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Partial sums of certain subclass of meromorphic univalent functions

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Abstract

In this paper we obtain some results concerning the partial sums of certain subclass of meromorphic univalent functions.

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1 Introduction

Let Σ denote the class of analytic and univalent functions in the punctured disc $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$ of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n . \tag{1}$$

A function $f(z)$ in Σ is said to belong to $\Sigma^*(\alpha)$, the class of meromorphically starlike functions of order α $(0 \leq \alpha < 1)$, if and only if

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{2}$$

A function $f(z)$ in Σ is said to belong to $\Sigma_K(\alpha)$, the class of meromorphically convex functions of order α $(0 \leq \alpha < 1)$, if and only if

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{3}$$

The classes $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$ were studied by Altintas et al. [1], Aouf [2, 3], Ganigi and Uralegaddi [7], Kulkarni and Joshi [9], Mogra et al. [12], Uralegaddi [16], Uralegaddi and Ganigi [17], Uralegaddi and Somanatha [18] and others.

Let Σ_p denote the class of analytic and univalent functions in the punctured disc $\mathbb{U}^*$ of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{4}$$

For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let $\Sigma(\alpha, \lambda)$ denote a subclass of Σ consisting of functions of the form (1) satisfying the condition that

$$\Re\left(\frac{zf'(z)}{(\lambda - 1)f(z) + \lambda zf'(z)}\right) > \alpha \quad (z \in \mathbb{U}^*). \tag{5}$$

Furthermore, we say that a function $f \in \Sigma_p(\alpha, \lambda)$, whenever $f(z)$ is of the form (4). A sufficient condition for a function $f(z)$ of the form (4) to be in $f \in \Sigma_p(\alpha, \lambda)$ (see [8]) is that

$$\sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(1 + n)] a_n \leq (1 - \alpha). \tag{6}$$

In [14] Silverman determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Also, Li and Owa [10] obtained the sharp radius which for the normalized univalent functions in $\mathbb{U}$, the partial sums of the well known Libera integral operator [11] imply starlikeness. Further, for various other interesting developments concerning partial sums of analytic univalent functions (see [5, 13, 14, 15, 19]).

Recently, Cho and Owa [6] and Aouf and Silverman [4, with $p = 1$] have investigated the ratio of function of the form (4) (with $\lambda = 0$) to its sequence of partial sums

$$f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n, \tag{7}$$

when the coefficients are sufficiently small to satisfy either condition (6) with $\lambda = 0$. Also Cho and Owa [6] and Aouf and Silverman [4, with $p = 1$] have determined sharp lower bounds for $\Re\left\{\frac{f(z)}{f_k(z)}\right\}$, $\Re\left\{\frac{f_k(z)}{f(z)}\right\}$, $\Re\left\{\frac{f'(z)}{f_k'(z)}\right\}$, and $\Re\left\{\frac{f_k'(z)}{f'(z)}\right\}$. In this paper, applying method used by Silverman [14] and Cho and Owa [6], we will investigate the ratio of a function of the form (4) to its sequence of partial sums given by (7), when the coefficient are sufficiently small to satisfy either condition (6). More precisely, we will determine sharp lower bounds for $\Re\left\{\frac{f(z)}{f_k(z)}\right\}$, $\Re\left\{\frac{f_k(z)}{f(z)}\right\}$, $\Re\left\{\frac{f'(z)}{f_k'(z)}\right\}$, and $\Re\left\{\frac{f_k'(z)}{f'(z)}\right\}$.

In the sequel, we will make use of the well-known result that $\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0$ $(z \in \mathbb{U})$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|w(z)| \leq |z|$. Unless otherwise stated, we will assume that f is of the form (4) and its sequence of partial sums is given by (7).

2 Main Results

Theorem 2.1 *Let the function $f(z)$ defined by (4) satisfies condition (6). Then*

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k+2\alpha-\alpha\lambda(2+k)}{k+1+\alpha-\alpha\lambda(2+k)} \quad (z \in \mathbb{U}). \tag{8}$$

The result is sharp for every k, with extremal function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{k+1+\alpha-\alpha\lambda(2+k)} z^{k+1} \quad (k \geq 0). \tag{9}$$

Proof. We may write

$$\begin{aligned} &\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\left[\frac{f(z)}{f_k(z)} - \frac{k+2\alpha-\alpha\lambda(2+k)}{k+1+\alpha-\alpha\lambda(2+k)}\right] \\ &= \frac{1+\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1} + \sum\limits_{n=1}^{k} a_n z^{n+1}}{1+\sum\limits_{n=1}^{k} a_n z^{n+1}} \\ &= \frac{1+A(z)}{1+B(z)}. \end{aligned}$$

Set $\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)}$, so that $w(z) = \frac{A(z)-B(z)}{2+A(z)+B(z)}$. Then

$$w(z) = \frac{\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{2+2\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}, \tag{10}$$

and

$$|w(z)| \leq \frac{\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n| - \left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}. \tag{11}$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum_{n=k+1}^{\infty}|a_n| \leq 2-2\sum_{n=1}^{k}|a_n|, \tag{12}$$

which is equivalent to

$$\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum_{n=k+1}^{\infty}|a_n| + \sum_{n=1}^{k}|a_n| \leq 1. \tag{13}$$

It suffices to show that the left hand side of (13) is bounded above by

$$\sum_{n=1}^{\infty}\frac{n+\alpha-\alpha\lambda(1+n)}{1-\alpha}|a_n|, \tag{14}$$

which is equivalent to

$$\sum_{n=1}^{k}\left(\frac{n+2\alpha-1-\alpha\lambda(1+n)}{1-\alpha}\right)|a_n| + \sum_{n=k+1}^{\infty}\left(\frac{n-k-1+\alpha\lambda(k-n+1)}{1-\alpha}\right)|a_n| \geq 0. \tag{15}$$

To see that the function f given by (9) gives the sharp result, we observe for $z = re^{\frac{\pi i}{(k+2)}}$ that

$$\begin{aligned}\frac{f(z)}{f_k(z)} &= 1+\frac{1-\alpha}{k+1+\alpha-\alpha\lambda(2+k)}z^{k+2} \to 1-\frac{1-\alpha}{k+1+\alpha-\alpha\lambda(2+k)}\\ &= \frac{k+2\alpha-\alpha\lambda(2+k)}{k+1+\alpha-\alpha\lambda(2+k)} when r \to 1^{-}.\end{aligned}$$

Therefore we complete the proof of Theorem 2.1.

Theorem 2.2 *Let the function $f(z)$ defined by (4) satisfies condition (6). Then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{k+1+\alpha-\alpha\lambda(2+k)}{2+k-\alpha\lambda(2+k)}. \tag{16}$$

Equalities hold in (16) for the functions given by (9).

Proof. We may write

$$\frac{2+k-\alpha\lambda(2+k)}{1-\alpha}\left[\frac{f_k(z)}{f(z)}-\frac{k+1+\alpha-\alpha\lambda(2+k)}{2+k-\alpha\lambda(2+k)}\right]$$

$$=\frac{1-\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}a_n\ z^{n+1}+\sum\limits_{n=1}^{k}a_n\ z^{n+1}}{1+\sum\limits_{n=1}^{\infty}a_n\ z^{n+1}}$$

$$=\frac{1+w(z)}{1-w(z)},$$

where

$$|w(z)|\leq\frac{\left(\frac{2+k-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n|-\left(\frac{2\alpha+k-\alpha\lambda(2+k)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}. \tag{17}$$

The last inequality is equivalent to

$$\left(\frac{k+1+\alpha-\alpha\lambda(2+k)}{1-\alpha}\right)\sum_{n=k+1}^{\infty}|a_n|+\sum_{n=1}^{k}|a_n|\leq 1. \tag{18}$$

Since the left hand side of (18) is bounded above by $\sum\limits_{n=1}^{\infty}\frac{n+\alpha-\alpha\lambda(1+n)}{1-\alpha}|a_n|$, the proof of Theorem 2.2 is completed.

We next turn to ratios involving derivatives.

Theorem 2.3 *Let the function $f(z)$ defined by (4) satisfies condition (6). Then*

$$\Re\left\{\frac{f'(z)}{f_k'(z)}\right\}\geq\frac{2(k+1)-\alpha(k+\lambda(2+k))}{k+1+\alpha-\alpha\lambda(2+k)}\quad(z\in\mathbb{U}), \tag{19}$$

$$\Re\left\{\frac{f_k'(z)}{f'(z)}\right\}\geq\frac{k+1+\alpha-\alpha\lambda(2+k)}{\alpha(k+2-\lambda(2+k))}\quad(z\in\mathbb{U};\alpha\neq 0). \tag{20}$$

The extremal function for the cases (19) and (20) is given by (9).

The proof of Theorem 2.3 follows the pattern of those in Theorems 2.1 and 2.2 so the details may be omitted.

Putting $\lambda=0$ in Theorem 2.3, we obtain the following corollary which corrects the result obtained by Cho and Owa [6, Theorem 2.4].

Corollary 2.4 *If f of the form (4) satisfies condition (6) with $\lambda=0$, then*

$$\Re\left\{\frac{f'(z)}{f_k'(z)}\right\}\geq\frac{2(k+1)-\alpha k}{k+1+\alpha}\quad(z\in\mathbb{U}), \tag{21}$$

$$\Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{k+1+\alpha}{\alpha(k+2)} \quad (z \in \mathbb{U}; \alpha \neq 0). \tag{22}$$

The extremal function for the case (21) is given by (9) (with $\lambda = 0$) and the extremal function for the case (22) is given by (9) (with $\lambda = 0$ and $\alpha \neq 0$).

Remarks. **(i)** Putting $\lambda = 0$ in Theorems 2.1, 2.2 and 2.3 we get the results obtained by Cho and Owa [6, Theorems 2.1, (a) of 2.3 and 2.4, respectively];

(ii) Putting $\lambda = 0$ in Theorems 2.1, 2.2 and 2.3 we get the results obtained by Aouf and Silverman [4, Theorems 2.1, (a) of 2.3 and 2.4 (with $p = 1$), respectively].

3 Open Problem

Is it possible to apply the idea of this paper on class Σ_p of meromorphic $p-$valent functions, where Σ_p denotes the class of analytic and $p-$valent functions in the punctured disc $\mathbb{U}^*$ of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{23}$$

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