\delta(p,q)\,z^{p-q}-\sum_{k=n+p}^{\infty}\delta(k,q)\,a_k z^{k-q},\qquad(9)$$

and

$$\delta(i,j)=\frac{i!}{(i-j)!}=\begin{cases}1 & (j=0)\\ i(i-1)\ldots(i-j+1) & (j\neq 0).\end{cases}\qquad(10)$$

The classes $\mathcal{S}_n(p,q,\alpha)$ and $\mathcal{C}_n(p,q,\alpha)$ were introduced and studied by Chen et al. [2]. It is noticed that

$$f^{(q)}\in\mathcal{C}_n(p,q,\alpha)\Leftrightarrow\frac{z\ f^{\ (1+q)}}{p-q}\in\mathcal{S}_n(p,q,\alpha).\qquad(11)$$

We note that

(i) $\mathcal{S}_n(p,0,\alpha)=\mathcal{T}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,0,\alpha)=\mathcal{C}_n(p,\alpha)$ (Owa [5]);
(ii) $\mathcal{S}_n(p,0,\alpha)=\mathcal{T}_\alpha(p,n)$ and $\mathcal{C}_n(p,0,\alpha)=\mathcal{CT}_\alpha(p,n)$ (Yamakawa [9]);
(iii) $\mathcal{S}_n(1,0,\alpha)=\mathcal{T}_n^*(1,\alpha)=\mathcal{T}_\alpha(1,n)=\mathcal{T}_\alpha(n)$ (Srivastava et al. [8]);
(iv) $\mathcal{C}_n(1,0,\alpha)=\mathcal{C}_n(1,\alpha)=\mathcal{CT}_\alpha(1,n)=\mathcal{C}_\alpha(n)$ (Srivastava et al. [8]);
(v) $\mathcal{S}_1(1,0,\alpha)=\mathcal{T}_\alpha(1)=\mathcal{T}^*(\alpha)$ (Silverman [7]);
(vi) $\mathcal{C}_1(1,0,\alpha)=\mathcal{C}_\alpha(1)=\mathcal{C}(\alpha)$ (Silverman [7]).

2 General Classes Associated with Coefficient Bounds

Unless otherwise mentioned, we shell assume in the reminder of this paper that, the parameters $0\leq\alpha<p-q$, $p>q$, n and $p\in\mathbb{N}$, m and $q\in\mathbb{N}_0$, $k\geq n+p$, $z\in\mathbb{U}$ and $\delta(i,j)$ is given by (10).

In order to prove our results for functions belonging to class $\mathcal{TC}_m(p,q,n,\alpha)$, we shall need the following lemmas given by Chen et al. [2].

Lemma 1 [2, Theorem 1] **.** Let the function $f(z)$ be in the class $\mathcal{T}(p,n)$. Then $f(z)$ is in the class $\mathcal{S}_n(p,q,\alpha)$ if and only if

$$\sum_{k=n+p}^{\infty}(k-q-\alpha)\,\delta(k,q)\,a_k\leq(p-q-\alpha)\,\delta(p,q).\qquad(12)$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{(p-q-\alpha)\,\delta(p,q)}{(n+p-q-\alpha)\,\delta(n+p,q)}z^{n+p}.\qquad(13)$$

Lemma 2 [2, Theorem 2]. Let the function $f(z)$ be in the class $\mathcal{T}(p,n)$. Then $f(z)$ is in the class $\mathcal{C}_n(p,q,\alpha)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{(k-q)}{(p-q)}(k-q-\alpha)\,\delta(k,q)\,a_k \leq (p-q-\alpha)\,\delta(p,q). \tag{14}$$

The result is sharp, the extremal function given by

$$f(z)=z^p-\frac{(p-q)(p-q-\alpha)\,\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\,\delta(n+p,q)}z^{n+p}. \tag{15}$$

Definition 1. A function $f(z)$ defined by (1) and belonging to the class $\mathcal{T}(p,n)$ is said to be in the class $\mathcal{TC}_m(p,q,n,\alpha)$ if it also satisfies the coefficient inequality:

$$\sum_{k=n+p}^{\infty}\left(\frac{k-q}{p-q}\right)^m(k-q-\alpha)\,\delta(k,q)\,a_k \leq (p-q-\alpha)\,\delta(p,q). \tag{16}$$

It is easily to observe that

$$\mathcal{TC}_0(p,q,n,\alpha)=\mathcal{S}_n(p,q,\alpha) \quad \textit{and} \quad \mathcal{TC}_1(p,q,n,\alpha)=\mathcal{C}_n(p,q,\alpha). \tag{17}$$

3 Distortion Theorems

Theorem 1. Let the function $f(z)$ defined by (1) be in the class $\mathcal{TC}_m(p,q,n,\alpha)$. Then for $|z|=r<1$ and $k\geq n+p$, we have

$$\left|f^{(l)}(z)\right| \leq \left[\delta(p,l)+\frac{(p-q)^m(p-q-\alpha)\,\delta(p,q)\,\delta(n+p,l)}{(n+p-q)^m(n+p-q-\alpha)\,\delta(n+p,q)}r^n\right]r^{p-l} \tag{18}$$

and

$$\left|f^{(l)}(z)\right| \geq \left[\delta(p,l)-\frac{(p-q)^m(p-q-\alpha)\,\delta(p,q)\,\delta(n+p,l)}{(n+p-q)^m(n+p-q-\alpha)\,\delta(n+p,q)}r^n\right]r^{p-l}. \tag{19}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{(p-q)^m(p-q-\alpha)\,\delta(p,q)}{(n+p-q)^m(n+p-q-\alpha)\,\delta(n+p,q)}z^{n+p} \quad (z\in\mathbb{U}). \tag{20}$$

Proof. Under the hypothesis of Theorem 1, we find from the assertion (16) of Definition 1 that

$$\sum_{k=n+p}^{\infty}k!a_k \leq \frac{(n+p)!\,(p-q)^m(p-q-\alpha)\,\delta(p,q)}{(n+p-q)^m(n+p-q-\alpha)\,\delta(n+p,q)} \quad (n,p\in\mathbb{N};q,m\in\mathbb{N}_0;p>q). \tag{21}$$

Now, the inequalities (18) and (19) would follow readily when we make use of (21) in conjunction with the series expansion for $f^{(l)}(z)$ $(l \in \mathbb{N}_0)$ given by (11).

Putting $l = 0$ in Theorem 1, we obtain the following corollary.

Corollary 1. Let the function $f(z)$ defined by (1) be in the class $\mathcal{TC}_m(p, q, n, \alpha)$. Then for $|z| = r < 1$ and $k \geq n + p$, we have

$$|f(z)| \leq \left[1 + \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)\,\delta(n+p,q)} r^n\right] r^p \tag{22}$$

and

$$|f(z)| \geq \left[1 - \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)\,\delta(n+p,q)} r^n\right] r^p. \tag{23}$$

The result is sharp for the function $f(z)$ given by (20).

Putting $l = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{TC}_m(p, q, n, \alpha)$. Then for $|z| = r < 1$ and $k \geq n + p$, we have

$$\left|f'(z)\right| \leq \left[p + \frac{(p-q)^m (p-q-\alpha)(n+p)\,\delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)\,\delta(n+p,q)} r^n\right] r^{p-1} \tag{24}$$

and

$$\left|f'(z)\right| \geq \left[p - \frac{(p-q)^m (p-q-\alpha)(n+p)\,\delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)\,\delta(n+p,q)} r^n\right] r^{p-1}. \tag{25}$$

The result is sharp for the function $f(z)$ given by (20).

Putting $m = 0$ in Theorem 1, we obtain the following corollary.

Corollary 3 [2, Theorem 1]**.** Let the function $f(z)$ defined by (1) be in the class $\mathcal{S}_n(p, q, \alpha)$. Then for $|z| = r < 1$ and $k \geq n + p$, we have

$$\left|f^{(l)}(z)\right| \leq \left[\delta(p,l) + \frac{(p-q-\alpha)\,\delta(p,q)\,\delta(n+p,l)}{(n+p-q-\alpha)\,\delta(n+p,q)} r^n\right] r^{p-l} \tag{26}$$

and

$$\left|f^{(l)}(z)\right| \geq \left[\delta(p,l) - \frac{(p-q-\alpha)\,\delta(p,q)\,\delta(n+p,l)}{(n+p-q-\alpha)\,\delta(n+p,q)} r^n\right] r^{p-l}. \tag{27}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\,\delta(p,q)}{(n+p-q-\alpha)\,\delta(n+p,q)} z^{n+p} \quad (z \in \mathbb{U}). \tag{28}$$

Putting $m = 1$ in Theorem 1, we obtain the following corollary.

Corollary 4 [2, Theorem 8]**.** Let the function $f(z)$ defined by (1) be in the class $\mathcal{C}_n(p,q,\alpha)$. Then for $|z| = r < 1$ and $k \geq n+p$, we have

$$\left|f^{(l)}(z)\right| \leq \left[\delta(p,l) + \frac{(p-q)(p-q-\alpha)\delta(p,q)\delta(n+p,l)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)} r^n\right] r^{p-l} \tag{29}$$

and

$$\left|f^{(l)}(z)\right| \geq \left[\delta(p,l) - \frac{(p-q)(p-q-\alpha)\delta(p,q)\delta(n+p,l)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)} r^n\right] r^{p-l}. \tag{30}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)} z^{n+p} \quad (z \in \mathbb{U}). \tag{31}$$

4 Closure Theorems

Let the functions $f_v(z)$ $(v = 1,2,...s)$ be defined by

$$f_v(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0). \tag{32}$$

We shall prove the following results for the closure functions in the class $\mathcal{TC}_m(p,q,n,\alpha)$.

Theorem 2. Let the function $f_v(z)$ $(v = 1,2,...s)$ defined by (32) be in the class $\mathcal{TC}_m(p,q,n,\alpha)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{v=1}^{s} c_v f_v(z) \quad (c_v \geq 0) \tag{33}$$

is also in the class $\mathcal{TC}_m(p,q,n,\alpha)$, where

$$\sum_{v=1}^{s} c_v = 1.$$

Proof According to the definition of $h(z)$, it can be written as

$$h(z) = \sum_{v=1}^{s} c_v \left(z^p - \sum_{k=n+p}^{\infty} a_{k,v} z^k\right) = \sum_{v=1}^{s} c_v z^p - \sum_{v=1}^{s}\sum_{k=n+p}^{\infty} c_v a_{k,v} z^k$$

$$= z^p - \sum_{k=n+p}^{\infty}\sum_{v=1}^{s} c_v a_{k,v} z^k. \tag{34}$$

Furthermore, since the function $f_v(z)$ $(v = 1, 2, ...s)$ are in the class $\mathcal{TC}_m(p, q, n, \alpha)$, then

$$\sum_{k=n+p}^{\infty} \left(\frac{k-q}{p-q}\right)^m (k-q-\alpha)\,\delta(k,q)\,a_{k,v} \leq (p-q-\alpha)\,\delta(p,q)\,.$$

Hence

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \left(\frac{k-q}{p-q}\right)^m (k-q-\alpha)\,\delta(k,q) \left(\sum_{v=1}^{s} c_v a_{k,v}\right) \\ = &\sum_{v=1}^{s} c_v \left[\sum_{k=n+p}^{\infty} \left(\frac{k-q}{p-q}\right)^m (k-q-\alpha)\,\delta(k,q)\,a_{k,v}\right] \leq (p-q-\alpha)\,\delta(p,q) \end{aligned}$$

which implies that $h(z)$ be in the class $\mathcal{TC}_m(p, q, n, \alpha)$.

Corollary 5. Let the function $f_v(z)\,(v = 1, 2)$ defined by (32) be in the class $\mathcal{TC}_m(p, q, j, \alpha)$. Then the function $h(z)$ defined by

$$h(z) = (1-\eta)\,f_1(z) + \eta f_2(z) \quad (0 \leq \eta \leq 1) \tag{35}$$

is also in the class $\mathcal{TC}_m(p, q, n, \alpha)$.

5 Extreme points

Theorem 3. Let $f_p(z) = z^p$ and

$$f_k(z) = z^p - \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(k-q)^m (k-q-\alpha)\,\delta(k,q)} z^k \quad (k \geq n+p; n, p \in \mathbb{N}; q, m \in \mathbb{N}_0; p > q\ ). \tag{36}$$

Then the function $f(z)$ is in the class $\mathcal{TC}_m(p, q, n, \alpha)$ if and only if it can be expressed in the form:

$$f(z) = \lambda_p\ z^p + \sum_{k=n+p}^{\infty} \lambda_k\ f_k(z) \tag{37}$$

where $(\lambda_p \geq 0, \lambda_k \geq 0, k \geq n+p)$ and $\lambda_p + \sum\limits_{k=n+p}^{\infty} \lambda_n = 1$.

Proof. Suppose that $f(z)$ is expressed in the form (37). Then

$$\begin{aligned} f(z) &= \lambda_p z^p + \sum_{k=n+p}^{\infty} \lambda_k \left[z^p - \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(k-q)^m (k-q-\alpha)\,\delta(k,q)} z^k\right] \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(k-q)^m (k-q-\alpha)\,\delta(k,q)} \lambda_k z^k. \end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)} \cdot \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(k-q)^m (k-q-\alpha)\,\delta(k,q)} \lambda_k \\
= &\sum_{k=n+p}^{\infty} \lambda_k = 1 - \lambda_p \leq 1.
\end{aligned}$$

Then, $f(z) \in \mathcal{TC}_m(p,q,n,\alpha)$.

Conversely, suppose that $f(z) \in \mathcal{TC}_m(p,q,n,\alpha)$. We may set

$$\lambda_k = \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)} a_k,$$

where a_k is given by (16). Then

$$\begin{aligned}
f(z) &= z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(k-q)^m (k-q-\alpha)\,\delta(k,q)} \lambda_k z^k \\
&= z^p - \sum_{k=n+p}^{\infty} \left[z^p - f_k(z)\right] \lambda_k = \left(1 - \sum_{k=n+p}^{\infty} \lambda_k\right) z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z) \\
&= \lambda_p z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z).
\end{aligned}$$

This completes the proof of Theorem 3.

6 Modified Hadamard Products

Let the function $f_v(z)$ $(v = 1, 2)$ defined by (32). The modified Hadamard product of the function $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1} a_{k,2} z^k. \tag{38}$$

Theorem 4. Let the function $f_v(z)\,(v = 1, 2)$ defined by (32) be in the class $\mathcal{TC}_m(p,q,n,\alpha)$ and $(k \geq n+p)$. Then we have $(f_1 * f_2)(z) \in \mathcal{TC}_m(p,q,n,\beta)$, where

$$\beta = p-q-\frac{n(p-q)^m (p-q-\alpha)^2 \delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)^2 \delta(n+p,q) - (p-q)^m (p-q-\alpha)^2 \delta(p,q)}. \tag{39}$$

The result is sharp for the functions $f_v(z)\,(v=1,2)$ given by

$$f_v(z) = z^p - \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)\,\delta(n+p,q)} z^{n+p}. \tag{40}$$

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to fined the largest $\beta = \beta(p,q,n,\alpha)$ such that

$$\sum_{k=n+p}^{\infty} \frac{(k-q)^m (k-q-\beta)\,\delta(k,q)}{(p-q)^m (p-q-\beta)\,\delta(p,q)} a_{k,1}a_{k,2} \leq 1. \tag{41}$$

Since the function $f_v(z)$ $(v=1,2)$ belong to the class $\mathcal{TC}_m(p,q,n,\alpha)$, then from Definition 1, we have

$$\sum_{k=n+p}^{\infty} \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)} a_{k,v} \leq 1. \tag{42}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=n+p}^{\infty} \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{43}$$

Thus, it is sufficient to show that

$$\frac{(k-q-\beta)}{(p-q-\beta)}\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)},$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-q-\alpha)(p-q-\beta)}{(p-q-\alpha)(k-q-\beta)}. \tag{44}$$

But from (43) we have

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-q)^m (p-q-\alpha)\,\delta(p,q)}{(k-q)^m (k-q-\alpha)\,\delta(k,q)}. \tag{45}$$

Consequently, we need only to prove that

$$\frac{(p-q-\alpha)(k-q-\beta)}{(k-q-\alpha)(p-q-\beta)} \leq \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)},$$

or, equivalently, that

$$\beta \leq (p-q) - \frac{(p-q)^m (k-p)(p-q-\alpha)^2\,\delta(p,q)}{(k-q)^m (k-q-\alpha)^2\,\delta(k,q) - (p-q)^m (p-q-\alpha)^2\,\delta(p,q)}. \tag{46}$$

Since the right hand side of (46) is an increasing function of k $(k \geq n+p)$. Hence, we have

$$\beta = p-q-\frac{n(p-q)^m (p-q-\alpha)^2 \delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)^2 \delta(n+p,q) - (p-q)^m (p-q-\alpha)^2 \delta(p,q)}.$$

This completes the proof of Theorem 4.

Putting $m=0$ in Theorem 4, we obtain the following corollary.

Corollary 6. Let the functions $f_v(z)$ $(v=1,2)$ defined by (32) be in the class $\mathcal{S}_n(p,q,\alpha)$ and $k \geq n+p$. Then we have $(f_1 * f_2)(z) \in \mathcal{S}_n(p,q,\beta)$, where

$$\beta = p-q-\frac{n(p-q-\alpha)^2 \delta(p,q)}{(n+p-q-\alpha)^2 \delta(n+p,q) - (p-q-\alpha)^2 \delta(p,q)}. \tag{47}$$

The result is sharp.

Remark 1. We note that the result obtained by Chen et al. [2, Theorem 5] is not correct. The correct result is given by (47).

Putting $m=1$ in Theorem 4, we obtain the following corollary.

Corollary 7. Let the functions $f_v(z)$ $(v=1,2)$ defined by (32) be in the class $\mathcal{C}_n(p,q,\alpha)$ and $(k \geq n+p)$. Then we have $(f_1 * f_2)(z) \in \mathcal{C}_n(p,q,\beta)$, where

$$\beta = (p-q)-\frac{n(p-q)(p-q-\alpha)^2 \delta(p,q)}{(n+p-q)(n+p-q-\alpha)^2 \delta(n+p,q) - (p-q)(p-q-\alpha)^2 \delta(p,q)}. \tag{48}$$

The result is sharp.

Remark 2. We note that the result obtained by Chen et al. [2, Theorem 6] is not correct. The correct result is given by (48).

Theorem 5. Let the functions $f_v(z)$, $(v=1,2)$ defined by (32) be in the class $\mathcal{TC}_m(p,q,n,\alpha)$. Then the function

$$h(z) = z^p - \sum_{k=n+p}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2\right) z^k \tag{49}$$

is in the class $\mathcal{TC}_m(p,q,n,\alpha)$, where

$$\beta = p-q-\frac{2n(p-q)^m (p-q-\alpha)^2 \delta(p,q)}{(n+p-q)^m (n+p-q-\alpha)^2 \delta(n+p,q) - 2(p-q)^m (p-q-\alpha)^2 \delta(p,q)}. \tag{50}$$

The result is sharp for the function $f_v(z)$ $(v=1,2)$ given by (40).

Proof. From Definition 1, we have

$$\sum_{k=n+p}^{\infty} \left[\frac{(k-q)^m (k-q-\alpha)\delta(k,q)}{(p-q)^m (p-q-\alpha)\delta(p,q)}\right]^2 a_{k,v}^2 \tag{51}$$

$$\leq \left[\sum_{k=n+p}^{\infty} \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)} a_{k,v} \right]^2 \leq 1 \ (v=1,2). \tag{52}$$

It follow that

$$\sum_{k=n+p}^{\infty} \frac{1}{2} \left[\frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)} \right]^2 \left(a_{k,1}^2 + a_{k,2}^2 \right) \leq 1. \tag{53}$$

Therefore, we need to fined the largest β such that

$$\frac{(k-q-\beta)}{(p-q-\beta)} \leq \frac{1}{2} \frac{(k-q)^m (k-q-\alpha)^2 \delta(k,q)}{(p-q)^m (p-q-\alpha)^2 \delta(p,q)} \quad (k \geq n+p),$$

that is, that

$$\beta = (p-q) - \frac{2(k-p)(p-q)^m (p-q-\alpha)^2 \delta(p,q)}{(k-q)^m (k-q-\alpha)^2 \delta(k,q) - 2(p-q)^m (p-q-\alpha)^2 \delta(p,q)}. \tag{54}$$

Since the right hand side of (54) is an increasing function of k $(k \geq n+p)$, then, setting $k = n+p$ in (54), we have (50). This completes the proof of Theorem 5.

Putting $m = 0$ in Theorem 5, we obtain the following corollary.

Corollary 8. Let the functions $f_v(z)$ $(v = 1,2)$ defined by (32) be in the class $\mathcal{S}_n(p,q,\alpha)$ and $(k \geq n+p)$. Then the function $h(z)$ defined by (49) belongs to the class $\mathcal{S}_n(p,q,\sigma)$, where

$$\sigma = p-q - \frac{2n(p-q-\alpha)^2 \delta(p,q)}{(n+p-q-\alpha)^2 \delta(n+p,q) - 2(p-q-\alpha)^2 \delta(p,q)}.$$

Putting $m = 1$ in Theorem 5, we obtain the following corollary.

Corollary 9. Let the functions $f_v(z)$ $(v = 1,2)$ defined by (32) be in the class $\mathcal{C}_n(p,q,\alpha)$ and $(k \geq n+p)$. Then the function $h(z)$ defined by (49) belongs to the class $\mathcal{C}_n(p,q,\rho)$, where

$$\rho = p-q - \frac{2n(p-q)(p-q-\alpha)^2 \delta(p,q)}{(n+p-q)(n+p-q-\alpha)^2 \delta(n+p,q) - 2(p-q)(p-q-\alpha)^2 \delta(p,q)}.$$

7 Radii of close-to-convexity, starlikeness and convexity

Theorem 6. Let the function $f(z)$ defined by (1) be in the class $\mathcal{TC}_m(p,q,n,\alpha)$, then

(i) $f(z)$ is $p-$valently close-to-convex of order δ $(0 \leq \delta < p)$ in $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)\,(p-\delta)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)\,k} \right\}^{\frac{1}{k-p}}. \tag{55}$$

(ii) $f(z)$ is $p-$valently starlike of order δ $(0 \leq \delta < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)\,(p-\delta)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)\,(k-\delta)} \right\}^{\frac{1}{k-p}}. \tag{56}$$

(iii) $f(z)$ is $p-$valently convex of order δ $(0 \leq \delta < p)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{(k-q)^m (k-q-\alpha)\,\delta(k,q)\,p\,(p-\delta)}{(p-q)^m (p-q-\alpha)\,\delta(p,q)\,k\,(k-\delta)} \right\}^{\frac{1}{k-p}}. \tag{57}$$

Each of these results is sharp for the function $f(z)$ given by (20).

Proof. It is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta \quad (|z| < r_1; 0 \leq \delta < p; p \in \mathbb{N}), \tag{58}$$

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \ \ (|z| < r_2; 0 \leq \delta < p; p \in \mathbb{N}), \tag{59}$$

and that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \delta \quad (|z| < r_3; 0 \leq \delta < p; p \in \mathbb{N}) \tag{60}$$

for a function $f \in \mathcal{TC}_m(p,q,n,\alpha)$, where r_1, r_2,and r_3 are defined by (55), (56) and (57), respectively.

Putting $m = 0$ in Theorem 6, we obtain the following corollary.

Corollary 10. Let the function $f(z)$ defined by (1) be in the class $\mathcal{S}_n(p,q,\alpha)$, then

(i) $f(z)$ is $p-$valently close-to-convex of order δ $(0 \leq \delta < p)$ in $|z| < r_1$,where

$$r_1 = \inf_k \left\{ \frac{(k-q-\alpha)\,\delta(k,q)\,(p-\delta)}{(p-q-\alpha)\,\delta(p,q)\,k} \right\}^{\frac{1}{k-p}},$$

(ii) $f(z)$ is $p-$valently starlike of order δ $(0 \leq \delta < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(k-q-\alpha)\,\delta(k,q)\,(p-\delta)}{(p-q-\alpha)\,\delta(p,q)\,(k-\delta)} \right\}^{\frac{1}{k-p}},$$

(iii) $f(z)$ is $p-$valently convex of order δ $(0 \leq \delta < p)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{(k-q-\alpha)\,\delta(k,q)\,p\,(p-\delta)}{(p-q-\alpha)\,\delta(p,q)\,k\,(k-\delta)} \right\}^{\frac{1}{k-p}}.$$

Each of these results is sharp for the function $f(z)$ given by (13).

Putting $m = 1$ in Theorem 6, we obtain the following corollary.

Corollary 11. Let the function $f(z)$ defined by (1) be in the class $\mathcal{C}_n(p, q, \alpha)$, then

(i) $f(z)$ is $p-$valently close-to-convex of order δ $(0 \leq \delta < p)$ in $|z| < r_1$,where

$$r_1 = \inf_k \left\{ \frac{(k-q)\,(k-q-\alpha)\,\delta(k,q)\,(p-\delta)}{(p-q)\,(p-q-\alpha)\,\delta(p,q)\,k} \right\}^{\frac{1}{k-p}},$$

(ii) $f(z)$ is $p-$valently starlike of order δ $(0 \leq \delta < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(k-q)^m\,(k-q-\alpha)\,\delta(k,q)\,(p-\delta)}{(p-q)^m\,(p-q-\alpha)\,\delta(p,q)\,(k-\delta)} \right\}^{\frac{1}{k-p}},$$

(iii) $f(z)$ is $p-$valently convex of order δ $(0 \leq \delta < p)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{(k-q)\,(k-q-\alpha)\,\delta(k,q)\,p\,(p-\delta)}{(p-q)\,(p-q-\alpha)\,\delta(p,q)\,k\,(k-\delta)} \right\}^{\frac{1}{k-p}}.$$

Each of these results is sharp for the function $f(z)$ given by (15).

Remarks. (i) Putting $m = 0$ in Theorem 1, Corollary 1, 2, Theorem 2, Corollary 5 and Theorem 3, 4 and 5 we obtain the results of Aouf and Mostafa [1, with $b_k = 1$, Theorem 3, Corallary 2, 3, Theorem 5, Corallary 6, Theorem 6, 8 and 10, respectively];

(ii) Putting $m = 1$ in Theorem 1, Corollary 1, 2, Theorem 2, Corollary 5 and Theorem 3, 4 and 5 we obtain the results of Aouf and Mostafa [1, with $b_k = 1$, Theorem 4, Corallary 4, 5, Theorem 7, Corallary 7, Corallary 8 and Theorem 9 and 11, respectively].

8 Open problem

The authors suggest solving this idea by using $f * g$ instead of the function f which leads to many other classes.

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