Int. J. Open Problems Complex Analysis, Vol. 7, No. 1, March 2015
ISSN 2074-2827; Copyright ©ICSRS Publication, 2015
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General Class of $p-$Valent Functions of Higher Order

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Received 9 July 2014; Accepted 9 November 2014

Abstract

In this paper, we define the general class $TVn(g; p; q; \alpha; \gamma)$ ***of p-valent functions of higher and derive distortion theorems and modified Hadamard products for functions belonging to this class.***

Keywords: *Analytic functions, p-valent functions, distortion theorems, Hadamard product.*
2000 Mathematical Subject Classification: 30C45.

1 Introduction

Let$T_p(n)$ be the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \ (a_k \geq 0, p \in \mathbb{N}, \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in T_p(n)$ is said to be p-valently starlike of order α if it satisfies the inequality:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha(z \in \mathbb{U}, 0 \leq \alpha < p, p \in \mathbb{N}). \tag{1.2}$$

We denote by $S_p^*(n,\alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T_p(n)$ is said to be p-valently convex of order α if it satisfies the inequality:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha(z \in \mathbb{U}, 0 \le \alpha < p, p \in \mathbb{N}). \tag{1.3}$$

We denote by $C_p(n,\alpha)$ the class of all p-valently convex functions of order α.We note that (see for example Duren [6] and Goodman [8]:

$$f(z) \in C_p(n,\alpha) \Leftrightarrow \frac{zf'(z)}{p} S_p^*(n,\alpha).$$

The classes $S_p^*(n,\alpha)$ and $C_p(n,\alpha)$ were studied by Owa [10].

Let $(f*g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ that is $f(z)$ is given by (1.1) and $g(z)$ is given by:

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k (b_k \ge 0). \tag{1.4}$$

Then

$$(f*g)(z) = z^p - \sum_{k=p+n}^{\infty} a_k b_k z^k = (g*f)(z). \tag{1.5}$$

For functions $f(z)$ defined by (1.1) and $g(z)$ defined by (1.4), Aouf and Mostafa [2] defined the classes $TS_n^*(g,p,q,\alpha)$ and $TC_n(g,p,q,\alpha)$ as follows:

Definition 1 [2]. Let $g(z)$ with $(b_k > 0)$ be defined by (1.4). The function $f(z)$ of the form (1.1) is said to be in the class $TS_n^*(g,p,q,\alpha)$ if and only if

$$Re\left(\frac{z\left((f*g)(z)\right)^{(q+1)}}{\left((f*g)(z)\right)^{(q)}}\right) > \alpha \quad (0 \le \alpha < p-q, p \in \mathbb{N}, q \in \mathbb{N}_0; p > q), \tag{1.6}$$

and is in the class $TC_n(g,p,q,\alpha)$ if and only if

$$Re\left(1+\frac{z\left((f*g)(z)\right)^{(q+2)}}{\left((f*g)(z)\right)^{(q+1)}}\right) > \alpha \quad (0 \le \alpha < p-q, p \in \mathbb{N}, q \in \mathbb{N}_0; p > q), \tag{1.7}$$

where

$$f^{(q)}(z) = \delta(p,q) z^{p-q} - \sum_{k=p+n}^{\infty} \delta(k,q) a_k z^{k-q}, \tag{1.8}$$

and

$$\delta(i,h) = \frac{i!}{(i-h)!} = \begin{cases} i(i-1)\dots(i-h+1) & (h \ne 0) \\ 1 & (h=0). \end{cases} \tag{1.9}$$

It follows from (1.6) and (1.7) that

$$f^{(q)}(z)\in TC_n\left(g,p,q,\alpha\right)\Longleftrightarrow\frac{zf^{(q+1)}(z)}{p-q}\in TS_n^*\left(g,p,q,\alpha\right). \tag{1.10}$$

Several well-known subclasses of functions are special cases of the classes $TS_n^*\left(g,p,q,\alpha\right)$ and $TC_n\left(g,p,q,\alpha\right)$ for suitable choices of $g\left(z\right),q$ and n. For example:
(i) for $q=0$ and replacing $n+p$ by m, we have, $TS_n^*\left(g,p,0,\alpha\right)=TC_n\left(g,m,\alpha\right)$ (Ali et al.[1]);

$(ii)\, TS_n^*\left(\frac{z^p}{(1-z)},p,0,\alpha\right)=\begin{cases} T_n^*\left(p,\alpha\right) & \text{(Owa [10])} \\ T_\alpha\left(p,n\right) & \text{(Yamakawa [15]);} \end{cases}$

$(iii)\, TC_n\left(\frac{z^p}{(1-z)},p,0,\alpha\right)=\begin{cases} C_n\left(p,\alpha\right) & \text{(Owa [10])} \\ CT_\alpha\left(p,n\right) & \text{(Yamakawa [15]);} \end{cases}$

We also have the following new classes:
(i) for $b_k=(\frac{k}{p})^\sigma$, we get the class:

$$TS_n^*\left(p,q,\alpha,\sigma\right)=\left\{f\left(z\right)\in T_p\left(n\right):Re\left(\frac{z\left(D_p^\sigma f\left(z\right)\right)^{(q+1)}}{\left(D_p^\sigma f\left(z\right)\right)^{(q)}}\right)>\alpha\right\},$$

where $0\leq\alpha<p-q,p\in\mathbb{N},q,\sigma\in\mathbb{N}_0,p>q$ and the operator D_p^σ was introduced and studied by Kamali and Orhan [9] and Aouf and Mostafa [3];
(ii) for $b_k=(\frac{p+\ell+\lambda(k-p)}{\ell+p})^m$, we get the class:

$$TS_n^*\left(p,q,\alpha,\lambda,\ell,m\right)=\left\{f\left(z\right)\in T_p\left(n\right):Re\left(\frac{z\left(I_p^m\left(\lambda,\ell\right)f\left(z\right)\right)^{(q+1)}}{\left(I_p^m\left(\lambda,\ell\right)f\left(z\right)\right)^{(q)}}\right)>\alpha\right\},$$

where $0\leq\alpha<p-q,p\in\mathbb{N},q,m\in\mathbb{N}_0,p>q,\ell\geq0,\lambda\geq0$ and the operator $I_p^m\left(\lambda,\ell\right)$ was introduced and studied by Catas [4];
(iii) for $b_k=\binom{\eta+k-1}{k-p}$ $(\eta>-p)$ we get the class:

$$TS_n^*\left(p,q,\alpha,\eta\right)=\left\{f\left(z\right)\in T_p\left(n\right):Re\left(\frac{z\left(\Omega^{p,\eta}f\left(z\right)\right)^{(q+1)}}{\left(\Omega^{p,\eta}f\left(z\right)\right)^{(q)}}\right)>\alpha\right\},$$

where $0\leq\alpha<p-q,p\in\mathbb{N},q\in\mathbb{N}_0;\eta>-p;p>q$ and the operator $\Omega^{p,\eta}$ is the extended Ruscheweyh derivative of order η which was investigated by Raina and Srivastava [11];
(iv) for $b_k=\frac{(\alpha_1)_{k-p}(\alpha_2)_{k-p}...(\alpha_r)_{k-p}}{(\beta_1)_{k-p}(\beta_2)_{k-p}...(\beta_s)_{k-p}(k-p)!}\geq0(\alpha_j\in\mathbb{C},j=1,2,...r;\beta_i\in\mathbb{C}\setminus\{0,-1,-2,...\};\ i=1,2,...s)$ we get the class:

$$TS_n^*\left(p,q,\alpha\right)=\left\{f\left(z\right)\in T_p\left(n\right):Re\left(\frac{z\left(H_{r,s}\left(\alpha_1\right)f\right)^{(q+1)}\left(z\right)}{\left(H_{r,s}\left(\alpha_1\right)f\right)^{(q)}\left(z\right)}\right)>\alpha\right\},$$

$$\left(0\leq\alpha<p-q;z\in U;r\leq s+1;q,r,s\in\mathbb{N}_0;p\in\mathbb{N};p>q\right),$$

the operator $H_{r,s}(\alpha_1)$ is the Dziok-Srivastava operator (see for details [7]).

2. General Classes Associated with Coefficient Bounds

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \alpha < p - q, n, p \in \mathbb{N}, p \in \mathbb{N}, q \in \mathbb{N}_0, p > q, \gamma \geq 0$ and $\delta(i,h)$ *is given by* (1.9).

In order to prove our results for functions belonging to the class $TV_n(g,p,q,\alpha,\gamma)$, we shall need the following lemmas given by Aouf and Mostafa [2].

Lemma 1[2, Theorem 1]. *Let a function $f(z)$ be in the class $T_p(n)$. Then $f \in TS_n^*(g,p,q,\alpha)$ if and only if*

$$\sum_{k=p+n}^{\infty} (k-q-\alpha)\,\delta(k,q)\,b_k a_k \leq \delta(p,q)(p-q-\alpha). \tag{2.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\delta(p,q)(p-q-\alpha)}{\delta(n+p,q)(n+p-q-\alpha)\,b_{n+p}} z^{n+p}. \tag{2.2}$$

Lemma 1[2, Theorem 2]**.** *Let a function $f(z)$ be in the class $T_p(n)$. Then $f(z) \in TC_n(g,p,q,\alpha)$ if and only if*

$$\sum_{k=p+n}^{\infty} \frac{(k-q)}{(p-q)}(k-q-\alpha)\,\delta(k,q)\,b_k a_k \leq \delta(p,q)(p-q-\alpha). \tag{2.3}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\delta(p,q)(p-q-\alpha)(p-q)}{\delta(n+p,q)(n+p-q-\alpha)(n+p-q)\,b_k} z^{n+p}. \tag{2.4}$$

Definition 2. *A function $f(z)$ defined by* (1.1) *is said to be in the class $TV_n(g,p,q,\alpha,\gamma)$ $(\gamma \geq 0)$ if it satisfies the coefficient inequality:*

$$\sum_{k=p+n}^{\infty} \left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k a_k \leq \delta(p,q)(p-q-\alpha). \tag{2.5}$$

It is easily to observe that

$$TV_n(g,p,q,\alpha,0) = TS_n^*(g,p,q,\alpha) \text{ and } TV_n(g,p,q,\alpha,1) = TC_n(g,p,q,\alpha). \tag{2.6}$$

3. Growth and distortion theorems

Theorem 1. *If a function $f(z)$ defined by* (1.1) *is in the class* $TV_n(g,p,q,\alpha,\gamma)$, *then*

$$\left(\delta(p,j)-\frac{\delta(p,q)(n+p-q)!(p-q-\alpha)}{(n+p-j)!(n+p-q-\alpha)\left(1-\gamma+\gamma\frac{n+p-q}{p-q}\right)b_{n+p}}|z|^n\right)|z|^{p-j}\le\left|f^{(j)}(z)\right|$$

$$\le\left(\delta(p,j)+\frac{\delta(p,q)(n+p-q)!(p-q-\alpha)}{(n+p-j)!(n+p-q-\alpha)\left(1-\gamma+\gamma\frac{n+p-q}{p-q}\right)b_{n+p}}|z|^n\right)|z|^{p-j}. \tag{3.1}$$

The result is sharp for the function $f(z)$*given by*

$$f(z)=z^p-\frac{\delta(p,q)(p-q-\alpha)}{\delta(n+p,q)(n+p-q-\alpha)\left(1-\gamma+\gamma\frac{n+p-q}{p-q}\right)b_{n+p}}z^{n+p}. \tag{3.2}$$

Proof. In view of Definition 1, we have

$$\frac{\left(1-\gamma+\gamma\frac{n+p-q}{p-q}\right)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}{\delta(p,q)(p-q-\alpha)(n+p)!}\sum_{k=p+n}^{\infty}k!a_k$$
$$\le\sum_{k=p+n}^{\infty}\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\delta(k,q)b_k}{\delta(p,q)(p-q-\alpha)}a_k\le 1,$$

which readily yields

$$\sum_{k=p+n}^{\infty}k!a_k\le\frac{\delta(p,q)(n+p-q)!(p-q-\alpha)}{(n+p-q-\alpha)\left(1-\gamma+\gamma\frac{n+p-q}{p-q}\right)b_{n+p}}. \tag{3.3}$$

Now, by differentiating both sides of (1.1) j times, we have

$$f^{(j)}(z)=\frac{p!}{(p-j)!}z^{p-j}-\sum_{k=p+n}^{\infty}\frac{k!}{(k-j)!}a_kz^{k-j}$$
$$(k\ge n+p;p,n\in\mathbb{N};q,j\in\mathbb{N}_0;p>\max\{q,j\}). \tag{3.4}$$

Theorem 1 would follow from (3.3) and (3.4) , respectively.
Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$given by (3.2) .
Remark 1. (i) Putting $\gamma=0$ and $b_k=1$ in Theorem 1 , we obtain the result obtained by Chen et al. [5, Theorem 7] ;
(ii) Putting $\gamma=1$ and $b_k=1$ in Theorem 1, we obtain the result obtained by Chen et al. [5, Theorem 8] .

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $TV_n(g,p,q,\alpha,\gamma)$, *then*
(i) $f(z)$ *is* $p-$*valently close-to-convex of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_{k\geq n+p}\left\{\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,p)\,b_k}{\delta(p,q)(p-q-\alpha)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}}, \tag{4.1}$$

(ii) $f(z)$ *is* $p-$*valently starlike of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_2$, *where*

$$r_2=\inf_{k\geq n+p}\left\{\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,p)\,b_k}{\delta(p,q)(p-q-\alpha)}\left(\frac{p-\varphi}{k-\varphi}\right)\right\}^{\frac{1}{k-p}}, \tag{4.2}$$

(iii) $f(z)$ *is* $p-$*valently convex of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_3$, *where*

$$r_3=\inf_{k\geq n+p}\left\{\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,p)\,b_k}{\delta(p,q)(p-q-\alpha)}.\frac{p(p-\varphi)}{k(k-\varphi)}\right\}^{\frac{1}{k-p}}. \tag{4.3}$$

Each of these results is sharp for the function $f(z)$ *given by* (3.2).
Proof. We prove (i). It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\varphi\quad(|z|<r_1;0\leq\varphi<p), \tag{4.4}$$

where r_1 is given by (4.1), Indeed we find, again from (1.1) that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq\sum_{k=p+n}^{\infty}ka_k\,|z|^{k-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\varphi$$

if

$$\sum_{k=p+n}^{\infty}\frac{k}{(p-\varphi)}a_k\,|z|^{k-p}\leq 1. \tag{4.5}$$

But, by (2.5), (4.5) will be true if

$$\frac{k}{(p-\varphi)}\left|z\right|^{k-p}\leq\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)},$$

that is, if

$$r_1=\inf_{k\geq p+n}\left\{\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}},$$

the proof of (i) is completed. The proof of (ii) and (iii) is similar to (i) and will be omitted.

Putting $\gamma=0$ in Theorem 2, we obtain the following corollary.
Corollary 1. *Let the function* $f\,(z)$ *defined by* (1.1) *be in the class* $TS_n^*\,(g,p,q,\alpha)$, *then*
(i) $f\,(z)$ *is* $p-$*valently close-to-convex of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_{k\geq p+n}\left\{\frac{(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}},$$

(ii) $f\,(z)$ *is* $p-$*valently starlike of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_2$, *where*

$$r_2=\inf_{k\geq p+n}\left\{\frac{(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)}\left(\frac{p-\varphi}{k-\varphi}\right)\right\}^{\frac{1}{k-p}},$$

(iii) $f\,(z)$ *is* $p-$*valently convex of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_3$, *where*

$$r_3=\inf_{k\geq p+n}\left\{\frac{(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)}.\frac{p\,(p-\varphi)}{k\,(k-\varphi)}\right\}^{\frac{1}{k-p}}.$$

Each of these results is sharp for the function $f\,(z)$ *given by* (2.2).

Putting $\gamma=1$ in Theorem 2, we obtain the following corollary.
Corollary 2. *Let the function* $f\,(z)$ *defined by* (1.1) *be in the class* $TC_n\,(g,p,q,\alpha)$, *then*
(i) $f\,(z)$ *is* $p-$*valently close-to-convex of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_{k\geq p+n}\left\{\frac{\frac{k-q}{p-q}\,(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}},$$

(ii) $f\,(z)$*is* $p-$*valently starlike of order* $\varphi\,(0\leq\varphi<p)$ *in* $|z|<r_2$, *where*

$$r_2=\inf_{k\geq p+n}\left\{\frac{\frac{k-q}{p-q}\,(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)}\left(\frac{p-\varphi}{k-\varphi}\right)\right\}^{\frac{1}{k-p}},$$

(iii) $f(z)$ *is* $p-$*valently convex of order* $\varphi\,(0 \leq \varphi < p)$ *in* $|z| < r_3$, *where*

$$r_3 = \inf_{k \geq p+n} \left\{ \frac{\frac{k-q}{p-q}\,(k-q-\alpha)\,\delta\,(k,p)\,b_k}{\delta\,(p,q)\,(p-q-\alpha)} . \frac{p\,(p-\varphi)}{k\,(k-\varphi)} \right\}^{\frac{1}{k-p}} .$$

Each of these results is sharp for the function $f(z)$ *given by* (2.4).

5. Closure theorems

Let the functions $f_v(z)\,(v = 1, 2, ...\imath)$ be defined by

$$f_\nu(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,\nu} z^k (a_{k,\nu} \geq 0; \nu = 1, 2, ...l). \tag{5.1}$$

We shall prove the following results for the closure functions in the class $TV_n\,(g, p, q, \alpha, \gamma)$.

Theorem 3. *Let the functions* $f_v(z)\,(v = 1, 2, ...\imath)$ *defined by* (5.1) *be in the class* $TV_n\,(g, p, q, \alpha, \gamma)$. *Then the function* $h(z)$ *defined by*

$$h(z) = \sum_{v=1}^{\imath} c_v f_v(z) \qquad (c_v \geq 0), \tag{5.2}$$

is also in the class $TV_n\,(g, p, q, \alpha, \gamma)$, *where*

$$\sum_{v=1}^{\imath} c_v = 1.$$

Proof. According to the definition of $h(z)$, it can be written as

$$\begin{aligned} h(z) &= \sum_{v=1}^{\imath} c_v \left[z^p - \sum_{k=p+n}^{\infty} a_{k,v} z^k \right] \\ &= \sum_{v=1}^{\imath} c_v z^p - \sum_{v=1}^{\imath} \sum_{k=p+n}^{\infty} c_v a_{k,v} z^k \\ &= z^p - \sum_{k=p+n}^{\infty} \sum_{v=1}^{\imath} c_v a_{k,v} z^k. \end{aligned} \tag{5.3}$$

Furthermore, since the functions $f_v(z)\,(v = 1, 2, ...\imath)$ are in the class $TV_n\,(g, p, q, \alpha, \gamma)$, then

$$\sum_{k=p+n}^{\infty} \left(1 - \gamma + \gamma \frac{k-q}{p-q}\right) (k-q-\alpha)\,\delta\,(k,q)\,b_k a_{k,v} \leq \delta\,(p,q)\,(p-q-\alpha).$$

Hence

$$\sum_{k=p+n}^{\infty}\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k\sum_{v=1}^{\imath}c_v a_{k,v}$$
$$=\sum_{v=1}^{\imath}c_v\left\{\sum_{k=p+n}^{\infty}\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k a_{k,v}\right\}\leq\delta(p,q)(p-q-\alpha),$$

which implies that $h(z)$ be in the class $TV_n(g,p,q,\alpha,\gamma)$.
Theorem 4. *Let* $f_p(z)=z^p$ *and*

$$f_k(z)=z^p-\frac{\delta(p,q)(p-q-\alpha)}{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}z^k\ (k\geq n+p). \quad (5.4)$$

Then the function $f(z)$ *is in the clsss* $TV_n(g,p,q,\alpha,\gamma)$ *if and only if it can be expressed in the form:*

$$f(z)=\lambda_p z^p+\sum_{k=p+n}^{\infty}\lambda_k f_k(z), \quad (5.5)$$

where $(\lambda_p\geq 0,\lambda_k\geq 0,k\geq n+p)$ and $\lambda_p+\sum\limits_{k=p+n}^{\infty}\lambda_k=1.$
Proof. Suppose that $f(z)$ is expressed in the form (5.5). Then

$$f(z)=\lambda_p z^p+\sum_{k=p+n}^{\infty}\lambda_k\left[z^p-\frac{\delta(p,q)(p-q-\alpha)}{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}z^k\right]$$
$$=z^p-\sum_{k=p+n}^{\infty}\frac{\delta(p,q)(p-q-\alpha)}{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}\lambda_k z^k\ .$$

Hence

$$\sum_{k=p+n}^{\infty}\frac{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}{\delta(p,q)(p-q-\alpha)}.\frac{\delta(p,q)(p-q-\alpha)}{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}\lambda_k$$
$$=\sum_{k=p+n}^{\infty}\lambda_k=1-\lambda_p\leq 1.$$

Then, $f(z)\,TV_n(g,p,q,\alpha,\gamma)$.
Conversely, suppose that $f(z)\in TV_n(g,p,q,\alpha,\gamma)$. We may set

$$\lambda_k=\frac{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}{\delta(p,q)(p-q-\alpha)}a_k\quad(k\geq p+n),$$

where a_k is given by (2.5). Then

$$\begin{aligned} f(z) &= z^p - \sum_{k=p+n}^{\infty} a_k z^k \\ &= z^p - \sum_{k=p+n}^{\infty} \frac{\delta(p,q)(p-q-\alpha)}{\delta(k,q)(k-q-\alpha)\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)b_k}\lambda_k z^k \\ &= z^p - \sum_{k=p+n}^{\infty} [z^p - f_k(z)]\lambda_k \\ &= (1-\lambda_p)z^p - \sum_{k=p+n}^{\infty} \lambda_k f_k(z) \\ &= \lambda_p z^p + \sum_{k=p+n}^{\infty} \lambda_k f_k(z) = f(z). \end{aligned}$$

This completes the proof of Theorem 4.

6. Modified Hadamard product

For the functions $f_v(z)\,(v=1,2)$ *defined by* (5.1), *we denote by* $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions f_1 and f_2

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1}a_{k,2}z^k. \tag{6.1}$$

Theorem 5. *Let each of the functions* $f_v(z)\,(v=1,2)$ *defined by* (5.1) *be in the class* $TV_n(g,p,q,\alpha,\gamma)$. *Then* $(f_1 * f_2)(z) \in TV_n(g,p,q,\beta,\gamma)$, *where*

$$\beta = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(1-\gamma+\gamma\frac{n+p-q}{p-q})(n+p-q-\alpha)^2\delta(n+p,q)b_{n+p} - (p-q-\alpha)^2\delta(p,q)}. \tag{6.2}$$

The result is sharp for the functions $f_v(z)$ $(v=1,2)$ *given by*

$$f_v(z) = z^p - \frac{\delta(p,q)(p-q-\alpha)}{\delta(n+p,q)(n+p-q-\alpha)\left(1-\gamma+\gamma\frac{n+p-q}{p-q}\right)b_{n+p}}z^{n+p} \quad (v=1,2). \tag{6.3}$$

Proof. Emloying the technique used earlier by Schild and Silverman [12], we need to find the largest β such that

$$\sum_{k=p+n}^{\infty} \frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\beta)\delta(k,q)b_k}{\delta(p,q)(p-q-\beta)}a_{k,1}a_{k,2} \le 1. \tag{6.4}$$

Since $f_v(z) \in TV_n(p,q,\beta,\gamma)$ $(v=1,2)$, we readily see that

$$\sum_{k=p+n}^{\infty} \frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k}{\delta(p,q)\,(p-q-\alpha)} a_{k,v} \leq 1 \quad (v=1,2). \tag{6.5}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p+n}^{\infty} \frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k}{\delta(p,q)\,(p-q-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{6.6}$$

Thus we only need to show that

$$\frac{(k-q-\beta)}{(p-q-\beta)} a_{k,1}a_{k,2} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)} \sqrt{a_{k,1}a_{k,2}}, \tag{6.7}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-q-\alpha)\,(p-q-\beta)}{(p-q-\alpha)\,(k-q-\beta)}. \tag{6.8}$$

Hence, by the inequality (6.6), it is sufficient to prove that

$$\frac{\delta(p,q)\,(p-q-\alpha)}{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k} \leq \frac{(k-q-\alpha)\,(p-q-\beta)}{(p-q-\alpha)\,(k-q-\beta)}. \tag{6.9}$$

It follows from (6.9) that

$$\beta \leq (p-q) - \frac{(k-p)\,(p-q-\alpha)^2\,\delta(p,q)}{(1-\gamma+\gamma\frac{k-q}{p-q})(k-q-\alpha)^2\delta(k,q)\,b_k - (p-q-\alpha)^2\,\delta(p,q)}. \tag{6.10}$$

Defining the function $\Phi(k)$ by

$$\Phi(k) = (p-q) - \frac{(k-p)\,(p-q-\alpha)^2\,\delta(p,q)}{(1-\gamma+\gamma\frac{k-q}{p-q})(k-q-\alpha)^2\delta(k,q)\,b_k - (p-q-\alpha)^2\,\delta(p,q)}, \tag{6.11}$$

we see that $\Phi(k)$ is an increasing function of $k\,(k \geq n+p)$. Therefore, we conclude from (6.10) that

$$\begin{aligned}\beta \leq \Phi(n+p) &= (p-q)\\ &- \frac{n\,(p-q-\alpha)^2\,\delta(p,q)}{(1-\gamma+\gamma\frac{n+p-q}{p-q})(n+p-q-\alpha)^2\delta(n+p,q)\,b_{n+p} - (p-q-\alpha)^2\,\delta(p,q)},\end{aligned} \tag{6.12}$$

which completes the proof of the main assertion of Theorem 5.

Putting $\gamma = 0$ and $g(z) = \frac{z^p}{(1-z)}$ $(p \in \mathbb{N})$ in Theorem 5, we obtain the following corollary.

Corollary 3. *Let the functions* $f_v(z)\,(v = 1,2)$ *defined by* (5.1) *be in the class* $TS_n^*(p,q,\alpha)$. *Then* $(f_1 * f_2)(z) \in TS_n^*(p,q,\beta)$, *where*

$$\beta = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q) - (p-q-\alpha)^2\delta(p,q)}. \tag{6.13}$$

The result is sharp.

Remark 2. We note that the result obtained by Chen et al. [5, Theorem 5] is not correct. The correct result is given by (6.13).

Putting $\gamma = 1$ and $g(z) = \frac{z^p}{(1-z)}$ $(p \in \mathbb{N})$ in Theorem 5, we obtain the following corollary.

Corollary 4. *Let the functions* $f_v(z)\,(v = 1,2)$ *defined by* (5.1) *be in the class* $TC_n(p,q,\alpha)$. *Then* $(f_1 * f_2)(z) \in TC_n(p,q,\beta)$, *where*

$$\beta = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{\left(\frac{n+p-q}{p-q}\right)(n+p-q-\alpha)^2\delta(n+p,q) - (p-q-\alpha)^2\delta(p,q)}. \tag{6.14}$$

The result is sharp

Remark 3. We note that the result obtained by Chen et al. [5, Theorem 6] is not correct. The correct result is given by (6.14).

Theorem 6. *Let the functions* $f_v(z)\,(v = 1,2)$ *defined by* (5.1) *be in the class* $TV_n(g,p,q,\alpha,\gamma)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z^p - \sum_{k=p+n}^{\infty}\left(a_{k,1}^2 + a_{k,2}^2\right)z^k, \tag{6.15}$$

belongs to the class $TV_n(g,p,q,\xi,\gamma)$, *where*

$$\xi = (p-q) - \frac{2n(p-q-\alpha)^2\delta(p,q)}{(1-\gamma+\gamma\frac{n+p-q}{p-q})(n+p-q-\alpha)^2\delta(n+p,q)\,b_{n+p} - 2(p-q-\alpha)^2\delta(p,q)}. \tag{6.16}$$

The result is sharp for the functions $f_v(z)\,(v = 1,2)$ *given by* (6.3).

Proof. Noting that

$$\sum_{k=p+n}^{\infty}\left[\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k}{\delta(p,q)\,(p-q-\alpha)}\right]^2 a_{k,v}^2$$
$$\leq\left[\sum_{k=p+n}^{\infty}\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k}{\delta(p,q)\,(p-q-\alpha)}a_{k,v}\right]^2\leq 1, \tag{6.17}$$

for $f_v(z)\in TV_n(g,p,q,\alpha,\gamma)\ \ (v=1,2)$, then we have

$$\sum_{k=p+n}^{\infty}\frac{1}{2}\left[\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)\,\delta(k,q)\,b_k}{\delta(p,q)\,(p-q-\alpha)}\right]^2\left(a_{k,1}^2+a_{k,2}^2\right)\leq 1. \tag{6.18}$$

Thus we need to find the largest ξ such that

$$\frac{(k-q-\xi)}{(p-q-\xi)}\leq\frac{1}{2}\frac{\left(1-\gamma+\gamma\frac{k-q}{p-q}\right)(k-q-\alpha)^2\,\delta(k,q)\,b_k}{\delta(p,q)\,(p-q-\alpha)^2}, \tag{6.19}$$

that is, that

$$\xi\leq(p-q)-\frac{2\,(k-p)\,(p-q-\alpha)^2\,\delta(p,q)}{(1-\gamma+\gamma\frac{k-q}{p-q})(k-q-\alpha)^2\delta(k,q)\,b_k-2\,(p-q-\alpha)^2\,\delta(p,q)}. \tag{6.20}$$

Defining the function $\Theta(k)$ by

$$\Theta(k)=(p-q)-\frac{2\,(k-p)\,(p-q-\alpha)^2\,\delta(p,q)}{(1-\gamma+\gamma\frac{k-q}{p-q})(k-q-\alpha)^2\delta(k,q)\,b_k-2\,(p-q-\alpha)^2\,\delta(p,q)}, \tag{6.21}$$

we observe that $\Theta(k)$ is an increasing function of $k\ (k\geq n+p)$. Therefore, we conclude from (6.20) that

$$\xi\leq\Theta(n+p)=(p-q)$$
$$-\frac{2n\,(p-q-\alpha)^2\,\delta(p,q)}{(1-\gamma+\gamma\frac{n+p-q}{p-q})(n+p-q-\alpha)^2\delta(n+p,q)\,b_{n+p}-2\,(p-q-\alpha)^2\,\delta(p,q)}, \tag{6.22}$$

which completes the proof of Theorem 6.

Putting $\gamma=0$, and $g(z)=\frac{z^p}{(1-z)}$ $(p\in\mathbb{N})$ in Theorem 6, we obtain the following corollary

Corollary 5. *Let the functions* $f_v(z)\,(v=1,2)$ *defined by* (5.1) *be in the class* $TS_n^*(p,q,\alpha)$. *Then the function* $h(z)$ *defined by* (6.16) *belongs to the class* $TS_n^*(p,q,\sigma)$, *where*

$$\sigma=(p-q)-\frac{2n\,(p-q-\alpha)^2\,\delta\,(p,q)}{(n+p-q-\alpha)^2\delta\,(n+p,q)-2\,(p-q-\alpha)^2\,\delta\,(p,q)}.$$

Putting $\gamma=1$ and $g(z)=\frac{z^p}{(1-z)}$ $(p\in\mathbb{N})$ in Theorem 6, we obtain the following corollary.

Corollary 6. *Let the functions* $f_v(z)\,(v=1,2)$ *defined by* (5.1) *be in the class* $TC_n(p,q,\alpha)$. *Then the function* $h(z)$ *defined by* (6.15) *belongs to the class* $TC_n(p,q,\rho)$, *where*

$$\rho=(p-q)-\frac{2n\,(p-q-\alpha)^2\,\delta\,(p,q)}{(\frac{n+p-q}{p-q})(n+p-q-\alpha)^2\delta\,(n+p,q)-2\,(p-q-\alpha)^2\,\delta\,(p,q)}.$$

Remarks 4. (i) Putting $\gamma=0$ in Theorems 1, 3, 4, 5 and 6, respectively, we obtain the results obtained by Aouf and Mostafa [2, Theorems 3, 5, 6, 8 and 10, respectively];

(ii) Putting $\gamma=1$ in Theorems 1, 4, 5 and 6, respectively, we obtain the results obtained by Aouf and Mostafa [2, Theorems 4, 7, 9 and 11, respectively].

7. Open problem

The authors suggest to study:

$$\sum_{k=n+p}^{\infty}\left(\frac{k-q}{p-q}\right)^{\gamma}(k-q-\alpha)\delta(k,q)b_ka_k\le(p-q-\alpha)\delta(p,q),$$

where $0\le\alpha<p-q, n,p\in\mathbb{N}, p\in\mathbb{N}, q\in\mathbb{N}_0, p>q, \gamma\ge0$ and $\delta(i,h)$ *is given by* (1.9).

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