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Inclusion Relations for Uniformly Certain Classes of Analytic Functions

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Abstract

In this paper we investigate a family of linear operator defined on the space of univalent functions. By making use of this linear operator, we introduce and investigate some new subclasses of uniformly starlike, uniformly convex, uniformly close-to-convex and uniformly quasi-convex univalent functions. Also we establish some inclusion relationships associated with the aforementioned linear operator. Some interesting integral-preserving properties are also considered.

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1 Introduction

Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Let $f \in \mathcal{A}$ be given by (1.1) and g be given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.2}$$

The Hadamard product (or convolution) $(f * g)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

For two functions $f(z)$ and $F(z)$, analytic in U, we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z) \prec F(z)(z \in U),$$

if there exists a Schwarz function $\omega(z) \in \Omega$, which (by definition) is analytic in U with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in U)$$

such that

$$f(z) = F(\omega(z))(z \in U).$$

Indeed it is known that

$$f(z) \prec F(z)(z \in U) \implies f(0) = F(0) \text{ and } f(U) \subset F(U).$$

In particular, if the function $F(z)$ is univalent in U, we have the following equivalence

$$f(z) \prec F(z)(z \in U) \iff f(0) = F(0) \text{ and } f(U) \subset F(U).$$

Further let S denote the subclass of $\mathcal{A}$ consisting of analytic and univalent functions f in U. A function f in S is said to be starlike of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1; z \in U). \tag{1.4}$$

We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Also a function f in S is said to be convex of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1; z \in U). \tag{1.5}$$

We denote by $C(\alpha)$ the class of all convex functions of order α.

We note that:

$$f(z) \in C(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha), \tag{1.6}$$

$$\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^* \text{ and } C(\alpha) \subseteq C(0) \equiv C.$$

The classes $\mathcal{S}^*, C,$ $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ were first introduced by Robertson [26] and the classes $S^*(\alpha)$ and $K(\alpha)$ were studied subsequently by MacGregor [21] Schild [29], Pinchuk [24], Jack [14] and others.

A function f in $\mathcal{A}$ is said to be uniformly convex in U if f is a univalent convex function along with the property that, for every circular arc γ contained in U, with center ξ also in U, the image curve $f(\gamma)$ is a convex arc. The class of uniformly convex functions is denoted by UCV (see [12]). The corresponding classUST is defined by the relation that $f \in UCV$ if, and only if, $zf' \in UST$. It is well known [22] that $f \in UCV$ if, and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in U).$$

Uniformly starlike and convex functions were first introduced by Goodman [13] and then studied by various other authors (e.g. [5]).
A function $f \in \mathcal{A}$ is said to be in the class of close-to-convex and quasi-convex in U respectively, if satisfies

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0 \quad (z \in U),$$

for some $g \in S^*$

$$Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0 \quad (z \in U)$$

for some $g \in C$ (see [11]).

Also, a function $f \in \mathcal{A}$ is said to be in the class of uniformly convex functions of order γ and type β denoted by $UC(\beta, \gamma)$ (see [1] and [2]) if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma \quad (\beta \geq 0, 0 \leq \gamma < 1, \beta+\gamma \geq 0; z \in U), \tag{1.7}$$

and is said to be in a corresponding class denoted by $SP(\beta, \gamma)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma \quad (\beta \geq 0, 0 \leq \gamma < 1,\ \beta+\gamma \geq 0; z \in U). \tag{1.8}$$

Also, a function $f \in \mathcal{A}$ is said to be in the class of uniformly close-to-convex functions of order γ and type β denoted by $UKC(\beta,\gamma)$ if

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} \geq \beta\left|\frac{zf'(z)}{g(z)}-1\right|+\gamma \quad (\beta \geq 0, 0 \leq \gamma < 1,\ \beta+\gamma \geq 0; z \in U), \tag{1.9}$$

for some $g \in SP(\beta,\gamma)$ (see [1] and [2]),and it is said to be in the class of uniformly quasi-convex functions of order γ and type β denoted by $UQC(\beta,\gamma)$ if

$$Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} \geq \beta\left|\frac{(zf'(z))'}{g'(z)}-1\right|+\gamma \quad (\beta \geq 0, 0 \leq \gamma < 1,\ \beta+\gamma \geq 0; z \in U), \tag{1.10}$$

for some $g \in UKC(\beta,\gamma)$ (see [1] and [19]).
We note that

$$f(z) \in UC(\beta,\gamma) \Leftrightarrow zf'(z) \in SP(\beta,\gamma) \tag{1.11}$$

and

$$f(z) \in UQC(\beta,\gamma) \Leftrightarrow zf'(z) \in UKC(\beta,\gamma). \tag{1.12}$$

Geometric interpretation. Let $f \in SP(\beta,\gamma)$ and $f \in UC(\beta,\gamma)$ if and only if $\frac{zf'(z)}{f(z)}$ and $1+\frac{zf''(z)}{f'(z)}$, respectively, take all the values in the conic domain $\mathcal{R}_{\beta,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{\gamma,\beta} = \left\{w = u+iv \in \mathbb{C} : u > \beta\sqrt{(u-1)^2+v^2}+\gamma, \beta \geq 0 \text{ and } \gamma \in [0,1)\right\}, \tag{1.13}$$

with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = 1+\frac{zf''(z)}{f'(z)}$ and considering the functions which map U onto conic domain $\mathcal{R}_{\beta,\gamma}$, such that $1 \in \mathcal{R}_{\beta,\gamma}$, we can write the conditions (1.7) or (1.8) in the form:

$$p(z) \prec P_{\beta,\gamma}(z) \tag{1.14}$$

Denote by $\mathcal{P}(P_{\beta,\gamma})$ $(\beta \geq 0, 0 \leq \gamma < 1)$, the family of functions p, such that $p \in \mathcal{P}$ and $p \prec P_{\beta,\gamma}$ in U, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and the function $P_{\beta,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{\gamma,\beta}$ such that $1 \in \mathcal{R}_{\beta,\gamma}$ and $\partial\mathcal{R}_{\beta,\gamma}$ is a curve by the equality

$$\partial\mathcal{R}_{\gamma,\beta} = \left\{w = u+iv \in \mathbb{C} : u^2 = \left(\beta\sqrt{(u-1)^2+v^2}+\gamma\right)^2, \beta \geq 0 \text{ and } \gamma \in [0,1)\right\}.$$

From elementary computations we see that $\partial\mathcal{R}_{\beta,\gamma}$ represent the conic sections symmetric about the real axis. Thus $\mathcal{R}_{\beta,\gamma}$ is an elliptic domain for $\beta > 1$, a parabolic domain for $\beta = 1$, a hyperbolic domain for $0 < \beta < 1$ and a right half plain $u > \gamma$ for $\beta = 0$.

The functions that play the role of extremal functions of the class $\mathcal{P}(P_{\beta,\gamma})$, where obtained in [1] as follows:

$$P_{\beta,\gamma}(z)=\begin{cases}\frac{1+(1-2\gamma)z}{1-z}, & \beta=0,\\ 1+\frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta=1,\\ \frac{1-\gamma}{1-\beta^2}\cos\left\{\left(\frac{2}{\pi}\cos^{-1}\beta\right)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{\beta^2-\gamma}{1-\beta^2}, & 0<\beta<1,\\ \frac{1-\gamma}{1-\beta^2}\sin\left(\frac{\pi}{2K(t)}\right)\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}dx+\frac{\beta^2-\gamma}{1-\beta^2}, & \beta>1,\end{cases} \tag{1.15}$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}, t\in(0,1), z\in U$ and t is chosen such that $\beta=\cosh\frac{\pi K'(t)}{4K(t)}$, $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is the complementary integral of $K(t)$.
For $\beta=0$ obviously $P_{0,\gamma}(z)=1+2(1-\gamma)z+2(1-\gamma)z^2+...$, for $\beta=1$ (compare [22] and [27]) $P_{1,\gamma}(z)=1+\frac{8}{\pi^2}(1-\gamma)z+\frac{16}{3\pi^2}(1-\gamma)z^2+...$, by comparing Taylor series expansion in [16], we have for $0<\beta<1$

$$P_{\beta,\gamma}(z)=1+\frac{1-\gamma}{1-\beta^2}\sum_{n=1}^{\infty}\left[\sum_{\ell=1}^{2n}2^{\ell}\binom{B}{\ell}\binom{2n-1}{2n-\ell}\right]z^n,$$

where $B=\cos^{-1}\beta$ and for $\beta>1$,

$$P_{\beta,\gamma}(z)=1+\frac{\pi^2(1-\gamma)}{4\sqrt{t}(\beta^2-1)K^2(t)\,(1+t)}\times\left[z+\frac{4K^2(t)\,(t^2+6t+1)-\pi^2}{24\sqrt{t}K^2(t)\,(1+t)}z^2+...\right].$$

From (1.13) and the properties of the domain $\mathcal{R}_{\beta,\gamma}$ we have

$$Re\,(p(z))>Re\,(P_{\beta,\gamma}(z))>\frac{\beta+\gamma}{\beta+1}. \tag{1.16}$$

Let for $m\in\mathbb{Z}=\{...,-2,-1,0,1,2,...\}$ and $\ell>-1,\lambda>0$ a linear operator $\mathcal{J}^m_{\lambda,\ell}:\mathcal{A}\longrightarrow\mathcal{A}$ be defined by

$$\begin{aligned}\mathcal{J}^m_{\lambda,\ell}f(z) &= f(z), m=0,\\ &= \frac{\ell+1}{\lambda}z^{1-\frac{\ell+1}{\lambda}}\int_0^z t^{\frac{\ell+1}{\lambda}-2}\mathcal{J}^{m-1}_{\lambda,\ell}f(z)dt,\quad m=1,2,...\\ &= \frac{\ell+1}{\lambda}z^{2-\frac{\ell+1}{\lambda}}\frac{d}{dz}\left(z^{\frac{\ell+1}{\lambda}-1}\mathcal{J}^{m-1}_{\lambda,\ell}f(z)\right),\quad m=1,2,...\end{aligned}$$

Let $A>0, a, c\in\mathbb{C}$ be such that $Re(c-a)\geq 0$ an Erdelyi-Kober type [17] integral operator $\mathcal{I}^{a,c}_A:\mathcal{A}\longrightarrow\mathcal{A}$ be defined for $Re(c-a)>0$ and $Re(a)>-A$ by

$$\mathcal{I}_A^{a,c} f(z) = \frac{\Gamma(c+A)}{\Gamma(a+A)\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} f(zt^A)dt,$$

and

$$\mathcal{I}_A^{a,a} f(z) = f(z).$$

By convolution of the linear operators (defined above), the operator $\mathcal{J}_{\lambda,\ell}^m(a,c,A) : \mathcal{A} \longrightarrow \mathcal{A}$ was introduced by Raina and Sharma [25] as follows:

$$\begin{aligned} \mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z) &= \mathcal{J}_{\lambda,\ell}^m\left(\mathcal{I}_A^{a,c} f(z)\right) = \mathcal{I}_A^{a,c}\left(\mathcal{J}_{\lambda,\ell}^m f(z)\right) \\ &= z + \frac{\Gamma(c+A)}{\Gamma(a+A)} \sum_{n=2}^{\infty} \left(1 + \frac{\lambda(n-1)}{1+\ell}\right)^m \frac{\Gamma(a+nA)}{\Gamma(c+nA)} a_n z^n, \end{aligned} \tag{1.17}$$

where $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $A > 0, \lambda \geq 0, \ell > -1, a, c \in \mathbb{C}$ be such that $Re(c-a) > 0$ and $Re(a) > -A$.

We may point out here that some of the special cases of the operator defined by (1.17) can be found in [3], [4], [6], [7], [9], [10], [15], [18], [28], [30].

Now, we will use the operator $\mathcal{J}_{\lambda,\ell}^m(a,c,A)$ in the form:

$$\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z) = z + \frac{\Gamma(c+A)}{\Gamma(a+A)} \sum_{n=2}^{\infty} \left(\frac{1+\ell}{1+\ell+\lambda(n-1)}\right)^m \frac{\Gamma(a+nA)}{\Gamma(c+nA)} a_n z^n \tag{1.18}$$

where $m \in \mathbb{Z}^+ = \{0, 1, 2, ...\}$ and $A > 0, \lambda \geq 0, \ell > -1, a, c \in \mathbb{C}$ be such that $Re(c-a) > 0$ and $Re(a) > -A$.
It is readily verified from (1.18) that

$$\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z) = (1 - \frac{\lambda}{1+\ell})\mathcal{J}_{\lambda,\ell}^{m+1}(a,c,A)f(z) + \frac{\lambda}{1+\ell} z\left(\mathcal{J}_{\lambda,\ell}^{m+1}(a,c,A)f(z)\right)' \tag{1.19}$$

and

$$\mathcal{J}_{\lambda,\ell}^m(a+1,c,A)f(z) = \frac{a}{a+A}\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z) + \frac{A}{a+A} z\left(\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z)\right)'. \tag{1.20}$$

Now, we define new subclasses of univalent functions by the linear operator $\mathcal{J}_{\lambda,\ell}^m(a,c,A)(z)$ as follows:

Definition 2. Let $f(z) \in \mathcal{A}$. Then $f(z) \in SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$ if and only if $\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z) \in SP(\beta,\gamma)$.

Definition 3. Let $f(z) \in \mathcal{A}$. Then $f(z) \in UC_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$ if and only if $\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z) \in UC(\beta,\gamma)$.

Definition 4. Let $f(z) \in \mathcal{A}$. Then $f(z) \in UKC^m_{\lambda,\ell}(a, c, A, \beta, \gamma)$ if and only if $\mathcal{J}^m_{\lambda,\ell}(a, c, A)f(z) \in UKC(\beta, \gamma)$.

Definition 5. Let $f(z) \in \mathcal{A}$. Then $f(z) \in UQC^m_{\lambda,\ell}(a, c, A, \beta, \gamma)$ if and only if $\mathcal{J}^m_{\lambda,\ell}(a, c, A)f(z) \in UQC(\beta, \gamma)$.

We note that

$$f(z) \in UC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \Leftrightarrow zf'(z) \in SP^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \tag{1.21}$$

$$f(z) \in UQC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \Leftrightarrow zf'(z) \in UKC^m(a, c, A, \beta, \gamma) \tag{1.22}$$

2 Preliminaries Results

In order to prove our results, we need the following lemmas.

Lemma 1. (see [8]). Let a and b be complex nuber and h be univalent convex function in U with $h(0) = c$ and $Re(ah(z) + b) > 0$. Let $g(z) = c + \sum_{n=1}^{\infty} b_n z^n$ be analytic in U. Then

$$g(z) + \frac{zg'(z)}{ag(z) + b} \prec h(z) \Rightarrow g(z) \prec h(z).$$

Lemma 2. (see [23]). Let h be convex function in U and let $D > 0$.Suppose $E(z)$ is analytic in U with $Re\{E(z)\} > D$. If g be analytic in U and $g(0) = h(0)$.Then

$$Dzg''(z) + E(z)zg'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

3 Inclusion Relations

In the following results we will study inclusion relations.
Unless otherwise mentioned we shall assume throughout the paper that

$m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}, \beta \geq 0, 0 \leq \gamma < 1,\ A > 0, \lambda \geq 0, \ell > -1, a, c \in \mathbb{C}$ be such that $Re(c - a) > 0$ and $Re(a) > -A$.

Theorem 1. Let $\frac{Re(a)}{A} > -\frac{\beta+\gamma}{\beta+1}$, $\frac{1+\ell}{\lambda} > \frac{1-\gamma}{\beta+1}$ and $f \in \mathcal{A}$. Then

$$SP^m_{\lambda,\ell}(a + 1, c, A, \beta, \gamma) \subset SP^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \subset SP^{m+1}_{\lambda,\ell}(a, c, A, \beta, \gamma).$$

Proof. To prove the first part of Theorem 1, let $f \in SP^m_{\lambda,\ell}(a + 1, c, A, \beta, \gamma)$, and set

$$\frac{z\left(\mathcal{J}^m(a, c, A)f(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a, c, A)f(z)} = p(z) \quad (z \in U), \tag{3.1}$$

where $p(z)=1+p_1z+p_2z^2+...$is analytic in U, with $p(0)=1$ and $p(z)\neq 0$, for all $z\in U$.
From (1.20), we can write

$$\frac{z\left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)f(z)\right)'}{\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)f(z)}=\left(\frac{a+A}{A}\right)\frac{\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)f(z)}{\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)f(z)}-\frac{a}{A}, \tag{3.2}$$

$$\left(\frac{a+A}{A}\right)\frac{\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)f(z)}{\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)f(z)}=p(z)+\frac{a}{A}. \tag{3.3}$$

By logarithmically differentiating both sides of the equation (3.3), we get

$$\begin{aligned}\frac{z\left(\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)f(z)\right)'}{\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)f(z)} &= \frac{z\left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)f(z)\right)'}{\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)f(z)}+\frac{zp'(z)}{p(z)+\frac{a}{A}}\\ \frac{z\left(\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)f(z)\right)'}{\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)f(z)} &= p(z)+\frac{zp'(z)}{p(z)+\frac{a}{A}}.\end{aligned}$$

Form this equation and the argument given by (1.14), we may write

$$p(z)+\frac{zp'(z)}{p(z)+\frac{a}{A}}\prec P_{\beta,\gamma}(z).$$

Therefore, $f\in SP_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma)$ by Lemma 1 and condition (1.14), since $P_{\beta,\gamma}(z)$ is univalent and convex in U and $Re(P_{\beta,\gamma}(z))>\frac{\beta+\gamma}{\beta+1}$.
To prove the second inclusion relationship asserted by Theorem 1, let
$f\in SP_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma)$, and put

$$\frac{z\left(\mathcal{J}^{m+1}(a,c,A)f(z)\right)'}{\mathcal{J}_{\lambda,\ell}^{m+1}(a,c,A)f(z)}=q(z)\quad(z\in U),$$

where the function $q(z)$ is analytic in U with $q(0)=1$. Then by arguments similar to those detailed above with (1.19) it follows that $q(z)\prec P_{\beta,\gamma}(z)$ in U, which implies that

$$f\in SP_{\lambda,\ell}^{m+1}(a,c,A,\beta,\gamma).$$

The proof of Theorem 1 is completed.

Theorem 2. Let $\frac{Re(a)}{A}>-\frac{\beta+\gamma}{\beta+1}$, $\frac{1+\ell}{\lambda}>\frac{1-\gamma}{\beta+1}$ and $f\in\mathcal{A}$. Then

$$UC_{\lambda,\ell}^{m}(a+1,c,A,\beta,\gamma)\subset UC_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma)\subset UC_{\lambda,\ell}^{m+1}(a,c,A,\beta,\gamma).$$

Proof. Applying (1.11) and Theorem 1, we observe that

$$\begin{aligned} f(z) &\in UC_{\lambda,\ell}^{m}(a+1,c,A,\beta,\gamma) \Leftrightarrow \mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,f(z) \in UC(\beta,\gamma) \\ &\Leftrightarrow z\left(\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,f(z)\right)' \in SP(\beta,\gamma) \\ &\Leftrightarrow \mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,(zf'(z)) \in SP(\beta,\gamma) \\ &\Leftrightarrow zf'(z) \in SP_{\lambda,\ell}^{m}(a+1,c,A,\beta,\gamma) \\ &\Rightarrow zf'(z) \in SP_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma) \\ &\Leftrightarrow \mathcal{J}_{\lambda,\ell}^{m}(a,c,A)\,(zf'(z)) \in SP(\beta,\gamma) \\ &\Leftrightarrow z\left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,A)\,f(z)\right)' \in SP(\beta,\gamma) \\ &\Leftrightarrow \mathcal{J}_{\lambda,\ell}^{m}(a,c,A)\,f(z) \in UC(\beta,\gamma) \\ &\Leftrightarrow f(z) \in UC_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma) \end{aligned}$$

and

$$\begin{aligned} f(z) &\in UC_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma) \Leftrightarrow zf'(z) \in SP_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma) \\ &\Rightarrow zf'(z) \in SP_{\lambda,\ell}^{m+1}(a,c,A,\beta,\gamma) \\ &\Leftrightarrow z\left(\mathcal{J}_{\lambda,\ell}^{m+1}(a,c,A)\,f(z)\right)' \in SP(\beta,\gamma) \\ &\Leftrightarrow f(z) \in UC_{\lambda,\ell}^{m+1}(a,c,A,\beta,\gamma) \end{aligned}$$

which evidently proves Theorem 2.

Theorem 3. Let $\frac{Re(a)}{A} > -\frac{\beta+\gamma}{\beta+1}$, $\frac{1+\ell}{\lambda} > \frac{1-\gamma}{\beta+1}$ and $f \in \mathcal{A}$. Then

$$UKC_{\lambda,\ell}^{m}(a+1,c,A,\beta,\gamma) \subset UKC_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma) \subset UKC_{\lambda,\ell}^{m+1}(a,c,A,\beta,\gamma).$$

Proof. We begin by proving that

$$UKC_{\lambda,\ell}^{m}(a+1,c,A,\beta,\gamma) \subset UKC_{\lambda,\ell}^{m}(a,c,A,\beta,\gamma).$$

Let $f \in UKC_{\lambda,\ell}^{m}(a+1,c,A,\beta,\gamma)$. Then, in view of the definition, we can write

$$Re\left\{\frac{z(\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,f(z))'}{\psi(z)}\right\} \prec P_{\beta,\gamma} \quad (z \in U),$$

for some $\psi(z) \in SP(\beta,\gamma)$. Choose the function $g(z)$ such that $\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,g(z) = \psi(z)$, so we have

$$Re\left\{\frac{z(\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,f(z))'}{\mathcal{J}_{\lambda,\ell}^{m}(a+1,c,A)\,g(z)}\right\} \prec P_{\beta,\gamma} \quad (z \in U). \tag{3.4}$$

Now, we set

$$\frac{z(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,f(z))'}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)}=p(z), \tag{3.5}$$

where $p(z)=1+p_1z+p_2z^2+...$is analytic in U, with $p(0)=1$ and $p(z)\neq 0$, for all $z\in U$.

Using the identity (1.20), we have

$$\begin{aligned}\frac{z(\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,f(z))'}{\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,g(z)} &= \frac{\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,(zf'(z))}{\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,g(z)}\\ &= \frac{z\left(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,(zf'(z))\right)'+(\frac{a}{A})\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,(zf'(z))}{z\left(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)\right)'+(\frac{a}{A})\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)}\\ &= \frac{\frac{z\left(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)(zf'(z))\right)'}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)}+\frac{(\frac{a}{A})\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)(zf'(z))}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)}}{\frac{z\left(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)\right)'}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)}+(\frac{a}{A})}.3.6 \qquad (2)\end{aligned}$$

Since $g(z)\in SP^{m}_{\lambda,\ell}(a+1,c,A,\beta,\gamma)$, and by Theorem 1, we can write $\frac{z\left(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)\right)'}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)}=r(z)$, where $Re\{r(z)\}>0$, $(z\in U)$,

$$\frac{z(\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,f(z))'}{\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,g(z)}=\frac{\frac{z\left(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)(zf'(z))\right)'}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)g(z)}+(\frac{a}{A})p(z)}{r(z)+(\frac{a}{A})}. \tag{3.7}$$

From (3.5), we consider that

$$z(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,f(z))'=\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)p(z) \tag{3.8}$$

differentiating both sides of (3.8) with respect to z and divide by $\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)$, we have

$$\begin{aligned}\left[z(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,f(z))'\right]' &= \mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)(zp'(z))+p(z)z(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z))'\\ \frac{\left[z(\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,f(z))'\right]'}{\mathcal{J}^{m}_{\lambda,\ell}(a,c,A)\,g(z)} &= zp'(z)+r(z)p(z)3.9 \qquad (3)\end{aligned}$$

Using (3.7) and (3.9), we obtain

$$\begin{aligned}\frac{z(\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,f(z))'}{\mathcal{J}^{m}_{\lambda,\ell}(a+1,c,A)\,g(z)} &= \frac{zp'(z)+r(z)p(z)+(\frac{a}{A})p(z)}{r(z)+(\frac{a}{A})}\\ &= p(z)+\frac{zp'(z)}{r(z)+(\frac{a}{A})}.3.10 \qquad (4)\end{aligned}$$

From (3.4) and (3.10), we conclude that

$$p(z) + \frac{1}{r(z) + (\frac{a}{A})} zp'(z) \prec P_{\beta,\gamma}. \tag{3.11}$$

Putting $D = 0$ and $E(z) = \frac{1}{r(z)+(\frac{a}{A})}$, we obtain

$$Re\{E(z)\} = \frac{1}{\left|r(z) + (\frac{a}{A})\right|^2} Re\left[r(z) + (\frac{a}{A})\right] > 0,$$

therefore, the inequality (3.11) satisfies the conditions required by Lemma 2. Hence $p(z) \prec P_{\beta,\gamma}$, so that $f \in UKC^m_{\lambda,\ell}(a, c, A, \beta, \gamma)$.

For the second inclusion relationship asserted by the Theorem 3, using arguments similar to those detailed above with equation (1.19), we obtain $UKC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \subset UKC^{m+1}_{\lambda,\ell}(a, c, A, \beta, \gamma)$.
Thus, we have completed the proof of Theorem 3

Theorem 4. Let $\frac{Re(a)}{A} > -\frac{\beta+\gamma}{\beta+1}$, $\frac{1+\ell}{\lambda} > \frac{1-\gamma}{\beta+1}$ and $f \in \mathcal{A}$. Then

$$UQC^m_{\lambda,\ell}(a+1, c, A, \beta, \gamma) \subset UQC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \subset UQC^{m+1}_{\lambda,\ell}(a, c, A, \beta, \gamma). \tag{3.12}$$

Proof. Applying (1.12), (1.22) and Theorem 3, we observe that

$$\begin{aligned} f(z) \quad &\in \quad UQC^m_{\lambda,\ell}(a+1, c, A, \beta, \gamma) \Leftrightarrow \mathcal{J}^m_{\lambda,\ell}(a+1, c, A) f(z) \in UQC(\beta, \gamma) \\ &\Leftrightarrow \quad z\left(\mathcal{J}^m_{\lambda,\ell}(a+1, c, A) f(z)\right)' \in UKC(\beta, \gamma) \\ &\Leftrightarrow \quad \mathcal{J}^m_{\lambda,\ell}(a+1, c, A)(zf'(z)) \in UKC^m_{\lambda,\ell}(a+1, c, A, \beta, \gamma) \\ &\Leftrightarrow \quad zf'(z) \in UKC^m_{\lambda,\ell}(a+1, c, A, \beta, \gamma) \\ &\Rightarrow \quad zf'(z) \in UKC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \\ &\Leftrightarrow \quad \mathcal{J}^m_{\lambda,\ell}(a, c, A)\left(zf'(z)\right) \in UKC(\beta, \gamma) \\ &\Leftrightarrow \quad z\left(\mathcal{J}^m_{\lambda,\ell}(a, c, A) f(z)\right)' \in UKC(\beta, \gamma) \\ &\Leftrightarrow \quad \mathcal{J}^m_{\lambda,\ell}(a, c, A) f(z) \in UQC(\beta, \gamma) \\ &\Leftrightarrow \quad f(z) \in UQC^m_{\lambda,\ell}(a, c, A, \beta, \gamma). \end{aligned}$$

and

$$\begin{aligned} f(z) \quad &\in \quad UQC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \Leftrightarrow zf'(z) \in UKC^m_{\lambda,\ell}(a, c, A, \beta, \gamma) \\ &\Rightarrow \quad zf'(z) \in UKC^{m+1}_{\lambda,\ell}(a, c, A, \beta, \gamma) \\ &\Leftrightarrow \quad z\left(\mathcal{J}^{m+1}_{\lambda,\ell}\,(a, c, A)\, f(z)\right)' \in UKC(\beta, \gamma) \\ &\Leftrightarrow \quad f(z) \in UQC^{m+1}_{\lambda,\ell}(a, c, A, \beta, \gamma) \end{aligned}$$

which evidently proves Theorem 4.

The proof is completed.

Next, we study the closure properties of generalized Bernardi integral operator (see [4] and [20]) defined by:

$$\mathcal{L}_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (f \in \mathcal{A}\ , c > -1)\,. \tag{3.13}$$

Theorem 5. Let $c > -\frac{\beta+\gamma}{\beta+1}$ and $f \in \mathcal{A}$. If $\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \in SP(\beta,\gamma)$, then

$$\mathcal{L}_c \left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \right) \in SP(\beta,\gamma).$$

Proof. From (3.13), and the linear operator $\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)$, we have

$$z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c f(z)\right)' = (c+1)\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) - c\mathcal{J}^m_{\lambda,\ell}(a,c,A)\left(\mathcal{L}_c f(z)\right). \tag{3.14}$$

Setting

$$\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c f(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)\left(\mathcal{L}_c f(z)\right)} = p(z),$$

where $p(z) = 1 + p_1 z + p_2 z^2 + ...$is analytic in U, with $p(0) = 1$ and $p(z) \neq 0$, for all $z \in U$.
From (3.14), we have

$$p(z) = (c+1)\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)\left(\mathcal{L}_c f(z)\right)} - c. \tag{3.15}$$

By logarithmically differentiating both sides of the equation (3.15), we have

$$\begin{aligned}
\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)} &= \frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\left(\mathcal{L}_c f(z)\right)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)\left(\mathcal{L}_c f(z)\right)} + \frac{zp'(z)}{p(z)+c}, \\
\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)} &= p(z) + \frac{zp'(z)}{p(z)+c}.
\end{aligned}$$

Since $p(z) \prec P_{\beta,\gamma}$ and $Re\left\{P_{\beta,\gamma} + c\right\} > 0$, then by applying Lemma 2, we have the result.

Theorem 6. Let $c > -\frac{\beta+\gamma}{\beta+1}$ and $f \in \mathcal{A}$. If $\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \in UC(\beta,\gamma)$, then

$$\mathcal{L}_c \left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \right) \in UC(\beta,\gamma).$$

Proof. Consider the following

$$\begin{aligned} f(z) &\in UC^m_{\lambda,\ell}(\beta,\gamma) \Leftrightarrow \mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \in UC(\beta,\gamma) \\ &\Leftrightarrow z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)\right)' \in SP(\beta,\gamma) \\ &\Leftrightarrow \mathcal{J}^m_{\lambda,\ell}(a,c,A)(zf'(z)) \in SP(\beta,\gamma) \\ &\Leftrightarrow zf'(z) \in SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma) \\ &\Rightarrow \mathcal{L}_c\left(zf'(z)\right) \in SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma) \\ &\Leftrightarrow \mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c\left(zf'(z)\right) \in SP(\beta,\gamma) \\ &\Leftrightarrow z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c f(z)\right)' \in SP(\beta,\gamma) \\ &\Leftrightarrow \mathcal{L}_c\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \in UC(\beta,\gamma) \end{aligned}$$

the proof is completed.

Theorem 7. Let $c > -\frac{\beta+\gamma}{\beta+1}$ and $f \in \mathcal{A}$. If $\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \in UKC(\beta,\gamma)$, then

$$\mathcal{L}_c\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)\right) \in UKC(\beta,\gamma).$$

Proof. Let $f \in UKC^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$. Then, in view of the definition, we can write

$$Re\left\{\frac{z(\mathcal{J}^m_{\lambda,\ell}(a,c,A,f(z))'}{\psi(z)}\right\} \prec P_{\beta,\gamma} \quad (z \in U),$$

for some $\psi(z) \in SP(\beta,\gamma)$. Choose the function $g(z)$ such that $\mathcal{J}^m_{\lambda,\ell}(a,c,A)g(z) = \psi(z)$, so we have

$$Re\left\{\frac{z(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z))'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)g(z)}\right\} \prec P_{\beta,\gamma} \quad (z \in U). \tag{3.16}$$

Now, we set

$$\frac{z(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z))'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)g(z)} = p(z), \tag{3.17}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + ...$is analytic in U, with $p(0) = 1$ and $p(z) \neq 0$, for all $z \in U$.

Using the identity (1.14), we have

$$\begin{aligned} \frac{z(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z))'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)g(z)} &= \frac{\mathcal{J}^m_{\lambda,\ell}(a,c,A)(zf'(z))}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)g(z)} \\ &= \frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c(zf'(z))\right)' + c\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c(zf'(z))}{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c g(z)\right)' + c\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c g(z)} \\ &= \frac{\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c(zf'(z))\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c g(z)} + \frac{c\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c(zf'(z))}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c g(z)}}{\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c g(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)\mathcal{L}_c g(z)} + c} .3.18 \end{aligned} \tag{5}$$

Since $g(z) \in SP^m_{\lambda,\ell}(a, c, A, \beta, \gamma)$, and by Theorem 5, we have $\mathcal{L}_c g(z) \in SP^m_{\lambda,\ell}(a, c, A, \beta, \gamma)$.

Setting

$$\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a, c, A)\mathcal{L}_c g(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a, c, A)\mathcal{L}_c g(z)} = H(z).$$

Also, we can define h by

$$z\left(\mathcal{J}^m_{\lambda,\ell}(a, c, A)\mathcal{L}_c f(z)\right)' = \mathcal{J}^m_{\lambda,\ell}(a, c, A)\mathcal{L}_c g(z)\left[h(z)\right], \tag{3.19}$$

differentiating both sides of (3.19) with respect to z, we have

$$\frac{z\left[z\left(\mathcal{J}^m_{\lambda,\ell}\mathcal{L}_c f(z)\right)'\right]'}{\mathcal{J}^m_{\lambda,\ell}(a, c, A)\mathcal{L}_c g(z)} = zh'(z) + h(z)H(z). \tag{3.20}$$

Using (3.18) and (3.20), we obtain

$$\begin{aligned} \frac{z(\mathcal{J}^m_{\lambda,\ell}(a, c, A)f(z))'}{\mathcal{J}^m_{\lambda,\ell}(a, c, A)g(z)} &= \frac{zh'(z) + h(z)H(z) + ch(z)}{H(z) + c} \\ &= h(z) + \frac{zh'(z)}{H(z) + c}.3.21 \qquad (6) \end{aligned}$$

This in conjunction with (3.16) leads to

$$h(z) + \frac{zh'(z)}{H(z) + c} \prec P_{\beta,\gamma}. \tag{3.22}$$

Putting $D = 0$ and $E(z) = \frac{1}{H(z)+c}zh'(z)$, we obtain $Re\left\{E(z)\right\} > 0$, if $c > -\frac{\beta-\gamma}{\beta+1}$, therefore, the inequality (3.22) satisfies the conditions required by Lemma 2. Hence $p(z) \prec q_{\beta,\gamma}$, so, the proof is completed.

Theorem 8. Let $c > -\frac{\beta+\gamma}{\beta+1}$ and $f \in \mathcal{A}$. If $\mathcal{J}^m_{\lambda,\ell}(a, c, A)f(z) \in UQC(\beta, \gamma)$, then

$$\mathcal{L}_c\left(\mathcal{J}^m_{\lambda,\ell}(a, c, A)f(z)\right) \in UQC(\beta, \gamma).$$

Proof. The proof of Theorem 8 similar that to the proof of Theorem 7, so the details are ometted.

Open problem

The authors sugeste to study the above inclusion problems for uniformly meromorphic functions.

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