

Generalized numbers and their holomorphic functions

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Abstract

The purpose of this paper is to introduce and develop the new concept of generalized numbers, which generalize at the same time complex, dual and hyperbolic numbers. We start by define, generalized numbers and some of their basic properties. In addition, we give another representation of generalized numbers basing on matrices. We extend the concept of holomorphicity to generalized functions. Moreover, generalized Cauchy-Riemann formulas were obtained.

Keywords: *Generalized numbers, generalized functions, holomorphicity, Cauchy-Riemann formulas.*

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1 Introduction

Alternative definitions of the imaginary unit i other than $i^2 = -1$ can give rise to interesting and useful complex number systems. The 16th-century Italian mathematicians G. Cardan (1501–1576) and R. Bombelli (1526–1572) are thought to be among the first to utilize the complex numbers, also said elliptic numbers, we know today by calculating with a quantity whose square is -1 .

Since then, various people have modified the original definition of the product of complex numbers. The English geometer W. Clifford (1845-1879) and the German geometer E. Study (1862-1930) added still another variant to the complex products [4, 13, 18], called dual numbers. The dual numbers arose from the convention that $\varepsilon^2 = 0$.

The ordinary, dual number is particular member of a two-parameter family of complex number systems often called binary number or generalized complex number. Which is two-component number of the form

$$z = x + y\varepsilon, \quad (1.1)$$

where $(x, y) \in \mathbb{R}^2$ and ε is an nilpotent number i.e. $\varepsilon^2 = 0$ and $\varepsilon \neq 0$.

Thus, the dual numbers, or which can be also called parabolic numbers, are elements of the 2-dimensional commutative associative algebra

$$\mathbb{D} = \mathbb{R}[\varepsilon] = \{z = x + y\varepsilon \mid (x, y) \in \mathbb{R}^2, \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0\}. \quad (1.2)$$

Another most basic situation appears if we replace the complex imaginary unit i with either the hyperbolic one $h^2 = +1$.

The set

$$\mathbb{H} = \mathbb{R}[h] = \{z = x + yh \mid (x, y) \in \mathbb{R}^2, h^2 = +1 \text{ and } h \neq -1 \text{ and } 1\}. \quad (1.3)$$

forms a 2-dimensional commutative associative algebra. They are called split-complex, duplex, hyperbolic or double numbers [16, 20]. The algebra has zero divisors $h_{\pm} = \frac{1}{2}(1 \pm h)$ satisfying the relations $h_{\pm}^2 = h_{\pm}$ and $h_+h_- = 0$. Thus, hyperbolic numbers are isomorphic to $\mathbb{R} \oplus \mathbb{R}$ —the direct sum of two copies of the real line spanned by h_+ and h_- . This explains the names of split-complex and double.

These nice concepts has lots of applications in many fields of fundamental sciences; such, algebraic geometry, Riemannian and Lorentzian geometry, quantum, mechanics and relativity, we refer the reader to [2, 3, 6, 9, 15, 17, 19].

Our interest is to develop, inspiring from the work of Anthony A. Harkin and Joseph B. Harkin [1], a general theory of complex product unified the three theories by introducing a new unit number by the relation $\sigma_{\alpha,\beta}^2 = \alpha\sigma_{\alpha,\beta} + \beta$ where $\alpha, \beta \in \mathbb{R}$.

Indeed, the main object of this work is to contribute to the development of generalized numbers and their holomorphic functions.

In the study of generalized functions, natural question arises whether it is possible to extend the concept of holomorphy to generalized functions and if it is possible to obtain Cauchy-Riemann formulas for generalized numbers unifying those known for complex, hyperbolic and dual cases ?.

We Begin by introducing the concept of generalized numbers and we give some of their basic properties. We define on the algebra of generalized numbers

some characteristics like index, conjugation as well as a matrices representation of generalized numbers. Relations which exist between on one hand generalized numbers and on the other hand complex, hyperbolic and dual numbers were shown.

We generalize the notion of holomorphicity to generalized functions. To do this, as in complex analysis. We start by study the differentiability. Cauchy-Riemann formulas have been also obtained for generalized functions.

2 Generalized Numbers

We introduce the concept of generalized numbers as follows.

A generalized number z is an ordered pair of real numbers (x, y) associated with the real unit 1 and generalized unit $\sigma_{\alpha,\beta}$, where it verifies $\sigma_{\alpha,\beta}^2 = \alpha\sigma_{\alpha,\beta} + \beta$ such that $\sigma_{\alpha,\beta}$ is different of the real roots, if there exist, of the equation $x^2 = \alpha x + \beta$, for $s \in \mathbb{R}$. A generalized number is usually denoted in the form

$$z = x + y\sigma_{\alpha,\beta}. \quad (2.1)$$

For $\alpha, \beta \in \mathbb{R}$, we denote by $\mathbb{C}_{\alpha,\beta}$ the set of generalized numbers defined by

$$\mathbb{C}_{\alpha,\beta} = \{z = x + y\sigma_{\alpha,\beta} \mid x, y \in \mathbb{R}\} \quad (2.2)$$

We will denote by $\text{real}(z)$ the real part of z given by

$$\text{real}(z) = x. \quad (2.3)$$

y is called the generalized part of the generalized number z .

There are many ways to choose the generalized unit number $\sigma_{\alpha,\beta}$. As simple example, we can take the real matrix

$$\sigma_{\alpha,\beta} = \begin{bmatrix} \alpha & 1 \\ \beta & 0 \end{bmatrix}. \quad (2.4)$$

Addition and multiplication of generalized numbers are defined for all $x_1 + y_1\sigma_{\alpha,\beta}$, $x_2 + y_2\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta}$ by

$$(x_1 + y_1\sigma_{\alpha,\beta}) + (x_2 + y_2\sigma_{\alpha,\beta}) = (x_1 + x_2) + (y_1 + y_2)\sigma_{\alpha,\beta}, \quad (2.5)$$

$$(x_1 + y_1\sigma_{\alpha,\beta}) \cdot (x_2 + y_2\sigma_{\alpha,\beta}) = (x_1x_2 + \alpha y_1y_2) + (x_1y_2 + x_2y_1 + \beta y_1y_2)\sigma_{\alpha,\beta}. \quad (2.6)$$

The power of $\sigma_{\alpha,\beta}$ can be computed as follows. Admit that there exists a sequences denoted a_n and b_n such that

$$\sigma_{\alpha,\beta}^n = a_n\sigma_{\alpha,\beta} + b_n. \quad (2.7)$$

This implies, making use relation (3.6), that the sequences are solution of the recursive Fibonacci equations

$$\begin{cases} a_0 = 0, a_1 = 1 \text{ and } a_{n+1} = \beta a_{n-1} + \alpha a_n, \\ b_0 = 1, b_1 = 0 \text{ and } b_{n+1} = \beta b_{n-1} + \alpha b_n. \end{cases} \quad (2.8)$$

Definition 1 1. We define the index of the set $\mathbb{C}_{\alpha,\beta}$ as the real number

$$\delta = \alpha^2 + 4\beta. \quad (2.9)$$

2. If $\delta < 0$ we say that $\mathbb{C}_{\alpha,\beta}$ is an elliptic set.
3. If $\delta = 0$ we say that $\mathbb{C}_{\alpha,\beta}$ is a parabolic set.
4. If $\delta > 0$ we say that $\mathbb{C}_{\alpha,\beta}$ is a hyperbolic set.

Complex, dual and hyperbolic conjugations play an important role both for algebraic and geometric properties of \mathbb{C} , \mathbb{D} and \mathbb{H} . For generalized numbers, we can also extend this notion. Let $z = x + y\sigma_{\alpha,\beta}$ a generalized number. Then we define the conjugate of z by the formula

$$\bar{z} = x' + y'\sigma_s, \quad (2.10)$$

where \bar{z} is characterized by the relation

$$z\bar{z} \in \mathbb{R}.$$

Under this definition we can verify that

$$\bar{z} = x + \alpha y - y\sigma_{\alpha,\beta}. \quad (2.11)$$

Thus, the inverse of z can be evaluated as follows

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} \\ &= \frac{x + \alpha y}{x^2 + \alpha xy - \beta y^2} - \frac{y}{x^2 + \alpha xy - \beta y^2} \sigma_{\alpha,\beta}. \end{aligned} \quad (2.12)$$

Generalized numbers form a commutative ring with characteristic 0. Moreover the inherited multiplication gives the generalized numbers the structure of 2-dimensional real Algebra.

The algebra $\mathbb{C}_{\alpha,\beta}$ is not a division algebra or field since if $x^2 + \alpha xy - \beta y^2 = 0$ the element $x + y\sigma_{\alpha,\beta}$ is not invertible. All elements of this form are zero divisors.

Hence, we can introduce the set $\mathcal{D}_{\alpha,\beta}$ of zero divisors of $\mathbb{C}_{\alpha,\beta}$, which can be called null part of $\mathbb{C}_{\alpha,\beta}$, as

$$\begin{aligned} \mathcal{D}_{\alpha,\beta} &= \{z = x + y\sigma_{\alpha,\beta} \mid x^2 + \alpha xy - \beta y^2 = 0\} \\ &= \{z = x + y\sigma_{\alpha,\beta} \mid (2x + \alpha y)^2 = \delta y^2\}. \end{aligned} \quad (2.13)$$

The following lemma gives some properties of the set $\mathcal{D}_{\alpha,\beta}$.

Lemma 1 1. The set $\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta}$ is a multiplicative abelian group.

2. The map

$$\begin{cases} \mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta} \longrightarrow \mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta}, \\ z \longmapsto \bar{z}, \end{cases} \quad (2.14)$$

is an automorphism of groups.

We define the positive part of $\mathbb{C}_{\alpha,\beta}$ as follows.

$$(\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_+ = \{z = x + y\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta} \mid x^2 + \alpha xy - \beta y^2 > 0\}. \quad (2.15)$$

Similarly, we can define the negative part of $\mathbb{C}_{\alpha,\beta}$ by

$$(\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_- = \{z = x + y\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta} \mid x^2 + \alpha xy - \beta y^2 < 0\}. \quad (2.16)$$

The lemma below immediately follows.

Lemma 2 1. The set $(\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_+$ is subgroup of $\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta}$.

2. If $\delta \leq 0$ then

$$(\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_+ = \mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta} \text{ and } (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_- = \emptyset. \quad (2.17)$$

3. If $z_1, z_2 \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_-$ then $z_1 z_2 \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_+$.

4. If $z_1 \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_{\pm}$ and $z_2 \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_{\mp}$ then $z_1 z_2 \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_-$.

5. If $z \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_{\pm}$ then $\bar{z} \in (\mathbb{C}_{\alpha,\beta} \setminus \mathcal{D}_{\alpha,\beta})_{\pm}$.

Elsewhere, we know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in \mathbb{R}^2 . The analogue of this, for generalized numbers, is the following. Let $z = x + y\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta}$, then

$$|z|^2 = |z\bar{z}| = |x^2 + \alpha xy - \beta y^2|. \quad (2.18)$$

We can also introduce another type of conjugation, called anti-conjugate. To this aim, for all $z = x + y\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta}$ the anti-conjugate is defined as follows

$$z^{\#} = y - x\sigma_{\alpha,\beta}. \quad (2.19)$$

Making use this definition, it is easy to check that if $\alpha \neq 0$ or $\beta \neq -1$ we can compute x and y by the following formulas

$$\begin{cases} x = \frac{1}{1+\beta+\alpha\sigma_{\alpha,\beta}} (z - \sigma_{\alpha,\beta} z^{\#}), \\ y = \frac{1}{1+\beta+\alpha\sigma_{\alpha,\beta}} (\sigma_{\alpha,\beta} z + z^{\#}). \end{cases} \quad (2.20)$$

And if $\alpha = 0$ and $\beta = -1$, corresponding to the case $\sigma_{0,-1} = i$, the following identity holds

$$z = iz^{\#}. \quad (2.21)$$

It is also important to know that every generalized number has another representation, using matrices.

Denoting by $\mathcal{G}_{\alpha,\beta}$ the subset of $M_2(\mathbb{R})$ given by

$$\mathcal{G}_{\alpha,\beta} = \left\{ A \in M_2(\mathbb{R}) \mid A = \begin{bmatrix} x & \beta y \\ y & x + \alpha y \end{bmatrix} \right\}. \quad (2.22)$$

Thus, $\mathcal{G}_{\alpha,\beta}$ is a subring of $M_2(\mathbb{R})$ which forms a 2-dimensional real associative and commutative Algebra.

Under the additional condition $x^2 + \alpha xy - \beta y^2 \neq 0$, $\mathcal{G}_{\alpha,\beta}$ becomes a subgroup of $GL(2)$.

Let us now define the map

$$\left\{ \begin{array}{l} \mathcal{N}_{\alpha,\beta} : \mathbb{C}_{\alpha,\beta} \longrightarrow \mathcal{G}_{\alpha,\beta}, \\ \mathcal{N}_{\alpha,\beta}(x + y\sigma_{\alpha,\beta}) = \begin{bmatrix} x & \beta y \\ y & x + \alpha y \end{bmatrix} \end{array} \right. \quad (2.23)$$

The following result allows us to making a correspondence between the two algebras $\mathbb{C}_{\alpha,\beta}$ and $\mathcal{G}_{\alpha,\beta}$ via the map $\mathcal{N}_{\alpha,\beta}$.

Theorem 3 \mathcal{N}_s is an isomorphism of algebras.

The proof is an immediate consequence of the definitions of $\mathbb{C}_{\alpha,\beta}$, $\mathcal{G}_{\alpha,\beta}$ and $\mathcal{N}_{\alpha,\beta}$.

Elsewhere, we also note that

$$z \in \mathcal{D}_{\alpha,\beta} \text{ iff } \det(\mathcal{N}_{\alpha,\beta}(z)) = 0. \quad (2.24)$$

From now on we denote by $|\cdot|_{\alpha,\beta}$ the map

$$\left\{ \begin{array}{l} |\cdot|_{\alpha,\beta} : \mathbb{C}_{\alpha,\beta} \longrightarrow \mathbb{R}_+, \\ |z|_{\alpha,\beta} = |z\bar{z}|^{\frac{1}{2}} = |x^2 + \alpha xy - \beta y^2|^{\frac{1}{2}}. \end{array} \right. \quad (2.25)$$

The following properties hold

$$\left\{ \begin{array}{l} |z|_{\alpha,\beta} = |\det(\mathcal{N}_{\alpha,\beta}(z))|^{\frac{1}{2}} \quad \forall z \in \mathbb{C}_{\alpha,\beta} \\ |z_1 + z_2|_{\alpha,\beta} \leq |z_1|_{\alpha,\beta} + |z_2|_{\alpha,\beta} \quad \forall z_1, z_2 \in \mathbb{C}_{\alpha,\beta}, \\ |z_1 z_2|_{\alpha,\beta} = |z_1|_{\alpha,\beta} |z_2|_{\alpha,\beta} \quad \forall z_1, z_2 \in \mathbb{C}_{\alpha,\beta}, \\ |\lambda z|_{\alpha,\beta} = |\lambda| |z|_{\alpha,\beta} \quad \forall z \in \mathbb{C}_{\alpha,\beta}, \quad \forall \lambda \in \mathbb{R}, \\ |z|_{\alpha,\beta} = 0 \text{ iff } z \in \mathcal{D}_{\alpha,\beta}. \end{array} \right. \quad (2.26)$$

It means in particular that $|\cdot|_{\alpha,\beta}$ defines a pseudo-modulus on $\mathbb{C}_{\alpha,\beta}$. It induces a structure of pseudo-topology over the algebra $\mathbb{C}_{\alpha,\beta}$.

So, we construct the generalized disk and generalized sphere of centre $z_0 = x_0 + y_0\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta}$ and radius $r > 0$, respectively, by

$$D_{\alpha,\beta}(z_0, r) = \left\{ z = x + y\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta} \mid |z - z_0|_{\alpha,\beta} < r \right\}, \quad (2.27)$$

$$S_{\alpha,\beta}(z_0, r) = \left\{ z = x + y\sigma_{\alpha,\beta} \in \mathbb{C}_{\alpha,\beta} \mid |z - z_0|_{\alpha,\beta} = r \right\}. \quad (2.28)$$

Notice that the algebra $\mathbb{C}_{\alpha,\beta}$ equipped with the previous pseudo-topology is not a Hausdorff space.

Otherwise, it is important to see, keeping in mind the definition of the unit generalized number $\sigma_{\alpha,\beta}$, that we can write $(2\sigma_{\alpha,\beta} - \alpha)^2 = \delta$. This permits us to think about creating a relationship between on one hand the algebra $\mathbb{C}_{\alpha,\beta}$ and on the other hand the algebras \mathbb{C} , \mathbb{D} and \mathbb{H} .

Theorem 4 1. If $\delta > 0$ then $\mathbb{C}_{\alpha,\beta}$ is isometrically isomorphic to \mathbb{H} .

2. If $\delta = 0$ then $\mathbb{C}_{\alpha,\beta}$ is isometrically isomorphic to \mathbb{D} .

3. If $\delta < 0$ then $\mathbb{C}_{\alpha,\beta}$ is isometrically isomorphic to \mathbb{C} .

This result may be seen as the main object of this section, which also allows us to give sense to definition 11.

For the three assertions the isomorphism can be easily obtained. So, to achieve the proof it is enough to verify the isometry. To do this, we proceed as follows.

Proof. For $\delta > 0$, $\delta = 0$ and $\delta < 0$, respectively, we consider the maps

$$\begin{cases} T_1 : \mathbb{C}_{\alpha,\beta} \longrightarrow \mathbb{H} \\ T_1(z) = x + \frac{\alpha}{2}y + \frac{\sqrt{\delta}}{2}yh \end{cases},$$

$$\begin{cases} T_2 : \mathbb{C}_{\alpha,\beta} \longrightarrow \mathbb{D} \\ T_2(z) = x + \frac{\alpha}{2}y + \frac{y}{2}\varepsilon \end{cases},$$

$$\begin{cases} T_3 : \mathbb{C}_{\alpha,\beta} \longrightarrow \mathbb{C} \\ T_3(z) = x + \frac{\alpha}{2}y + \frac{\sqrt{-\delta}}{2}yi \end{cases}.$$

Hence, if $x_1 + y_1\sigma_{\alpha,\beta}$, $x_2 + y_2\sigma_{\alpha,\beta}$ are two element of $\mathbb{C}_{\alpha,\beta}$, we can infer, for $\delta > 0$, $\delta = 0$ and $\delta < 0$, respectively, keeping in mind the definition of the pseudo-moduli of \mathbb{H} and \mathbb{D} as well as the modulus of \mathbb{C} , see [1, 12, 16, 20]

$$\begin{aligned} |T_2(z_1) - T_2(z_2)|_{\mathbb{H}} &= \left| \left(x_1 - x_2 + \frac{\alpha}{2}(y_1 - y_2) \right)^2 - \frac{\delta}{4}(y_1 - y_2)^2 \right|^{\frac{1}{2}} \\ &= \left| (x_1 - x_2)^2 + \alpha(x_1 - x_2)(y_1 - y_2) + \frac{\alpha^2 - \delta}{4}(y_1 - y_2)^2 \right|^{\frac{1}{2}} \\ &= |z_1 - z_2|_{\alpha,\beta}, \end{aligned}$$

$$\begin{aligned}
|T_3(z_1) - T_3(z_2)|_{\mathbb{D}} &= \left| \left(x_1 - x_2 + \frac{\alpha}{2} (y_1 - y_2) \right)^2 \right|^{\frac{1}{2}} \\
&= \left| (x_1 - x_2)^2 + \alpha (x_1 - x_2) (y_1 - y_2) + \frac{\alpha^2}{4} (y_1 - y_2)^2 \right|^{\frac{1}{2}} \\
&= |z_1 - z_2|_{\alpha, \beta},
\end{aligned}$$

and

$$\begin{aligned}
|T_1(z_1) - T_1(z_2)|_{\mathbb{C}} &= \left| \left(x_1 - x_2 + \frac{\alpha}{2} (y_1 - y_2) \right)^2 + \frac{-\delta}{4} (y_1 - y_2)^2 \right|^{\frac{1}{2}} \\
&= \left| (x_1 - x_2)^2 + \alpha (x_1 - x_2) (y_1 - y_2) + \frac{\alpha^2 - \delta}{4} (y_1 - y_2)^2 \right|^{\frac{1}{2}} \\
&= |z_1 - z_2|_{\alpha, \beta},
\end{aligned}$$

■

3 holomorphicity of Generalized Functions

We discuss now about some properties of generalized functions (functions of generalized variable). We investigate the continuity of generalized functions and the differentiability in the generalized sense, which can be also called, as in complex, hyperbolic and dual cases, holomorphicity. Due to the fact that $\mathbb{C}_{\alpha, \beta}$ is not a Hausdorff space we will suppose from now on that $\mathbb{C}_{\alpha, \beta}$ is equipped with the usual topology of \mathbb{R}^2 .

Definition 2 *A generalized function is a mapping from a subset $\Omega \subset \mathbb{C}_{\alpha, \beta}$ to $\mathbb{C}_{\alpha, \beta}$.*

Let Ω be an open subset of $\mathbb{C}_{\alpha, \beta}$, $z_0 = x_0 + y_0\sigma_{\alpha, \beta} \in \Omega$ and $f : \Omega \longrightarrow \mathbb{C}_{\alpha, \beta}$ a generalized function.

Definition 3 *We say that the generalized function f is continuous at $z_0 = x_0 + y_0\sigma_{\alpha, \beta}$ if*

$$\lim_{z \rightarrow z_0} f(z) = f(z_0). \quad (3.1)$$

where the limit is calculated coordinate by coordinate, this means that

$$\lim_{z \rightarrow z_0} f(z) = \lim_{x \rightarrow x_0, y \rightarrow y_0} f(x + y\sigma_{\alpha, \beta}) = f(x_0 + y_0\sigma_{\alpha, \beta}). \quad (3.2)$$

Definition 4 *The function f is continuous in $\Omega \subset \mathbb{C}_{\alpha, \beta}$ if it is continuous at every point of Ω .*

Definition 5 The generalized function f is said to be differentiable in the generalized sense at $z_0 = x_0 + y_0\sigma_{\alpha,\beta}$ if the following limit exists

$$\frac{df}{dz}(z_0) = \lim_{x \rightarrow x_0, y \rightarrow y_0} \frac{f(z) - f(z_0)}{z - z_0}. \quad (3.3)$$

$\frac{df}{dz}(z_0)$ is called the derivative of f at the point z_0 .

If f is differentiable for all points in a neighbourhood of the point z then f is called holomorphic at z .

Definition 6 The function f is holomorphic in $\Omega \subset \mathbb{C}_{\alpha,\beta}$ if it is holomorphic at every point of Ω .

In the following results we generalize the Cauchy-Riemann formulas to generalized functions.

Theorem 5 Let f be a generalized function in $\Omega \subset \mathbb{C}_{\alpha,\beta}$. Suppose that the partial derivatives of f are \mathcal{C}^0 . Then, f is holomorphic in $\Omega \subset \mathbb{C}_{\alpha,\beta}$ if and only if it satisfies

$$\mathcal{D}_{\alpha,\beta}(f) = 0, \quad (3.4)$$

whither $\mathcal{D}_{\alpha,\beta}$ is the differential operator

$$\mathcal{D}_{\alpha,\beta}(f) = -\sigma_{\alpha,\beta} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}. \quad (3.5)$$

Proof. Let $z_0 = x_0 + y_0\sigma_{\alpha,\beta}$ be an arbitrary element of Ω .

We have

$$\begin{aligned} \frac{\partial f}{\partial x}(z_0) &= \lim_{x \rightarrow x_0, y \rightarrow y_0} \frac{f(x + y\sigma_{\alpha,\beta}) - f(x_0 + y_0\sigma_{\alpha,\beta})}{x - x_0} \\ &\quad + \lim_{x \rightarrow x_0, y \rightarrow y_0} \frac{f(x + y_0\sigma_{\alpha,\beta}) - f(x + y\sigma_{\alpha,\beta})}{x - x_0} \\ &= \left(\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \sigma_{\alpha,\beta} - \lim_{x \rightarrow x_0, y \rightarrow y_0} \frac{f(x + y\sigma_{\alpha,\beta}) - f(x + y_0\sigma_{\alpha,\beta})}{y - y_0} \right) \times \\ &\quad \frac{y - y_0}{x - x_0} + \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}. \end{aligned}$$

So, the limit exists if and only if it does not depend on limit of the bounded ratio $\frac{y-y_0}{x-x_0}$. Hence, we should impose the condition

$$\frac{df}{dz}(z_0) \sigma_{\alpha,\beta} = \frac{\partial f}{\partial y}(z_0).$$

Thus, we can infer

$$\begin{cases} \frac{df}{dz}(z_0) = \frac{\partial f}{\partial x}(z_0), \\ -\sigma_{\alpha,\beta} \frac{\partial f}{\partial x}(z_0) + \frac{\partial f}{\partial y}(z_0) = 0. \end{cases} \quad (3.6)$$

This permits us to achieve the proof. ■

Corollary 6 *Let f be a generalized function in $\Omega \subset \mathbb{C}_{\alpha,\beta}$, which can be written in term of its real and generalized parts as $f = p + q\sigma_{\alpha,\beta}$ and suppose that the partial derivatives of f are \mathcal{C}^0 . Then, f is holomorphic in $\Omega \subset \mathbb{C}_{\alpha,\beta}$ if and only if its real and generalized parts satisfy the following generalized Cauchy-Riemann equations,*

$$\begin{cases} \frac{\partial p}{\partial x} = -\alpha \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}, \\ \frac{\partial p}{\partial y} = \beta \frac{\partial q}{\partial x}. \end{cases} \quad (3.7)$$

Proof. We know that the total differential of f is given by

$$df = \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} \sigma_{\alpha,\beta} \right) dx + \left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} \sigma_{\alpha,\beta} \right) dy. \quad (3.8)$$

On the other hand, the total differential of f can be also written, making use (3.6)

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dz \\ &= \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} \sigma_{\alpha,\beta} \right) dx + \left(\frac{\partial p}{\partial x} \sigma_{\alpha,\beta} + \alpha \frac{\partial q}{\partial x} \sigma_{\alpha,\beta} + \beta \frac{\partial q}{\partial x} \right) dy \end{aligned} \quad (3.9)$$

Combining equations (3.8) and (3.9), we find

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} \sigma_s = \left(\frac{\partial p}{\partial x} + \alpha \frac{\partial q}{\partial x} \right) \sigma_{\alpha,\beta} + \beta \frac{\partial q}{\partial x},$$

which leads to

$$\begin{cases} \frac{\partial p}{\partial x} = -\alpha \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}, \\ \frac{\partial p}{\partial y} = \beta \frac{\partial q}{\partial x}. \end{cases}$$

■

Moreover, as in complex analysis, the Cauchy-Riemann equations can be also reformulated using the partial derivative with respect to the anti-conjugate. For this, we can write, taking into account the formula (2.20)

$$\begin{cases} dx = \frac{1}{1+\beta+\alpha\sigma_{\alpha,\beta}} (dz - \sigma_s dz^\#), \\ dy = \frac{1}{1+\beta+\alpha\sigma_{\alpha,\beta}} (\sigma_s dz + dz^\#), \end{cases} \quad (3.10)$$

Let f be a generalized function in $\Omega \subset \mathbb{C}_{\alpha,\beta}$ where its partial derivatives are supposed to be \mathcal{C}^0 . Making use (3.10) the total differential of f , becomes

$$df = \frac{1}{1 + \beta + \alpha\sigma_{\alpha,\beta}} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \sigma_{\alpha,\beta} \right) dz + \frac{1}{1 + \beta + \alpha\sigma_{\alpha,\beta}} \left(-\sigma_{\alpha,\beta} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) dz^\#.$$
(3.11)

This suggest us to introduce the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z^\#}$ which act on the function f as follows

$$\begin{cases} \frac{\partial f}{\partial z} = \frac{1}{1+\beta+\alpha\sigma_{\alpha,\beta}} \frac{\partial f}{\partial x} + \frac{\sigma_{\alpha,\beta}}{1+\beta+\alpha\sigma_{\alpha,\beta}} \frac{\partial f}{\partial y}, \\ \frac{\partial f}{\partial z^\#} = -\frac{\sigma_{\alpha,\beta}}{1+\beta+\alpha\sigma_{\alpha,\beta}} \frac{\partial f}{\partial x} + \frac{1}{1+\beta+\alpha\sigma_{\alpha,\beta}} \frac{\partial f}{\partial y}. \end{cases}$$
(3.12)

These can be also written

$$\begin{cases} \frac{\partial f}{\partial z} = \frac{1+\beta+\alpha^2-\alpha\sigma_{\alpha,\beta}}{(1+\beta)^2+\alpha^2} \frac{\partial f}{\partial x} + \frac{-\alpha\beta+(1+\beta)\sigma_{\alpha,\beta}}{(1+\beta)^2+\alpha^2} \frac{\partial f}{\partial y}, \\ \frac{\partial f}{\partial z^\#} = \frac{\alpha\beta-(1+\beta)\sigma_{\alpha,\beta}}{(1+\beta)^2+\alpha^2} \frac{\partial f}{\partial x} + \frac{1+\beta+\alpha^2-\alpha\sigma_{\alpha,\beta}}{(1+\beta)^2+\alpha^2} \frac{\partial f}{\partial y}. \end{cases}$$
(3.13)

We deduce from formula (3.13) that the generalized function f is a holomorphic in $\Omega \subset \mathbb{C}_{\alpha,\beta}$ if and only if it satisfies the particular compact equation

$$\frac{\partial f}{\partial z^\#} = 0.$$
(3.14)

Moreover, again formula (3.13) permits us to deduce that the derivative of verifies the equation

$$\frac{df}{dz} = \frac{\partial f}{\partial z}.$$
(3.15)

It is well known from standard results of complex analysis that if f is holomorphic function then the real and complex parts of f are harmonic, this result can be also extended to generalized functions. To this aim, the lemma below is fulfilled.

Lemma 7 *Let f be a generalized function in $\Omega \subset \mathbb{C}_{\alpha,\beta}$, which can be written in term of its real and generalized parts as $f = p + q\sigma_{\alpha,\beta}$ and suppose that the partial derivatives of f are \mathcal{C}^1 . If f is holomorphic in $\Omega \subset \mathbb{C}_{\alpha,\beta}$, then the real functions p and q satisfy the partial differential equations*

$$\begin{cases} \alpha \frac{\partial^2 p}{\partial x^2} + \beta \frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial y^2} = 0, \\ \alpha \frac{\partial^2 q}{\partial x^2} + \beta \frac{\partial^2 q}{\partial x \partial y} - \frac{\partial^2 q}{\partial y^2} = 0. \end{cases}$$
(3.16)

The proof is an immediate consequence of the Cauchy-Riemann formulas (3.7).

Furthermore, we remark that:

1. If $\delta > 0$ then equations are hyperbolic.
2. If $\delta = 0$ then equations are parabolic.
3. If $\delta < 0$ then equations are elliptic.

These permit us to give another justification to definition 1.

4 Open Problem

It is well-known that hyperbolic numbers play an important role in theoretical physical, since them model the transformations in Minkowski spaces. Also, quantum mechanics is direct application of complex analysis (Schrödinger equation). Moreover, dual numbers find applications in theoretical physics, where they constitute one of the simplest non-trivial examples of a superspace.

Our interest is to find physical applications of generalized numbers in unified format.

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