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Double subordination-preserving properties of analytic functions associated with generalized transformations

Trailokya Panigrahi

Department of Mathematics
School of Applied Sciences
KIIT University, Bhubaneswar-751024, Odisha, India
e-mail: trailokyap6@gmail.com

Abstract

In the present paper, the author investigates some subordination and superordination -preserving properties for analytic functions associated with generalized multiplier transformations defined on the space of normalized analytic functions in the open unit disk $\mathbb{U}$. Several Sandwich-type results associated with this transformations is also derived.

Keywords: *Analytic functions, Univalent functions, Cãtaş operator, Differential subordination and superordination.*

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1 Introduction

Let $\mathcal{H} := \mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in the *open* unit disc

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, let

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}. \tag{1}$$

We denote by $\mathcal{A}$ the subclass of the functions $f \in \mathcal{H}[a,1]$ *normalized* with the conditions $f(0) = f'(0) - 1 = 0$ and $f^{p+1}(0) \neq 0 \ \ (p \in \mathbb{N})$. A function $f(z)$ in $\mathcal{A}$ is said to be *univalent* in $\mathbb{U}$ if $f(z)$ is one to one in $\mathbb{U}$.

Let the functions f and g be members of $\mathcal{H}$. We say that f is *subordinate* to g [12] and write

$$f \prec g \text{ in } \mathbb{U} \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U})$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

It follows from the Schwarz lemma that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then (see, e.g; [17])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

We often say that g is the subordinating function and f is the subordinated function. Or equivalently, g is the dominant and f is the subordinant in the subordination.

We need the following definitions for our present investigation:

Definition 1.1. *(see [12]) Let $\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination:*

$$\psi(p(z), zp'(z)) \prec h(z), \tag{2}$$

then p is called a solution of the differential subordination (2). A univalent function q is called a dominant of the solutions of the differential subordination (2) or more simply, a dominant if $p(z) \prec q(z)$ for all p satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants q of (2) is said to be the best dominant of (2).

Definition 1.2. *(see [13]) Let $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\phi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination:*

$$h(z) \prec \phi(p(z), zp'(z)), \tag{3}$$

then p is called a solution of the differential superordination (3). An analytic function q is called a subordinant of the solutions of the differential superordination (3) or more simply, a subordinant if $q(z) \prec p(z)$ for all p satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants q of (3) is said to be the best subordinant of (3).

Definition 1.3. *(see [12], Definition 2.2b, p. 21) We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\bar{\mathbb{U}} \setminus E(f)$, where*

$$E(f) = \{\xi : \xi \in \partial\mathbb{U} \quad and \quad \lim_{z \longrightarrow \xi} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(f)$.

Definition 1.4. *(see [13]) A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is called a subordination (or a Löwner) chain if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 < t_2$ and $z \in \mathbb{U}$.*

Let $\mathcal{A}$ denote the family of *normalized* functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{4}$$

which are analytic in $\mathbb{U}$. Let $f, g \in \mathcal{A}$, where $f(z)$ is defined by (4) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{5}$$

then the Hadamard product (or convolution) of f and g denoted by $f * g$ is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{6}$$

For any real numbers k and λ, Cãtaş [5] defined the multiplier transformation $I(\delta, \lambda, l)$ on $\mathcal{A}$ by the following infinite series:

$$\mathrm{I}(\delta, \lambda, l) f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + \lambda(n-1) + l}{1 + l}\right]^{\delta} a_n z^n \quad (n \in \mathbb{N},\ \delta,\ \lambda,\ l \geq 0;\ z \in \mathbb{U}). \tag{7}$$

Finding sufficient conditions using differential subordination for generalized multiplier transformation is an important topic of research in Geometric Function Theory. In recent years, several authors (see, for details, [6], [8]) have obtained various basic properties such as inclusion, subordination , superordination, convolution properties of the multiplier transformation defined by (7). Now using the convolution, we extend the multiplier transformation defined in (7) to more generalized class as follows:

Define

$$\phi^{\delta}(\lambda, l; z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + l}{1 + \lambda(n-1) + l}\right]^{\delta} z^n \quad (\delta,\ \lambda \in \mathbb{R},\ \delta,\ \lambda,\ l \geq 0;\ z \in \mathbb{U}). \tag{8}$$

Corresponding to the function $\phi^\delta(\lambda, l; z)$, we define the function $\phi^{\delta,\dagger}(\lambda, l; z)$, the generalized multiplicative inverse of $\phi^\delta(\lambda, 1; z)$ given by

$$\phi^\delta(\lambda, l; z) * \phi^{\delta,\dagger}(\lambda, l; z) = \frac{z}{(1-z)^\mu} \quad (\mu > 0). \tag{9}$$

If $\mu = 1$, the function $\phi^{\delta,\dagger}(\lambda, l; z)$ is the inverse of $\phi^\delta(\lambda, l; z)$ with respect to the Hadamard product. Using this function we define the family of transforms $\mathrm{I}^\delta(\lambda, \mu, l) : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\begin{aligned}\mathrm{I}^\delta(\lambda, \mu, l) f(z) &= \phi^{\delta,\dagger}(\lambda, l; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\mu)_{n-1}}{(1)_{n-1}} \left[\frac{1+\lambda(n-1)+l}{1+l}\right]^\delta a_n z^n \quad (\delta,\ \lambda,\ l \geq 0,\ \mu > 0;\ z \in \mathbb{U})\end{aligned} \tag{10}$$

where $(a)_n$ is the *Pochhammer symbol* (or the *shifted factorial*) defined (in terms of the familiar Gamma function) by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0) \\ a(a+1)\cdots(a+n-1) & (n \in \mathbb{N}). \end{cases}$$

The transformation $\mathrm{I}^\delta(\lambda, \mu, l)$ generalizes several previously studied familiar operators. The following are the some of the interesting particular cases:

- for $\delta = m$ ($m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), $\lambda \geq 0$, $l = 0$, $\mu = 1$, the operator $\mathrm{I}^m(\lambda, 1, 0) = D_\lambda^m$ was introduced and studied by Al-Oboudi [2];
- for $\delta = m$, $\lambda = 1$, $\mu = 1, l = 0$, the operator $\mathrm{I}^m(1, 1, 0) = D^m$ was introduced and studied in [16];
- for $\delta = m$, $\lambda = 1$, $\mu = 1$, the operator $\mathrm{I}^m(1, 1, l) = \mathcal{I}_l^m$ was studied by Cho and Srivastava [7];
- for $\delta = m$, $\lambda = \mu = l = 1$, the operator $\mathrm{I}^m(1, 1, 1) = \mathcal{I}_m$ was studied by Uralegaddi and Somanatha [18];
- for $\delta = m$, $\mu = 1$, $l = 0$, the operator $\mathrm{I}^m(\lambda, 1, 0) = D_\lambda^m$ was introduced and studied by Acu and Owa [1].

Note that

$$\mathrm{I}^0(1, 1, 0) f(z) = f(z) \quad \text{and} \quad \mathrm{I}^1(1, 1, 0) f(z) = z f'(z) \tag{11}$$

$$\mathrm{I}^{\delta+1}(1, 1, 0) f(z) = \mathrm{I}^\delta(1, 2, 0) f(z) = D^{\delta+1} f(z). \tag{12}$$

It can be easily shown from (10) that

$$(1+l)\mathrm{I}^{\delta+1}(\lambda,\mu,l)f(z)=(1-\lambda+l)\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)+\lambda z[\mathrm{I}^{\delta}(\lambda,\mu,l)f((z)]' \quad (13)$$

and

$$\mu\mathrm{I}^{\delta}(\lambda,\mu+1,l)f(z)=z[\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)]'+(\mu-1)\mathrm{I}^{\delta}(\lambda,\mu,l)f(z). \quad (14)$$

Recently, several authors obtained many interesting results involving various integral operators associated with differential subordination and superordination. For example, using the principle of subordination between analytic functions, Miller et al.[10] and Owa and Srivastava [14] investigated some subordination- preserving properties for certain integral operators while Bulboaca [3, 4] investigated the subordination as well as superordination- preserving properties of certain non linear integral operators.

Motivated by aforementioned work, in this paper the author obtains the subordination and superordination- preserving properties associated with the operator $\mathrm{I}^{\delta}(\lambda,\mu,l)$ defined in (10). Several Sandwich-type results involving this operator are also derived.

2 Preliminaries Lemmas

The proof of the theorems proceed through a number of steps, stated below as lemmas.

Lemma 2.1. *(see [9, 12]) Suppose that the function $H:\mathbb{C}^2\longrightarrow\mathbb{C}$ satisfies the condition:*

$$\Re H(is,t)\leq 0,$$

for all real s and $t\leq\frac{-n(1+s^2)}{2}$, where n is a positive integer. If the function $p(z)=1+p_nz^n+\cdots$ is analytic in $\mathbb{U}$ and

$$\Re H(p(z),zp'(z))>0 \quad (z\in\mathbb{U}),$$

then $\Re\{p(z)\}>0$ in $\mathbb{U}$.

Lemma 2.2. *(see [11]) Let $\beta,\gamma\in\mathbb{C}$ with $\beta\neq 0$, and let $h\in\mathcal{H}(\mathbb{U})$ with $h(0)=c$. If $\Re\{\beta h(z)+\gamma\}>0$ for $z\in\mathbb{U}$, then the solution of the differential equation:*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z) \quad (z\in\mathbb{U};\ q(0)=c)$$

is analytic in $\mathbb{U}$ and satisfies $\Re\{\beta q(z)+\gamma\}>0 \quad (z\in\mathbb{U})$.

Lemma 2.3. *(see [12]) Let $q \in \mathcal{Q}$ with $q(0) = a$ and let $p(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $p(z) \neq a$ and $n \geq 1$. If p is not subordinate to q, then there exists the points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(f)$ for which $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$,*

$$p(z_0) = q(\xi_0), \quad \text{and} \quad z_0 p'(z_0) = m\xi_0 q'(\xi_0) \quad (m \geq n \geq 1)$$

where $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Lemma 2.4. *(see [13]) Let $\mathcal{H}[a,1] = \{f \in \mathcal{H} : f(0) = a, f'(0) \neq 0\}$ and $q \in \mathcal{H}[a,1]$, $\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$. Also set $\psi(q(z), zq'(z)) \equiv h(z)$. If $L(z,t) = \psi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, then*

$$h(z) \prec \psi(p(z), zp'(z)) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

Furthermore, if the differential equation $\psi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

Lemma 2.5. *(see [15]) The function $L(z,t) : \mathbb{U} \times [0, \infty) \longrightarrow \mathbb{C}$ of the form*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$$

with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \longrightarrow \infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\Re\left[\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] > 0 \quad (z \in \mathbb{U}; 0 \leq t < \infty).$$

3 Main Results

Theorem 3.1 contains subordination results for the integral operator $\mathrm{I}^{\delta}(\lambda, \mu, l)$ defined by equation (10).

Theorem 3.1. *Let $f, g \in \mathcal{A}$ and Suppose that*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\eta \quad (z \in \mathbb{U}) \tag{15}$$

where

$$\phi(z) := \left[\frac{\mathrm{I}^{\delta+1}(\lambda, \mu, l)g(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}{z}\right]^{\alpha} \tag{16}$$

and η is given by

$$\eta = \frac{\lambda^2 + \alpha^2(1+l)^2 - |\lambda^2 - \alpha^2(1+l)^2|}{4\alpha\lambda(1+l)} \quad (\alpha > 0, l \geq 0, \lambda > 0). \tag{17}$$

Then the subordination condition:

$$\left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)f(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)g(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \tag{18}$$

implies that

$$\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}. \tag{19}$$

Moreover, the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}$ *is the best dominant.*

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by

$$F(z) := \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \quad \text{and} \quad G(z) := \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \quad (z \in \mathbb{U}) \tag{20}$$

respectively. Now, we show that, if the function $q(z)$ is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{21}$$

then

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Taking the logarithmic differentiation on both sides of the second equation in (20) and using the identity (13) for $g \in \mathcal{A}$ in the resulting equation, we get

$$\phi(z) = G(z) + \frac{\lambda z G'(z)}{\alpha(1+l)} \tag{22}$$

where the function $\phi(z)$ is defined in (16).
Differentiating both sides of (22) with respect to z gives

$$\phi'(z) = \left(1 + \frac{\lambda}{\alpha(1+l)}\right) G'(z) + \frac{\lambda}{\alpha(1+l)} z G'(z). \tag{23}$$

From (21) and (23) after simplification yields

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} =& 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + \frac{\alpha(1+l)}{\lambda}} \\ =& q(z) + \frac{zq'(z)}{q(z) + \frac{\alpha(1+l)}{\lambda}} \equiv h(z) \quad (z \in \mathbb{U}). \end{aligned} \tag{24}$$

Therefore, it follows from (15) and (24) that

$$\Re\left\{h(z) + \frac{\alpha(1+l)}{\lambda}\right\} > 0 \quad (z \in \mathbb{U}). \tag{25}$$

Hence by Lemma 2.2 we deduce that the differential equation (24) has a solution $q \in \mathcal{H}(\mathbb{U})$ with $h(0) = q(0) = 1$.
Let us define the function

$$\mathcal{H}(u, v) = u + \frac{v}{u + \frac{\alpha(1+l)}{\lambda}} + \eta, \tag{26}$$

where η is given by (17). From (15), (24) and (26) it follows that

$$\Re\{\mathcal{H}\left(q(z), zq'(z)\right)\} > 0 \quad (z \in \mathbb{U}).$$

Now we proceed to prove that

$$\Re\{\mathcal{H}(is, t)\} \leq 0 \ \ \textit{for all real} \ \ s \ \ \textit{and} \ \ t \leq -\frac{(1+s^2)}{2}. \tag{27}$$

From (26), we have

$$\begin{aligned}\Re\{\mathcal{H}(is, t)\} &= \Re\left\{is + \frac{t}{is + \frac{\alpha(1+l)}{\lambda}} + \eta\right\} \\ &= \frac{t\left(\frac{\alpha(1+l)}{\lambda}\right)}{\left|is + \frac{\alpha(1+l)}{\lambda}\right|^2} + \eta \\ &\leq -\frac{\psi_\eta(s)}{2\left|is + \frac{\alpha(1+l)}{\lambda}\right|^2}\end{aligned} \tag{28}$$

where

$$\psi_\eta(s) = \left(\frac{\alpha(1+l)}{\lambda} - 2\eta\right)s^2 - \frac{\alpha(1+l)}{\lambda}\left(2\eta\frac{\alpha(1+l)}{\lambda} - 1\right). \tag{29}$$

For η given by (17), we observe that the coefficient of s^2 in the quadratic expression $\psi_\eta(s)$ given by (29) is positive or equal to zero. Moreover, the discriminant Δ of $\psi_\eta(s)$ in (29) is given by

$$\frac{\Delta}{4} = -4\frac{\alpha(1+l)}{\lambda}\eta^2 + 2\left[1 + \frac{\alpha^2(1+l)^2}{\lambda^2}\right]\eta - \frac{\alpha(1+\eta)}{\lambda}$$

which, for assumed value of η given by (17) gives

$$\Delta = 0,$$

and so that the quadratic expression for s in $\psi_\eta(s)$ given by (29) is a perfect square. Therefore, it follows from (28) that

$$\Re\{\mathcal{H}(is, t)\} \leq 0 \quad \left(s \in \mathbb{R}, t \leq -\frac{1+s^2}{2}\right). \tag{30}$$

Thus by application of Lemma 2.1, we conclude that

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

That is the function $G(z)$ defined by (20) is convex (univalent) in $\mathbb{U}$.
Next, we will prove that the subordination condition (18) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{31}$$

for the functions F and G defined by (20). Without loss of generality, we can assume that the function $G(z)$ is analytic, univalent on $\bar{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. Otherwise, we replace F and G by $F_r(z) = F(rz)$ and $G_r(z) = G(rz)$ respectively for $r \in (0,1)$. Then these new functions satisfy the conditions of the theorem on $\bar{\mathbb{U}}$. Thus we need to prove that $F_r(z) \prec G_r(z)$ for all $r \in (0,1)$, which the result (31) follows by letting $r \longrightarrow 1^-$.
To prove (31), let us define the function $L(z,t)$ by

$$L(z,t) := G(z) + \frac{1+t}{\frac{\alpha(1+l)}{\lambda}} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{32}$$

Since G is convex in $\mathbb{U}$ and $\frac{\alpha(1+l)}{\lambda} > 0$, we obtain

$$\frac{\partial L(z,t)}{\partial z}\Big|_{z=0} = G'(0)\left[1 + \frac{1+t}{\frac{\alpha(1+l)}{\lambda}}\right] = \left[1 + \frac{1+t}{\frac{\alpha(1+l)}{\lambda}}\right] \neq 0 \quad (z \in \mathbb{U};\quad 0 \leq t < \infty).$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots$$

satisfies the conditions $a_1(t) \neq 0$ for all $t \in [0,\infty)$ and $lim_{t \longrightarrow \infty}|a_1(t)| = \infty$. Furthermore,

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \Re\left\{\frac{\alpha(1+l)}{\lambda} + (1+t)q(z)\right\} > 0 \quad (z \in \mathbb{U}).$$

Thus, by virtue of Lemma 2.5, $L(z,t)$ is a subordination chain. Hence, it follows from Definition 1.4 that

$$\phi(z) = G(z) + \frac{zG'(z)}{\frac{\alpha(1+l)}{\lambda}} = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

This implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U}; 0 \leq t < \infty). \tag{33}$$

Now suppose that the function F is not subordinate to G, then by Lemma 2.3 there exists two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty). \tag{34}$$

Hence, we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + (1+t)\frac{\zeta_0 G'(\zeta_0)}{\frac{\alpha(1+l)}{\lambda}} \\ &= F(z_0) + \frac{z_0 F'(z_0)}{\frac{\alpha(1+l)}{\lambda}} \\ &= \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z_0)}{z_0}\right]^{\alpha}\left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)f(z_0)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z_0)}\right] \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (18). This contradicts to (33). Thus, the subordination condition (18) must imply the subordination given by (31). Considering $F = G$, we see that the function $G(z)$ is the best dominant. Thus, the prove of Theorem 3.1 is completed.

Letting $\delta = m,\ \lambda = \mu = 1$ in Theorem 3.1 we obtain the following result:

Corollary 3.2. *Let $f, g \in \mathcal{A}$ and suppose that*

$$\Re\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > -\eta_1, \tag{35}$$

where

$$\phi(z) = \left[\frac{\mathrm{I}_l^{m+1}g(z)}{\mathrm{I}_l^{m}g(z)}\right]\left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha} \tag{36}$$

and η_1 is given by

$$\eta_1 = \frac{1 + \alpha^2(1+l)^2 - |1 - \alpha^2(1+l)^2|}{4\alpha(1+l)}. \tag{37}$$

Then the subordination condition:

$$\left[\frac{\mathrm{I}_l^{m+1}f(z)}{\mathrm{I}_l^{m}f(z)}\right]\left[\frac{\mathrm{I}_l^{m}f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}_l^{m+1}g(z)}{\mathrm{I}_l^{m}g(z)}\right]\left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha} \tag{38}$$

implies

$$\left[\frac{\mathrm{I}_l^{m}f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha} \tag{39}$$

and the function $\left(\frac{\mathrm{I}_l^{m}g(z)}{z}\right)^{\alpha}$ is the best dominant.

Taking $l = 0$ in Corollary 3.2, we have

Corollary 3.3. *Let $f, g \in \mathcal{A}$ and suppose that*

$$\Re\left(1+\frac{z\phi''(z)}{\phi'(z)}\right) > -\eta_2, \tag{40}$$

where

$$\phi(z) = \left[\frac{\mathrm{D}^{m+1}g(z)}{\mathrm{D}^m g(z)}\right]\left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha \tag{41}$$

and η_2 is given by

$$\eta_2 = \frac{1+\alpha^2-|1-\alpha^2|}{4\alpha}. \tag{42}$$

Then the subordination condition

$$\left[\frac{\mathrm{D}^{m+1}f(z)}{\mathrm{D}^m f(z)}\right]\left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^{m+1}g(z)}{\mathrm{D}^m g(z)}\right]\left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha \tag{43}$$

implies

$$\left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha \tag{44}$$

and the function $\left(\frac{\mathrm{D}^m g(z)}{z}\right)^\alpha$ is the best dominant.

Putting $\delta = m = 0$ in Corollary 3.3 and using the relation (11) we get the following results:

Corollary 3.4. *Let $f, g \in \mathcal{A}$ and suppose that*

$$\Re\left(1+\frac{z\phi''(z)}{\phi'(z)}\right) > -\eta_3, \tag{45}$$

where

$$\phi(z) = \left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha \tag{46}$$

and η_3 is given by

$$\eta_3 = \frac{1+\alpha^2-|1-\alpha^2|}{4\alpha}. \tag{47}$$

Then the subordination condition:

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\alpha \prec \left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha \tag{48}$$

implies

$$\left[\frac{f(z)}{z}\right]^\alpha \prec \left[\frac{g(z)}{z}\right]^\alpha \tag{49}$$

and the function $\left(\frac{g(z)}{z}\right)^\alpha$ is the best dominant.

By employing the same technique as in the proof of Theorem 3.1 and using the identity (14) instead of (13), we obtain the following theorem.

Theorem 3.5. *Let $f, g \in \mathcal{A}$ and $\mu > 0$. Suppose that*

$$\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\sigma_1 \tag{50}$$

where

$$\psi(z) = \left[\frac{\mathrm{I}^{\delta}(\lambda, \mu+1, l)g(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}{z}\right]^{\alpha} \tag{51}$$

and σ_1 is given by

$$\sigma_1 = \frac{1 + \alpha^2\mu^2 - |1 - \alpha^2\mu^2|}{4\alpha\mu} \quad (\alpha > 0). \tag{52}$$

Then, the subordination condition:

$$\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu+1, l)f(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda, \mu+1, l)g(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}{z}\right]^{\alpha}, \tag{53}$$

implies that

$$\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}{z}\right]^{\alpha}. \tag{54}$$

Moreover, the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}$ is the best dominant.

Putting $\delta = m,\ \mu = \lambda = 1,\ l = 0$ in Theorem 3.5 and using the relation (12) we obtain the result of Corollary 3.3.

As we know if f is subordinate to h, then h is superordinate to f. Now we investigate a dual problem regarding Theorem 3.1 in the sense that the subordinations are replaced by superordinations.

Theorem 3.6. *Let $f, g \in \mathcal{A}$ and $\lambda > 0$. Suppose that*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\eta \tag{55}$$

where

$$\phi(z) = \left[\frac{\mathrm{I}^{\delta+1}(\lambda, \mu, l)g(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}{z}\right]^{\alpha} \tag{56}$$

and η is given by (17). If the function $\left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)g(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \in \mathcal{Q}$, then the superordination condition:

$$\left[\frac{\mathrm{I}^{\delta+1}(\lambda, \mu, l)g(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)g(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta+1}(\lambda, \mu, l)f(z)}{\mathrm{I}^{\delta}(\lambda, \mu, l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda, \mu, l)f(z)}{z}\right]^{\alpha}, \tag{57}$$

implies that

$$\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha}. \tag{58}$$

Moreover, the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}$ *is the best subordinant.*

Proof. The proof of the theorem follows the same lines as that of Theorem 3.1. We will give only main steps.

Let the functions F, G and q are defined by (20) and (21) respectively. As in the proof of Theorem 3.1, we have

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

That is G is defined by (20) is convex (univalent) in $\mathbb{U}$. Next, to arrive at our desired result, we show that $G \prec F$. For this purpose, we defined the function $L(z,t)$ as (32).

Since G is convex and $\frac{\alpha(1+l)}{\lambda} > 0$, by applying a similar method as in Theorem 3.1 we deduce that $L(z,t)$ is a subordination chain. Therefore, by using Lemma 2.4, we conclude that the superordination condition (57) must imply the superordination $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{zG'(z)}{\frac{\alpha(1+l)}{\lambda}} = \varphi(G(z), zG'(z))$$

has a univalent solution G, it is the best subordinant of the given differential superordination. This completes the proof of Theorem 3.6.

Letting $\delta = m,\ \lambda = \mu = 1$ in Theorem 3.6 we obtain the following:

Corollary 3.7. *Let $f, g \in \mathcal{A}$ and suppose that*

$$\Re\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > -\eta_1,$$

where

$$\phi(z) = \left[\frac{\mathrm{I}_l^{m+1}g(z)}{\mathrm{I}_l^{m}g(z)}\right]\left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha}$$

and η_1 is given by (37). If the function $\left[\frac{\mathrm{I}_l^{m+1}g(z)}{\mathrm{I}_l^{m}g(z)}\right]\left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha}$ *is univalent in $\mathbb{U}$ and* $\left[\frac{\mathcal{I}_l^{m}g(z)}{z}\right]^{\alpha} \in \mathcal{Q}$, *then the superordination condition:*

$$\phi(z) = \left[\frac{\mathrm{I}_l^{m+1}g(z)}{\mathrm{I}_l^{m}g(z)}\right]\left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}_l^{m+1}f(z)}{\mathrm{I}_l^{m}f(z)}\right]\left[\frac{\mathrm{I}_l^{m}f(z)}{z}\right]^{\alpha}$$

implies

$$\left[\frac{\mathrm{I}_l^{m}g(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}_l^{m}f(z)}{z}\right]^{\alpha}$$

and the function $\left(\frac{\mathrm{I}_l^{m}g(z)}{z}\right)^{\alpha}$ *is the best subordinant.*

Putting $l = 0$ in Corollary 3.7 we get

Corollary 3.8. *Let $f, g \in \mathcal{A}$ and suppose that*

$$\Re\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > -\eta_2,$$

where

$$\phi(z) = \left[\frac{\mathrm{D}^{m+1}g(z)}{\mathrm{D}^m g(z)}\right]\left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha$$

and η_2 is given by (42). If the function $\left[\frac{\mathrm{D}^{m+1}g(z)}{\mathrm{D}^m g(z)}\right]\left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathcal{D}^m g(z)}{z}\right]^\alpha \in \mathcal{Q}$, then the superordination condition:

$$\phi(z) = \left[\frac{\mathrm{D}^{m+1}g(z)}{\mathrm{D}^m g(z)}\right]\left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^{m+1}f(z)}{\mathrm{D}^m f(z)}\right]\left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha$$

implies

$$\left[\frac{\mathrm{D}^m g(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha$$

and the function $\left(\frac{\mathrm{D}^m g(z)}{z}\right)^\alpha$ is the best subordinant.

Letting $\delta = m = 0$ in Corollary 3.8 and using the relation (11) we get the following results:

Corollary 3.9. *Let $f, g \in \mathcal{A}$ and suppose that*

$$\Re\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > -\eta_3,$$

where

$$\phi(z) = \left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha$$

and η_3 is given by (47). If the function $\left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{g(z)}{z}\right]^\alpha \in \mathcal{Q}$, then the superordination condition:

$$\left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha \prec \left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\alpha$$

implies

$$\left[\frac{g(z)}{z}\right]^\alpha \prec \left[\frac{f(z)}{z}\right]^\alpha$$

and the function $\left(\frac{g(z)}{z}\right)^\alpha$ is the best subordinant.

By using the same technique as in the proof of Theorem 3.6 and using the identity (14) instead of (13), we obtain the following:

Theorem 3.10. *Let $f, g \in \mathcal{A}$ and $\mu > 0$. Suppose that*

$$\Re\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\sigma_1 \tag{59}$$

where

$$\psi(z) = \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)g(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \tag{60}$$

and σ_1 is given by (52). If the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)g(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \in \mathcal{Q}$, then the superordination condition:

$$\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)g(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)f(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha}, \tag{61}$$

implies that

$$\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha}. \tag{62}$$

Moreover, the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g(z)}{z}\right]^{\alpha}$ is the best subordinant.

Combining Theorems 3.1, 3.6 and Theorems 3.5, 3.10, we obtain the following "Sandwich-type theorems".

Theorem 3.11. *Let $f, g_k \in \mathcal{A}$ $(k = 1, 2)$ and suppose that*

$$\Re\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\eta \tag{63}$$

where

$$\phi_k(z) = \left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)g_k(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g_k(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_k(z)}{z}\right]^{\alpha} \quad (\mu,\ \lambda,\ \alpha > 0, l \geq 0; z \in \mathbb{U}) \tag{64}$$

and η is given by (17). If the function $\left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)f(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha}$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \in \mathcal{Q}$, then the condition:

$$\begin{aligned}\left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)g_1(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}{z}\right]^{\alpha} &\prec \left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)f(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \\ &\prec \left[\frac{\mathrm{I}^{\delta+1}(\lambda,\mu,l)g_2(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}{z}\right]^{\alpha}\end{aligned} \tag{65}$$

implies that

$$\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}{z}\right]^{\alpha}. \tag{66}$$

Moreover, the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}{z}\right]^{\alpha}$ *and* $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}{z}\right]^{\alpha}$ *are respectively the best subordinant and the best dominant.*

Theorem 3.12. *Let* $f, g_k \in \mathcal{A}\ \ (k = 1, 2)$ *and* $\mu > 0$. *Suppose that*

$$\Re\left\{1 + \frac{z\psi_k''(z)}{\psi_k'(z)}\right\} > -\sigma_1 \tag{67}$$

where

$$\psi_k(z) = \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)g_k(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g_k(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_k(z)}{z}\right]^{\alpha} \quad (\lambda,\ \alpha > 0, l \geq 0; z \in \mathbb{U}) \tag{68}$$

and σ_1 *is given by (52). If the function* $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)f(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha}$ *is univalent in* $\mathbb{U}$ *and* $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \in \mathcal{Q}$, *then the condition:*

$$\begin{aligned}\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)g_1(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}{z}\right]^{\alpha} &\prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)f(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \\ &\prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu+1,l)g_2(z)}{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}\right]\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}{z}\right]^{\alpha}\end{aligned} \tag{69}$$

implies that

$$\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}{z}\right]^{\alpha}. \tag{70}$$

Moreover, the function $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_1(z)}{z}\right]^{\alpha}$ *and* $\left[\frac{\mathrm{I}^{\delta}(\lambda,\mu,l)g_2(z)}{z}\right]^{\alpha}$ *are respectively the best subordinant and the best dominant.*

By taking $\delta = m$ and $\lambda = \mu = 1$ in Theorem **??** we obtain the following corollary.

Corollary 3.13. *Let* $f, g_k \in \mathcal{A}\ \ (k = 1, 2)$ *and suppose that*

$$\Re\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\eta_1$$

where

$$\phi_k(z) = \left[\frac{\mathrm{I}_l^{m+1}g_k(z)}{\mathrm{I}_l^{m}g_k(z)}\right]\left[\frac{\mathrm{I}_l^{m}g_k(z)}{z}\right]^{\alpha} \quad (z \in \mathbb{U})$$

and η_1 *is given by (37). If the function* $\left[\frac{\mathrm{I}_l^{m+1}f(z)}{\mathrm{I}_l^m f(z)}\right]\left[\frac{\mathrm{I}_l^m f(z)}{z}\right]^\alpha$ *is univalent in* $\mathbb{U}$ *and* $\left[\frac{\mathrm{I}_l^m f(z)}{z}\right]^\alpha \in \mathcal{Q}$*, then the condition:*

$$\left[\frac{\mathrm{I}_l^{m+1}g_1(z)}{\mathrm{I}_l^m g_1(z)}\right]\left[\frac{\mathrm{I}_l^m g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{I}_l^{m+1}f(z)}{\mathrm{I}_l^m f(z)}\right]\left[\frac{\mathrm{I}_l^m f(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{I}_l^{m+1}g_2(z)}{\mathrm{I}_l^m g_2(z)}\right]\left[\frac{\mathrm{I}_l^m g_2(z)}{z}\right]^\alpha$$

implies that

$$\left[\frac{\mathrm{I}_l^m g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{I}_l^m f(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{I}_l^m g_2(z)}{z}\right]^\alpha,$$

and the function $\left[\frac{\mathrm{I}_l^m g_1(z)}{z}\right]^\alpha$ *and* $\left[\frac{\mathrm{I}_l^m g_2(z)}{z}\right]^\alpha$ *are respectively the best subordinant and the best dominant.*

By putting $l = 0$ in Corollary 3.13, we have:

Corollary 3.14. *Let* $f, g_k \in \mathcal{A}$ $(k = 1, 2)$ *and suppose that*

$$\Re\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\eta_2$$

where

$$\phi_k(z) = \left[\frac{\mathrm{D}^{m+1}g_k(z)}{\mathrm{D}^m g_k(z)}\right]\left[\frac{\mathrm{D}^m g_k(z)}{z}\right]^\alpha \quad (z \in \mathbb{U})$$

and η_2 *is given by (42). If the function* $\left[\frac{\mathrm{D}^{m+1}f(z)}{\mathrm{D}^m f(z)}\right]\left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha$ *is univalent in* $\mathbb{U}$ *and* $\left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha \in \mathcal{Q}$*, then the condition:*

$$\left[\frac{\mathrm{D}^{m+1}g_1(z)}{\mathrm{D}^m g_1(z)}\right]\left[\frac{\mathrm{D}^m g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^{m+1}f(z)}{\mathrm{D}^m f(z)}\right]\left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^{m+1}g_2(z)}{\mathrm{D}^m g_2(z)}\right]\left[\frac{\mathrm{D}^m g_2(z)}{z}\right]^\alpha$$

implies that

$$\left[\frac{\mathrm{D}^m g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^m f(z)}{z}\right]^\alpha \prec \left[\frac{\mathrm{D}^m g_2(z)}{z}\right]^\alpha,$$

and the function $\left[\frac{\mathrm{D}^m g_1(z)}{z}\right]^\alpha$ *and* $\left[\frac{\mathrm{D}^m g_2(z)}{z}\right]^\alpha$ *are respectively the best subordinant and the best dominant.*

Further, by letting $\delta = m = 0$ in Corollary 3.14 and using the relation (11) we get the following result:

Corollary 3.15. *Let* $f, g_k \in \mathcal{A}$ $(k = 1, 2)$ *and suppose that*

$$\Re\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\eta_3$$

where

$$\phi_k(z) = \left[\frac{zg_k'(z)}{g_k(z)}\right]\left[\frac{g_k(z)}{z}\right]^{\alpha} \quad (z \in \mathbb{U})$$

and η_3 *is given by (47). If the function* $\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^{\alpha}$ *is univalent in* $\mathbb{U}$ *and* $\left[\frac{f(z)}{z}\right]^{\alpha} \in \mathcal{Q}$*, then the condition:*

$$\left[\frac{zg_1'(z)}{g_1(z)}\right]\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec \left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^{\alpha} \prec \left[\frac{zg_2'(z)}{g_2(z)}\right]\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies that

$$\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec \left[\frac{f(z)}{z}\right]^{\alpha} \prec \left[\frac{g_2(z)}{z}\right]^{\alpha},$$

and the function $\left[\frac{g_1(z)}{z}\right]^{\alpha}$ *and* $\left[\frac{g_2(z)}{z}\right]^{\alpha}$ *are respectively the best subordinant and the best dominant.*

4 Open Problem

The author suggests to introduce different operator on the function and study the above results in the context of the modified operator.

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