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Regularity of solutions of convolution equations related to the Jacobi-Cherednik operator on $\mathbb{R}$

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Abstract

In this paper we define the Hypergeometric Fourier transform and convolution product of distributions associated with the Jacobi-Cherednik operator on $\mathbb{R}$. By using these results we study the regularity of solutions of convolution equations on $\mathbb{R}$.

Keywords : *Jacobi-Cherednik operator, Hypergeometric Fourier transform of distributions, Jacobi-Cherednik convolution product of distributions.*
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1 Introduction

We consider a root system $\mathcal{R}$ in $\mathbb{R}^d$, $\mathcal{R}_+$ a fixed positive subsystem and k a non-negative multiplicity function defined on $\mathcal{R}$, the Cherednik operators $T_j, j = 1, 2, .., d$, (see [2]), are defined for f of class $\mathcal{C}^1(\mathbb{R}^d)$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \alpha_j}{1 - e^{-\langle \alpha, x \rangle}} (f(x) - f(\sigma_\alpha(x))) - \rho f(x),$$

where $\langle ., . \rangle$ is the usual scalar product, σ_α is the orthogonal reflection in the hyperplane orthogonal to α, $\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k_\alpha \alpha$ and k is invariant by the finite

reflection group W generated by the reflections $\sigma_\alpha, \alpha \in \mathcal{R}$.
For $d = 1$ and $W = \mathbb{Z}_2$ the root system is $\mathcal{R} = \{-2\alpha, \alpha, \alpha, 2\alpha\}$, with α the positive root. We will take the normalization $\alpha = 2$. The positive root system is $\mathcal{R}_+ = \{\alpha, 2\alpha\}$. The Cherednik operator is defined for f in $\mathcal{C}^1(\mathbb{R})$ by

$$T^{(k,k')}f(x) = f'(x) + (\frac{2k_\alpha}{1-e_{-2x}} + \frac{4k_{2\alpha}}{1-e^{-4x}})\{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k_\alpha + k_{2\alpha}$.
It is also equal to

$$T^{(k,k')}f(x) = f'(x)+((k_\alpha+k_{2\alpha})\coth(x)+k_{2\alpha}\tanh(x))\{f(x)-f(-x)\}-\rho f(-x), \tag{1}$$

with $k_\alpha \geq 0$ and $k_{2\alpha} \geq 0$. More precisely there are three cases

- 1st case : $k_\alpha > 0$, $k_{2\alpha} = 0$, which corresponds to the positive root system $\mathcal{R}_+ = \{\alpha\}$ (the reduced case).
- 2nd case : $k_\alpha = 0$, $k_{2\alpha} > 0$, which corresponds to the positive root system $\mathcal{R}_+ = \{2\alpha\}$.
- 3rd case : $k_\alpha > 0$, $k_{2\alpha} > 0$, which corresponds to the positive root system $\mathcal{R}_+ = \{\alpha, 2\alpha\}$.

The operator (1) is a particular case of the operator

$$T^{(k,k')}f(x) = f'(x) + (a_0\coth(x) + b_0\tanh(x))\{f(x) - f(-x)\} - \rho f(-x),$$

with $a_0 \geq 0, b_0 \geq 0$ and $\rho = a_0 + b_0$.
This operator is called the Jacobi-Cherednik operator.

In this paper, we recall the main results of the Harmonic Analysis associated with the Jacobi-Cherednik operator $T^{(k,k')}$, we introduce the Hypergeometric Fourier transform of distributions and we study the Jacobi-Cherednik convolution product on space of distributions (see [8,10]).
These results have permitted to study the regularity of the solution $\mathcal{U}$ of the convolution equation on the form

$$\mathcal{V} *_{k,k'} \mathcal{U} = \mathcal{W},$$

where $\mathcal{V}$ in $\mathcal{E}'(\mathbb{R})$ and $\mathcal{U}$,$\mathcal{W}$ in $\mathcal{D}'(\mathbb{R})$. More precisely, we will give a condition on the Hypergeometric Fourier transform of the distribution $\mathcal{V}$ in order that the distribution $\mathcal{U}$ will be given by a function $f\mathcal{A}$, $f \in \mathcal{E}(\mathbb{R})$, whenever $\mathcal{W}$ is given by a function $g\mathcal{A}$, g in $\mathcal{E}(\mathbb{R})$, where $\mathcal{A}$ is the weight function associated to the operator $T^{(k,k')}$ and $\mathcal{E}(\mathbb{R})$ is the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}$. In the case of the classical Fourier transform on $\mathbb{R}^d$, this regularity was first studied by L. Ehrenpreis (see [3]) and next by L.Hörmander (see [5]). In [7,9] the authors have studied this regularity in the cases of the Dunkl operators on $\mathbb{R}^d$ and the Jacobi-Dunkl operator on $\mathbb{R}$.

2 Preliminaries

The Jacobi-Cherednik operator is a differential-difference operator $T^{(k,k')}$, defined on $\mathbb{R}$ by

$$T^{(k,k')}f(x) = f'(x) + (k\coth(x) + k'\tanh(x))\frac{f(x)-f(-x)}{2} - \rho f(-x),$$

with $\rho = k + k'$.
We denote by $G_\lambda^{(k,k')}, \lambda \in \mathbb{C}$, the unique solution on $\mathbb{R}$ of the differential-difference equation

$$\begin{cases} T^{(k,k')}u(x) &= -i\lambda u(x) \\ u(0) &= 1 \end{cases}. \tag{1.1}$$

It is called the Jacobi-Cherednik kernel and is given by

$$\forall x \in \mathbb{R},\ G_\lambda^{(k,k')}(x) = \varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x) - \frac{1}{\rho - i\lambda}\frac{d}{dx}\varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x), \tag{1.2}$$

where $\varphi_\lambda^{(\alpha,\beta)}$ is the Jacobi function defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; \sinh^2 x),$$

with ${}_2F_1$ is the Gauss hypergeometric function (see [6]).
The function $G_\lambda^{(k,k')}$ is multiplicative on $\mathbb{R}$ in the sense

$$\forall x, y \in \mathbb{R},\ G_\lambda^{(k,k')}(x)G_\lambda^{(k,k')}(y) = \int_{\mathbb{R}} G_\lambda^{(k,k')}(z)d\mu_{x,y}^{(k,k')}(z), \tag{1.3}$$

where,

$$d\mu_{x,y}^{(k,k')}(z) = \begin{cases} \mathcal{K}_{k,k'}(x,y,z)\mathcal{A}(z)dz, & if\ xy \neq 0 \\ d\delta_x(z), & if\ y = 0 \\ d\delta_y(z), & if\ x = 0 \end{cases}, \tag{1.4}$$

with $d\delta_x$ the Dirac measure at x, $\mathcal{A}$ the function defined by

$$\mathcal{A}(z) = \mathcal{A}_{k-\frac{1}{2},k'-\frac{1}{2}}(z) = (\sinh|z|)^{2k}(\cosh|z|)^{2k'}, \tag{1.5}$$

and $\mathcal{K}_{k,k'}$ a continuous function on $]-|x|-|y|, -||x|-|y||[\cup]||x|-|y||, |x|+|y|[$, with support in $I_{xy} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$, defined by

$$\mathcal{K}_{k,k'}(x,y,z) = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k-k')\Gamma(k')}|\sinh x \sinh y \sinh z|^{-2k+1} \times$$

$$\int_0^\pi (1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos\chi)^{k-k'-1} \times$$

$$[1-\sigma^{\chi}_{x,y,z}+\sigma^{\chi}_{x,z,y}+\sigma^{\chi}_{z,y,x}+\frac{\rho}{k'}\coth x\coth y\coth z(\sin\chi)^2](\sin\chi)^{2k'-1}d\chi,$$

if $x, y, z \in \mathbb{R}\backslash\{0\}$ satisfying the triangular inequality $||x|-|y|| < |z| < |x|+|y|$, and $\mathcal{K}_{k,k'}(x, y, z) = 0$ otherwise and $\sigma^{\chi}_{x,y,z}$ is the function given, for $x, y, z \in \mathbb{R}$ and $\chi \in [0, \pi]$, by

$$\sigma^{\chi}_{x,y,z} = \begin{cases} \dfrac{\cosh x\cosh y - \cosh z\cos\chi}{\sinh x\sinh y}, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}.$$

(For more details refer to [1]).
The function $G^{(k,k')}_{\lambda}$ has the following Laplace type integral representation

$$\forall x \in \mathbb{R}\backslash\{0\},\ G^{(k,k')}_{\lambda}(x) = \int_{-|x|}^{|x|} K_{k,k'}(x, y)e^{-i\lambda y}dy, \tag{1.6}$$

where $K_{k,k'}(x, y)$ is the function given by

$$K_{k,k'}(x, y) = \frac{2^{k+k'-1}\Gamma(k-k'+\frac{1}{2})}{\sqrt{\pi}\Gamma(k-k')}(\sinh x)^{2(k'-k)}(\cosh x-\cosh y)^{k+k'-1}sg(x)(e^x-e^y). \tag{1.7}$$

Moreover, the function $G^{(k,k')}_{\lambda}$ satisfies

$$\forall x \in \mathbb{R}, \forall\lambda \in \mathbb{C},\ \overline{G}^{(k,k')}_{\lambda}(x) = G^{(k,k')}_{-\overline{\lambda}}(x). \tag{1.8}$$

$$\forall x \in \mathbb{R}, \forall\lambda \in \mathbb{R},\ |G^{(k,k')}_{\lambda}(x)| \leq M_0, \tag{1.9}$$

where M_0 is a positive constant.

Notations. We denote by :
- $\mathcal{E}(\mathbb{R})$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}$.
- $\mathcal{D}(\mathbb{R})$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}$ with compact support.
- $PW(\mathbb{C})$ the space of entire functions on $\mathbb{C}$ which are of exponential type and rapidly decreasing.

We provide these spaces with the classical topology.
We consider also the following spaces.
- $\mathcal{D}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$.
- $\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support.

The Jacobi-Cherednik intertwining operator $V_{k,k'}$ is defined on $\mathcal{E}(\mathbb{R})$ by

$$\forall x \in \mathbb{R}\backslash\{0\},\ V_{k,k'}(g)(x) = \int_{-|x|}^{|x|} K_{k,k'}(x, y)g(y)dy, \tag{1.10}$$

where the function $K_{k,k'}(x,y)$ is given by relation (1.7).
The operator $V_{k,k'}$ is a topological isomorphism from $\mathcal{E}(\mathbb{R})$ onto itself satisfying the transmutation relation

$$\forall x \in \mathbb{R},\ T^{(k,k')}V_{k,k'}(g)(x) = V_{k,k'}(\frac{d}{dx}g)(x),\ g \in \mathcal{E}(\mathbb{R}). \qquad (1.11)$$

The dual ${}^tV_{k,k'}$ of the Jacobi-Cherednik intertwining operator $V_{k,k'}$ is defined on $\mathcal{D}(\mathbb{R})$ by

$$\int_{\mathbb{R}} {}^tV_{k,k'}(f)(y)g(y)dy = \int_{\mathbb{R}} f(x)V_{k,k'}(g)(x)\mathcal{A}(x)dx. \qquad (1.12)$$

The operator ${}^tV_{k,k'}$ possesses the following integral representation

$$\forall y \in \mathbb{R},\ {}^tV_{k,k'}(f)(y) = \int_{|x|>|y|} f(x)K_{k,k'}(x,y)\mathcal{A}(x)dx, \qquad (1.13)$$

with $K_{k,k'}$ the function defined by (1.7).
The operator ${}^tV_{k,k'}$ is a topological isomorphism from $\mathcal{D}(\mathbb{R})$ onto itself satisfying the transmutation relation

$$\forall y \in \mathbb{R},\ {}^tV_{k,k'}(T^{(k,k')} + 2\rho S)(g) = \frac{d}{dx}(\ {}^tV_{k,k'}(g))(x),$$

where S is the operator on $\mathcal{D}(\mathbb{R})$ given by

$$\forall x \in \mathbb{R},\ S(g)(x) = g(-x). \qquad (1.14)$$

Definition 2.1 *The operator* ${}^tV_{k,k'}$ *is defined on* $\mathcal{E}'(\mathbb{R})$ *by*

$$\langle\ {}^tV_{k,k'}(\mathcal{V}), \phi\rangle = \langle \mathcal{V}, V_{k,k'}(\phi)\rangle,\ \phi \in \mathcal{E}(\mathbb{R}). \qquad (1.15)$$

This operator satisfies the following properties.

Proposition 2.1 **i)** *The transform* ${}^tV_{k,k'}$ *is a topological isomorphism from* $\mathcal{E}'(\mathbb{R})$ *onto itself. Its inverse is given by*

$$\langle\ {}^tV^{-1}_{k,k'}(\mathcal{V}), \phi\rangle = \langle \mathcal{V}, V^{-1}_{k,k'}(\phi)\rangle,\ \phi \in \mathcal{D}(\mathbb{R}). \qquad (1.16)$$

ii) *Let* $\mathcal{T}_{f\mathcal{A}}$ *be the distribution of* $\mathcal{E}'(\mathbb{R})$ *given by the function* $f\mathcal{A}$*, with* f *in* $\mathcal{D}(\mathbb{R})$ *and* $\mathcal{A}$ *is the function given by the relation (1.5). Then, we have*

$${}^tV_{k,k'}(\mathcal{T}_{f\mathcal{A}}) = \mathcal{T}_{{}^tV_{k,k'}(f)}. \qquad (1.17)$$

iii) *Let* $\mathcal{T}_g$ *be the distribution of* $\mathcal{E}'(\mathbb{R})$ *given by the function* g *in* $\mathcal{D}(\mathbb{R})$*. Then we have,*

$${}^tV^{-1}_{k,k'}(\mathcal{T}_g) = \mathcal{T}_{{}^tV^{-1}_{k,k'}(g)\mathcal{A}} \qquad (1.18)$$

Definition 2.2 *The Hypergeometric Fourier transform $\mathcal{H}$ is defined for f in $\mathcal{D}(\mathbb{R})$ by*

$$\forall\lambda \in \mathbb{C},\ \mathcal{H}(f)(\lambda) = \int_{\mathbb{R}} f(x) G_\lambda^{(k,k')}(x)\mathcal{A}(x)dx. \tag{1.19}$$

Theorem 2.1 *The transform $\mathcal{H}$ is a topological isomorphism between $\mathcal{D}(\mathbb{R})$ and $PW(\mathbb{C})$. The inverse transform is given by*

$$\forall x \in \mathbb{R},\ \mathcal{H}^{-1}(g)(x) = \int_{\mathbb{R}} g(\lambda) G_\lambda^{(k,k')}(-x)\mathcal{C}_{k,k'}(\lambda)d\lambda,$$

where,

$$\mathcal{C}_{k,k'}(\lambda)d\lambda = (1 - \frac{\rho}{i\lambda})\frac{d\lambda}{8\pi |c_{k,k'}(\lambda)|^2},$$

with,

$$c_{k,k'}(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(k+\frac{1}{2})\Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{k-k'+1+i\lambda}{2})}, \qquad \lambda \in \mathbb{C}\backslash\{i\mathbb{N}\}. \tag{1.20}$$

See ([1,8]).

Proposition 2.2 **i)** *For $\lambda \in \mathbb{C}$ and f in $\mathcal{D}(\mathbb{R})$, we have*

$$\mathcal{H}(f)(\lambda) = 2\mathcal{F}_{k,k'}(f_e)(\lambda) + 2(\rho + i\lambda)\mathcal{F}_{k,k'}(Jf_o)(\lambda),$$

where, f_e (resp. f_o) denotes the even (resp. the odd) part of f, J is the operator given by

$$Jf_o = \int_{-\infty}^{x} f_o(t)dt,$$

and $\mathcal{F}_{k,k'}$ is the Jacobi transform defined by

$$\forall\lambda \in \mathbb{R},\ \mathcal{F}_{k,k'}(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x)\mathcal{A}(x)dx. \tag{1.21}$$

ii) *We have*

$$\mathcal{H}(f) = \mathcal{F} \circ {}^tV_{k,k'}(f),\ f \in \mathcal{D}(\mathbb{R}), \tag{1.22}$$

where $\mathcal{F}$ is the classical Fourier transform of functions given by

$$\forall\lambda \in \mathbb{R},\ \mathcal{F}(h)(\lambda) = \int_{\mathbb{R}} h(x)e^{-i\lambda x}dx,\ h \in \mathcal{D}(\mathbb{R}).$$

Definition 2.3 *The Hypergeometric Fourier transform of a distribution $\mathcal{V}$ in $\mathcal{E}'(\mathbb{R})$ is defined by*

$$\forall\lambda \in \mathbb{R},\ \mathcal{H}(\mathcal{V})(\lambda) = \langle \mathcal{V}, G_\lambda^{(k,k')}\rangle. \tag{1.23}$$

Proposition 2.3 *For $\mathcal{V}$ in $\mathcal{E}'(\mathbb{R})$, the function $\mathcal{H}(\mathcal{V})$ belongs to $\mathcal{E}(\mathbb{R})$ and we have*

$$\mathcal{H}(\mathcal{V}) = \mathcal{F} \circ {}^tV_{k,k'}(\mathcal{V}), \tag{1.24}$$

where $\mathcal{F}$ is the classical Fourier transform of distributions defined for $\mathcal{U}$ in $\mathcal{E}'(\mathbb{R})$ by

$$\forall \lambda \in \mathbb{R},\ \mathcal{F}(\mathcal{U})(\lambda) = \langle \mathcal{U}_x, e^{-i\lambda x}\rangle.$$

Definition 2.4 *Let $x \in \mathbb{R}$ and f in $\mathcal{D}(\mathbb{R})$, the Jacobi-Cherednik translation operator $\tau_x^{(k,k')}$ is defined by*

$$\tau_x^{(k,k')}(f)(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^{(k,k')}(z), \tag{1.25}$$

where $d\mu_{x,y}^{(k,k')}$ is given by the relation (1.4).

Proposition 2.4 **i)** *We have*

$$\forall x, y \in \mathbb{R},\ \tau_x^{(k,k')} G_\lambda^{(k,k')}(y) = G_\lambda^{(k,k')}(x) G_\lambda^{(k,k')}(y). \tag{1.26}$$

ii) *For each $x \in \mathbb{R}$, the dual of the Jacobi-Cherednik translation operator τ_x is the operator ${}^t\tau_x$ given on $\mathcal{D}(\mathbb{R})$ by*

$$\forall y \in \mathbb{R},\ {}^t\tau_x(f)(y) = \tau_x(\check{f})(-y). \tag{1.27}$$

iii) *For f in $\mathcal{D}(\mathbb{R})$, we have*

$$\mathcal{H}(\ {}^t\tau_x^{(k,k')}(f)(\lambda) = G_\lambda^{(k,k')}(x)\mathcal{H}(f)(\lambda), \tag{1.28}$$

Definition 2.5 *The Jacobi-Cherednik convolution product of f and g in $\mathcal{D}(\mathbb{R})$ is defined by*

$$\forall y \in \mathbb{R},\ f *_{k,k'} g(y) = \int_{\mathbb{R}} {}^t\tau_x^{(k,k')}(f)(y) g(x) \mathcal{A}(x) dx. \tag{1.29}$$

Proposition 2.5 **i)** *The Jacobi-Cherednik convolution product is commutative and associative.*

ii) *Let f and g be in $\mathcal{D}(\mathbb{R})$ with support respectively in $[-a, a], a > 0$ and $[-b, b], b > 0$. Then the function $f *_{k,k'} g$ belongs to $\mathcal{D}(\mathbb{R})$ with support in $[-(a + b), a + b]$ and we have*

$$\forall \lambda \in \mathbb{R},\ \mathcal{H}(f *_{k,k'} g)(\lambda) = \mathcal{H}(f)(\lambda)\mathcal{H}(g)(\lambda). \tag{1.30}$$

Definition 2.6 *The Jacobi-Cherednik convolution product of a distribution $\mathcal{U}$ in $\mathcal{D}'(\mathbb{R})$ and a function ϕ in $\mathcal{D}(\mathbb{R})$ is the function $\mathcal{U} *_{k,k'} \phi$ in $\mathcal{E}(\mathbb{R})$ defined by*

$$\forall x \in \mathbb{R},\ \mathcal{U} *_{k,k'} \phi(x) = \langle \mathcal{U}_y,\ {}^t\tau_x^{(k,k')}(\phi)(y)\rangle. \tag{1.31}$$

Proposition 2.6 **i)** *If $\mathcal{U} = \mathcal{T}_{f\mathcal{A}}$ is the distribution in $\mathcal{D}'(\mathbb{R})$ given by the function $f\mathcal{A}$, with f in $\mathcal{E}(\mathbb{R})$, we have*

$$\forall y \in \mathbb{R},\ \mathcal{U} *_{k,k'} \phi(y) = f *_{k,k'} \phi(y). \tag{1.32}$$

ii) *If $\mathcal{U}$ is in $\mathcal{E}'(\mathbb{R})$ and ϕ in $\mathcal{D}(\mathbb{R})$, then, the function $\mathcal{U} *_{k,k'} \phi$ belongs to $\mathcal{D}(\mathbb{R})$.*

Definition 2.7 *Let $\mathcal{U}$ be in $\mathcal{D}'(\mathbb{R})$ and $\mathcal{V}$ in $\mathcal{E}'(\mathbb{R})$. The Jacobi-Cherednik convolution product of $\mathcal{U}$ and $\mathcal{V}$ is the distribution in $\mathcal{D}'(\mathbb{R})$ defined by*

$$\langle \mathcal{U} *_{k,k'} \mathcal{V}, \phi\rangle = \langle \mathcal{V}_y, \langle \mathcal{U}_x,\ {}^t\tau_{-x}^{(k,k')}(\phi)(y)\rangle\rangle = \langle \mathcal{U}_x, \langle \mathcal{V}_y,\ {}^t\tau_{-x}^{(k,k')}(\phi)(y)\rangle\rangle,\ \ \phi \in \mathcal{D}(\mathbb{R}). \tag{1.33}$$

The convolution product $*_{k,k'}$ satisfies the following properties.

Proposition 2.7 **i)** *Let $\mathcal{U}$ be in $\mathcal{E}'(\mathbb{R})$ and f in $\mathcal{D}(\mathbb{R})$. Then,*

$$\mathcal{U} *_{k,k'} \mathcal{T}_{f\mathcal{A}} = \mathcal{T}_{(\mathcal{U} *_{k,k'} f)\mathcal{A}}, \tag{1.34}$$

where $\mathcal{T}_{(\mathcal{U} *_{k,k'} f)\mathcal{A}}$ is the distribution in $\mathcal{E}'(\mathbb{R})$ given by the function $(\mathcal{U} *_{k,k'} f)\mathcal{A}$.

ii) *The Jacobi-Cherednik convolution product of distributions is commutative and associative.*

Proposition 2.8 *Let $\mathcal{U}$ and $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$. The distribution $\mathcal{U} *_{k,k'} \mathcal{V}$ belongs to $\mathcal{E}'(\mathbb{R})$ and we have*

$$\mathcal{H}(\mathcal{U} *_{k,k'} \mathcal{V}) = \mathcal{H}(\mathcal{U})\mathcal{H}(\mathcal{V}). \tag{1.35}$$

$${}^tV_{k,k'}(\mathcal{U} *_{k,k'} \mathcal{V}) = {}^tV_{k,k'}(\mathcal{U}) * {}^tV_{k,k'}(\mathcal{V}), \tag{1.36}$$

where $*$ is the classical convolution product of distributions on $\mathbb{R}$.

Remark 2.1 *For the Hypergeometric Fourier transform of distributions on $\mathbb{R}^d$ associated to the Cherednik operators and the Cherednik convolution product of distributions on $\mathbb{R}^d$, see [10].*

Definition 2.8 *The Jacobi-Cherednik operator $T^{(k,k')}$ is defined on $\mathcal{E}'(\mathbb{R})$ by*

$$\langle T^{(k,k')}\mathcal{U}, \phi\rangle = -\langle \mathcal{U}, (T^{(k,k')} + 2\rho S)\phi)\rangle, \tag{1.37}$$

with S is the operator given by (1.14).

3 Convolution equations

Let $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$, we consider the convolution equations of the form

$$\mathcal{V} *_{k,k'} \mathcal{U} = \mathcal{W}, \tag{2.1}$$

where $\mathcal{U}$ and $\mathcal{W}$ are in $\mathcal{D}'(\mathbb{R})$.
In this section we shall study the regularity of the solutions of the equation (2.1). More precisely, we will give a condition on the Hypergeometric Fourier transform of the distribution $\mathcal{V}$ in order that the distribution $\mathcal{U}$ will be given by a function $f\mathcal{A}$, $f \in \mathcal{E}(\mathbb{R})$, whenever $\mathcal{W}$ is given by a function $g\mathcal{A}$, g in $\mathcal{E}(\mathbb{R})$.

Definition 3.1 *We say that the distribution $\mathcal{V}$ in $\mathcal{E}'(\mathbb{R})$ satisfies the H-property if,*

i) *We have,*

$$\lim_{\substack{|z|\to\infty \\ z\in Z}} \frac{|Imz|}{\ln|z|} = +\infty, \text{ where } Z = \{z \in \mathbb{C}, \mathcal{H}(\mathcal{V})(z) = 0\}.$$

ii) *There exist $n, M > 0$ such that*

$$|\mathcal{H}(\mathcal{V})(\lambda)| \geq |\lambda|^{-n}, \ for\ all\ |\lambda| \geq M.$$

Theorem 3.1 *Let $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$ such that $Z = \{z \in \mathbb{C}, \mathcal{H}(\mathcal{V})(z) = 0\}$ is infinite and $\mathcal{W}$ given by a function $g\mathcal{A}$, $g \in \mathcal{E}(\mathbb{R})$. Then, the following assertions are equivalent :*

a) *$\mathcal{U}$ is given by a function $f\mathcal{A}$, $f \in \mathcal{E}(\mathbb{R})$.*

b) *$\mathcal{V}$ satisfies the H-property.*

We shall deduce the proof of this theorem from the following three propositions.

Proposition 3.1 *Let $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$ satisfying the H-property. Then the assertion **b)** implies the assertion **a)**.*

We need the following Lemma to show this Proposition.

Lemma 3.1 *Let $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$. If $\mathcal{V}$ satisfies the H-property then, there exists a parametrix for $\mathcal{V}$, that is, there exist $\mathcal{W}$ in $\mathcal{E}'(\mathbb{R})$ and ϕ in $\mathcal{D}(\mathbb{R})$ such that*

$$\delta = \mathcal{V} *_{k,k'} \mathcal{W} + \mathcal{T}_{\psi\mathcal{A}},$$

where δ is the Dirac distribution at 0.

Proof.

By using (1.24) the H-property can also be written in the form

i) We have,

$$\lim_{\substack{|z|\to\infty \\ z\in Z}} \frac{|Imz|}{\ln|z|} = +\infty, \ where\ Z = \{z \in \mathbb{C}, \mathcal{F}(\ {}^tV_{k,k'}(\mathcal{V}))(z) = 0\}.$$

ii) There exist $n, M > 0$ such that

$$|\mathcal{F}(\ {}^tV_{k,k'}(\mathcal{V}))(\lambda)| \geq |\lambda|^{-n},\ for\ all\ |\lambda| \geq M.$$

We see that the H-property is true for the distribution ${}^tV_{k,k'}(\mathcal{V})$ in $\mathcal{E}'(\mathbb{R})$ in the case of the classical Fourier transform $\mathcal{F}$. Then from [5], there exists a parametrix for ${}^tV_{k,k'}(\mathcal{V})$, that is, there exists $\mathcal{W}_0$ in $\mathcal{E}'(\mathbb{R})$ and ψ_0 in $\mathcal{D}(\mathbb{R})$ such that

$$\delta = \ {}^tV_{k,k'}(\mathcal{V}) * \mathcal{W}_0 + \mathcal{T}_{\psi_0} \tag{2.2}$$

As the operator ${}^tV_{k,k'}$ is a topological isomorphism from $\mathcal{E}'(\mathbb{R})$ onto itself, from (2.2),(1.18), we deduce that

$$\delta = \ {}^tV_{k,k'}(\mathcal{V}) * \ {}^tV_{k,k'}({}^tV_{k,k'}^{-1}(\mathcal{W}_0)) + \ {}^tV_{k,k'}({}^tV_{k,k'}^{-1}(\mathcal{T}_{\psi_0})),$$

thus,

$$\delta = \ {}^tV_{k,k'}(\mathcal{V}) * \ {}^tV_{k,k'}(\mathcal{W}) + \ {}^tV_{k,k'}(\mathcal{T}_{\psi\mathcal{A}}), \tag{2.3}$$

with $\mathcal{W} = \ {}^tV_{k,k'}^{-1}(\mathcal{W}_0)$ and $\psi = \ {}^tV_{k,k'}^{-1}(\psi_0)$, the distribution $\mathcal{W}$ and the function ψ belong respectively to $\mathcal{E}'(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$.

On the other hand from (1.36), the relation (2.3) can also be written in the form

$${}^tV_{k,k'}^{-1}(\delta) = \mathcal{V} *_{k,k'} \mathcal{W} + \mathcal{T}_{\psi\mathcal{A}}.$$

But,

$${}^tV_{k,k'}^{-1}(\delta) = \delta,$$

then,

$$\delta = \mathcal{V} *_{k,k'} \mathcal{W} + \mathcal{T}_{\psi\mathcal{A}}.$$

Proof of Proposition 2.1.

Let $\mathcal{U}$ be in $\mathcal{D}'(\mathbb{R})$. Assume that $\mathcal{V} *_{k,k'} \mathcal{U}$ is given by a function $g\mathcal{A}$, with g in $\mathcal{E}(\mathbb{R})$. From Lemma 2.1, we have

$$\delta = \mathcal{V} *_{k,k'} \mathcal{W} + \mathcal{T}_{\psi\mathcal{A}},$$

with $\mathcal{W}$ in $\mathcal{E}'(\mathbb{R})$ and ψ in $\mathcal{D}(\mathbb{R})$. Thus,

$$\begin{aligned} \mathcal{U} &= \mathcal{U} *_{k,k'} \delta \\ &= \mathcal{U} *_{k,k'} (\mathcal{V} *_{k,k'} \mathcal{W} + \mathcal{T}_{\psi\mathcal{A}}). \end{aligned}$$

By using the commutativity and the associativity of the Jacobi-Cherednik convolution product in $\mathcal{E}'(\mathbb{R})$, we obtain

$$\begin{aligned} \mathcal{U} &= \mathcal{U} *_{k,k'} (\mathcal{V} *_{k,k'} \mathcal{W}) + \mathcal{U} *_{k,k'} \mathcal{T}_{\psi\mathcal{A}} \\ &= \mathcal{W} *_{k,k'} (\mathcal{V} *_{k,k'} \mathcal{U}) + \mathcal{U} *_{k,k'} \mathcal{T}_{\psi\mathcal{A}}, \end{aligned}$$

Then,

$$\mathcal{U} = \mathcal{W} *_{k,k'} \mathcal{T}_{g\mathcal{A}} + \mathcal{U} *_{k,k'} \mathcal{T}_{\psi\mathcal{A}}.$$

From (1.34), we obtain

$$\mathcal{U} = \mathcal{T}_{(\mathcal{W} *_{k,k'} g + \mathcal{U} *_{k,k'} \psi)\mathcal{A}}.$$

As $\mathcal{U} \in \mathcal{D}'(\mathbb{R}),\ \psi \in \mathcal{D}(\mathbb{R}),\ \mathcal{W} \in \mathcal{E}'(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$, then

$$(\mathcal{W} *_{k,k'} g + \mathcal{U} *_{k,k'} \psi) \in \mathcal{E}(\mathbb{R}).$$

Proposition 3.2 *Let $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$ such that $Z = \{z \in \mathbb{C}, \mathcal{H}(\mathcal{V})(z) = 0\}$ is infinite and $\mathcal{W}$ given by a function $g\mathcal{A}$, $g \in \mathcal{E}(\mathbb{R})$. Then, the assertion* ***a)*** *implies that $\mathcal{V}$ satisfies the* ***i)*** *of the H-property.*

Proof.

Suppose that the i) of the H-property is not hold. Then, there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and a positive constant M such that

$$\forall n \in \mathbb{N},\ \mathcal{H}(\mathcal{V})(z_n) = 0 \textit{ and } |Imz_n| \leq M \ln|z_n|.$$

Let ϕ be in $\mathcal{D}(\mathbb{R})$. According to Theorem 1.1, there exists $b \in \mathbb{N}$ such that for every $p \in \mathbb{N}$, we find a constant $C_p > 0$, so that for all $z \in \mathbb{C}$ such that $|z| > 1$, we have

$$|\mathcal{H}(\phi)(z)| \leq C_p \exp(b|Imz| - p\ln|z|).$$

If we take $p \in \mathbb{N}$, such that $p > Mb + 1$, we get

$$\forall n \in \mathbb{N},\ |z_n||\mathcal{H}(\phi)(z_n)| \leq C_p. \tag{2.4}$$

Let $\{a_n\}_{n\in\mathbb{N}}$ be a complex sequence such that the series $\sum_{n=0}^{+\infty} |a_n|$ is convergent. We consider the sequence $\{\mathcal{X}_q\}_{q\in\mathbb{N}}$ of distributions in $\mathcal{D}'(\mathbb{R})$, given by

$$\mathcal{X}_q = \sum_{n=0}^{q} a_n\ \mathcal{T}_{|z_n|G^{(k,k')}_{z_n}\mathcal{A}}.$$

For all $q, r \in \mathbb{N}$, $q > r$, we have

$$\begin{aligned} \langle \mathcal{X}_q, \phi\rangle \; - \langle \mathcal{X}_r, \phi\rangle &= \langle \sum_{n=r+1}^{q} a_n \; \mathcal{T}_{|z_n| G_{z_n}^{(k,k')}\mathcal{A}}, \phi\rangle \\ &= \sum_{n=r+1}^{q} a_n \; |z_n| \; \mathcal{H}(\phi)(z_n). \end{aligned}$$

Thus, using (2.4) we obtain

$$|\langle \mathcal{X}_q, \phi\rangle - \langle \mathcal{X}_r, \phi\rangle| \leq C_p \sum_{n=r+1}^{q} |a_n| \underset{r\to+\infty}{\longrightarrow} 0. \tag{2.5}$$

Then, $\{\langle \mathcal{X}_q, \phi\rangle\}_{q\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and we have

$$\langle \mathcal{X}_q, \phi\rangle \underset{q\to+\infty}{\longrightarrow} L(\phi).$$

We deduce that L is a distribution $\mathcal{X}$ in $\mathcal{D}'(\mathbb{R})$ and $\mathcal{X}_q$ converges to $\mathcal{X}$ in $\mathcal{D}'(\mathbb{R})$ as q tends to infinity. Then,

$$\mathcal{X} = \sum_{n=0}^{+\infty} a_n \mathcal{T}_{|z_n| G_{z_n}^{(k,k')}\mathcal{A}}, \tag{2.6}$$

and from (2.5), we deduce

$$|\langle \mathcal{X}, \phi\rangle| \leq C_p \sum_{n=0}^{+\infty} \; |a_n|. \tag{2.7}$$

From (1.33),(2.6) and Proposition 1.4 ii), we have

$$\begin{aligned} \langle \mathcal{V} *_{k,k'} \mathcal{X}, \phi\rangle &= \langle \mathcal{V}_x, \langle \mathcal{X}_y, \; {}^t\tau_{-x}^{(k,k')}(\phi)(y)\rangle\rangle \\ &= \langle \mathcal{V}_x, \sum_{n=0}^{+\infty} a_n |z_n| G_{z_n}^{(k,k')}(-x) \mathcal{H}(\phi)(z_n)\rangle. \end{aligned}$$

From Definition 1.3, we obtain

$$\begin{aligned} \langle \mathcal{V} *_{k,k'} \mathcal{X}, \phi\rangle &= \sum_{n=0}^{+\infty} a_n |z_n| \mathcal{H}(\phi)(z_n) \langle \mathcal{V}_x, G_{z_n}^{(k,k')}(-x)\rangle. \\ &= \sum_{n=0}^{+\infty} a_n |z_n| \mathcal{H}(\phi)(z_n) \mathcal{H}(\check{\mathcal{V}})(z_n) \\ &= 0, \end{aligned}$$

where $\check{\mathcal{V}}$ is the distribution in $\mathcal{E}'(\mathbb{R})$ defined by

$$\langle \check{\mathcal{V}}, \varphi \rangle = \langle \mathcal{V}, \check{\varphi} \rangle, \quad \varphi \in \mathcal{E}(\mathbb{R}),$$

with

$$\forall x \in \mathbb{R},\ \check{\varphi}(x) = \varphi(-x).$$

Thus,

$$\mathcal{V} *_{k,k'} \mathcal{X} = 0. \tag{2.8}$$

As the distribution $\mathcal{X}$ is given by a function $f\mathcal{A}$, with f in $\mathcal{E}(\mathbb{R})$. Then,

$$\mathcal{X} = \mathcal{T}_{f\mathcal{A}}. \tag{2.9}$$

We have,

$$\forall n \in \mathbb{N},\ G_{z_n}^{(k,k')}(0) = 1.$$

Thus, for all $m > 0$, we have

$$\forall n \in \mathbb{N},\ \sup_{x \in [-m,m]} |G_{z_n}^{(k,k')}(x)| \geq 1. \tag{2.10}$$

On the other hand, by using (2.7),(2.9) we obtain

$$\sup_{x \in [-m,m]} |f(x)| \leq C_p \sum_{n=0}^{+\infty} |a_n|.$$

Thus,

$$\forall n \in \mathbb{N},\ |z_n| \sup_{x \in [-m,m]} |G_{z_n}^{(k,k')}(x)| \leq C_p.$$

From this relation and (2.10), we deduce that

$$\forall n \in \mathbb{N},\ |z_n| \leq C_p,$$

which is a contradiction with the choice of the sequence $\{z_n\}_{n \in \mathbb{N}}$.

Proposition 3.3 *Let $\mathcal{V}$ be in $\mathcal{E}'(\mathbb{R})$ and $\mathcal{W}$ given by a function $g\mathcal{A}$, $g \in \mathcal{E}(\mathbb{R})$. Then, the assertion **a)** implies that $\mathcal{V}$ satisfies the condition **ii)** of the H-property.*

Proof.

Assume that the ii) of the H-property is not hold. Then we can find a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $|\lambda_n|^{-n} \geq 2^n$ and

$$\forall n \in \mathbb{N},\ |\mathcal{H}(\mathcal{V})(\lambda_n)| \leq |\lambda_n|^{-n}. \tag{2.11}$$

We consider the sequence $\{\mathcal{U}_p\}_{p\in\mathbb{N}}$ of distributions in $\mathcal{D}'(\mathbb{R})$ given by

$$\mathcal{U}_p = \sum_{n=0}^{p} \mathcal{T}_{G^{(k,k')}_{\lambda_n}\mathcal{A}}.$$

Let ψ be in $\mathcal{D}(\mathbb{R})$. For all $p, q \in \mathbb{N}$ with $p > q$, we have

$$\langle \mathcal{U}_p, \psi\rangle - \langle \mathcal{U}_q, \psi\rangle = \sum_{n=q+1}^{p} \langle \mathcal{T}_{G^{(k,k')}_{\lambda_n}\mathcal{A}}, \psi\rangle$$

Thus,

$$\langle \mathcal{U}_p, \psi\rangle - \langle \mathcal{U}_q, \psi\rangle = \sum_{n=q+1}^{p} \mathcal{H}(\psi)(\lambda_n). \tag{2.12}$$

But from Theorem 1.1, the function $\mathcal{H}(\psi)$ is rapidly decreasing. Then, there exists a positive constant C such that

$$\forall \lambda \in \mathbb{R},\ |\mathcal{H}(\psi)(\lambda)| \leq \frac{C}{1+|\lambda|},$$

then,

$$\forall n \in \mathbb{N},\ |\mathcal{H}(\psi)(\lambda_n)| \leq \frac{C}{2^n}.$$

By applying this relation to (2.12), we obtain

$$|\langle \mathcal{U}_p, \psi\rangle - \langle \mathcal{U}_q, \psi\rangle| \leq C \sum_{n=q+1}^{p} \frac{1}{2^n} \underset{q\to+\infty}{\longrightarrow} 0.$$

Thus, $\{\langle \mathcal{U}_p, \psi\rangle\}_{p\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and we have

$$\langle \mathcal{U}_p, \psi\rangle \underset{p\to+\infty}{\longrightarrow} L(\psi).$$

We deduce that L is a distribution in $\mathcal{D}'(\mathbb{R})$ and $\mathcal{U}_p$ converges to $\mathcal{U}$ in $\mathcal{D}'(\mathbb{R})$ as p tends to infinity. Thus,

$$\mathcal{U} = \sum_{n=0}^{+\infty} \mathcal{T}_{G^{(k,k')}_{\lambda_n}\mathcal{A}}$$

and for all $\psi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle \mathcal{U}, \psi\rangle = \sum_{n=0}^{+\infty} \mathcal{H}(\psi)(\lambda_n). \tag{2.13}$$

Now, we shall prove that the distribution $\mathcal{V} *_{k,k'} \mathcal{U}$ of $\mathcal{D}'(\mathbb{R})$ is given by a function $f\mathcal{A}$, with f in $\mathcal{E}(\mathbb{R})$.
By making a proof similar to those which have given the relation (2.8), we obtain

$$\langle \mathcal{V} *_{k,k'} \mathcal{U}, \psi\rangle = \sum_{n=0}^{+\infty} \mathcal{H}(\check{\mathcal{V}})(\lambda_n) \int_{\mathbb{R}} \psi(t) G_{\lambda_n}^{(k,k')}(t)\mathcal{A}(t)dt.$$

By using (1.9), the fact that ψ is in $\mathcal{D}(\mathbb{R})$ and $\mathcal{H}(\check{\mathcal{V}})$ satisfies (2.11), we can interchange the series and the integral and we obtain

$$\langle \mathcal{V} *_{k,k'} \mathcal{U}, \psi\rangle = \int_{\mathbb{R}} [\sum_{n=0}^{+\infty} \mathcal{H}(\check{\mathcal{V}})(\lambda_n) G_{\lambda_n}^{(k,k')}(t)]\psi(t)\mathcal{A}(t)dt.$$

Thus, the distribution $\mathcal{V} *_{k,k'} \mathcal{U}$ is given by function $f\mathcal{A}$ with

$$\forall t \in \mathbb{R},\ f(t) = \sum_{n=0}^{+\infty} \mathcal{H}(\check{\mathcal{V}})(\lambda_n) G_{\lambda_n}^{(k,k')}(t).$$

From relations (2.11),(1.1) we deduce that f is in $\mathcal{E}(\mathbb{R})$.

In the following we want to show that the distribution $\mathcal{U}$ is not given by a function $g\mathcal{A}$, with g in $\mathcal{E}(\mathbb{R})$. Proceeding by contradiction, we take an even function χ in $\mathcal{D}(\mathbb{R})$ such that $\chi(0) = 1$ and $\mathcal{H}(\chi)$ is positive. For all $\mu \in \mathbb{R}$, we consider

$$\langle \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})} \mathcal{U}, \chi\rangle = \langle \mathcal{U}, \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})} \chi\rangle, \tag{2.14}$$

where $\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}$ is the Jacobi function. By using (2.13), we obtain

$$\langle \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})} \mathcal{U}, \chi\rangle = \sum_{n=0}^{+\infty} \mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})} \chi)(\lambda_n). \tag{2.15}$$

But, as χ and $\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}$ are even, then $\mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)$ is also even and from Proposition 1.2 i), we have

$$\forall \nu \in \mathbb{R},\ \mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\nu) = 2\mathcal{F}_{k,k'}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\nu).$$

Then,

$$\mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\nu) = 2\int_{\mathbb{R}_+} \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\varphi_\nu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\chi(x)\mathcal{A}(t)dt. \tag{2.16}$$

But from [6], the function $\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t), t \in \mathbb{R}_+$ admits the following product formula, with respect to the variable μ,

$$\forall \mu, \nu \in \mathbb{R}_+,\ \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\varphi_\nu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t) = \int_{\mathbb{R}_+} \varphi_\xi^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\mathfrak{U}(\mu,\nu,\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}, \tag{2.17}$$

where $c_{k,k'}$ is defined by (1.20) and $\mathfrak{U}(\mu,\nu,\xi)$ is an even positive function on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. It is given by the relation

$$\mathfrak{U}(\mu,\nu,\xi)=\int_{\mathbb{R}_+}\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}(x)\varphi_\nu^{(k-\frac{1}{2},k'-\frac{1}{2})}(x)\varphi_\xi^{(k-\frac{1}{2},k'-\frac{1}{2})}(x)\mathcal{A}(x)dx,$$

and satisfies

$$\int_{\mathbb{R}_+}\mathfrak{U}(\mu,\nu,\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}=1. \tag{2.18}$$

By using (2.17) and Fubini-Tonelli's theorem, the relation (2.16) can be written in the form,

$$\forall\nu\in\mathbb{R},\ \mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\nu)=2\int_{\mathbb{R}_+}\mathfrak{U}(\mu,\nu,\xi)(\int_{\mathbb{R}_+}\chi(t)\varphi_\xi^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\mathcal{A}(t)dt)\frac{d\xi}{|c_{k,k'}(\xi)|^2}.$$

Then,

$$\mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\nu)=\int_{\mathbb{R}_+}\mathfrak{U}(\mu,\nu,\xi)\mathcal{H}(\chi)(\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}. \tag{2.19}$$

Thus, for all $\mu,\nu\in\mathbb{R}$, the function $\mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\nu)$ is positive.

By taking $\nu=\lambda_n$ and by replacing $\mathcal{H}(\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi)(\lambda_n)$ by its expression (2.19), we get from (2.15)

$$\langle\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle=\sum_{n=0}^{+\infty}\int_{\mathbb{R}_+}\mathfrak{U}(\mu,\lambda_n,\xi)\mathcal{H}(\chi)(\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}.$$

As the function $\mu\mapsto\langle\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle$ and those of the second member are positive, then by applying (2.17),(2.15) and by using Fubini-Tonelli's theorem we deduce that

$$\begin{aligned}\int_{\mathbb{R}_+}\langle\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle\frac{d\mu}{|c_{k,k'}(\mu)|^2} &= \sum_{n=0}^{+\infty}\int_{\mathbb{R}_+}[\int_{\mathbb{R}_+}\mathfrak{U}(\mu,\lambda_n,\xi)\frac{d\mu}{|c_{k,k'}(\mu)|^2}]\mathcal{H}(\chi)(\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}\\ &= \sum_{n=0}^{+\infty}\int_{\mathbb{R}_+}\mathcal{H}(\chi)(\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}.\end{aligned}$$

But from Theorem 1.1, we have

$$\int_{\mathbb{R}_+}\mathcal{H}(\chi)(\xi)\frac{d\xi}{|c_{k,k'}(\xi)|^2}=\chi(0)=1.$$

Then,

$$\int_{\mathbb{R}_+}\langle\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle\frac{d\mu}{|c_{k,k'}(\mu)|^2}=+\infty. \tag{2.20}$$

On the other hand, as the distribution $\mathcal{U}$ is given by a function $g\mathcal{A}$, with g in $\mathcal{E}(\mathbb{R})$, then from (2.14), we have

$$\begin{aligned}\langle \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle &= \langle \mathcal{T}_{g\mathcal{A}},\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\chi\rangle \\ &= \int_{\mathbb{R}} g(t)\chi(t)\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\mathcal{A}(t)dt \\ &= \int_{\mathbb{R}} g_e(t)\chi(t)\varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}(t)\mathcal{A}(t)dt,\end{aligned}$$

when g_e is the even part of g.
Thus,

$$\langle \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle = \mathcal{H}(g_e\chi)(\mu).$$

By integrating the two members of the preceding relation with respect to the measure $\dfrac{d\mu}{|c_{k,k'}(\mu)|^2}$, we obtain from Theorem 1.1

$$\begin{aligned}\int_{\mathbb{R}}\langle \varphi_\mu^{(k-\frac{1}{2},k'-\frac{1}{2})}\mathcal{U},\chi\rangle\frac{d\mu}{|c_{k,k'}(\mu)|^2} &= \int_{\mathbb{R}}\mathcal{H}(g_e\chi)(\mu)\frac{d\mu}{|c_{k,k'}(\mu)|^2} \\ &= g_e(0)\chi(0) \\ &= g(0).\end{aligned}$$

This contradicts the result given by (2.20). Hence, the distribution $\mathcal{U}$ is not given by a function $g\mathcal{A}$, with g in $\mathcal{E}(\mathbb{R})$.

4 An example of regular solution of convolution equation

We consider the distribution $\mathcal{V}$ given by

$$\mathcal{V} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}(T+2\rho S)^{(2n)}(\delta), \tag{3.1}$$

where $T = T^{(k,k')}$ is the Jacobi-Cherednik operator, δ is the Dirac distribution at 0 and S the operator defined on $\mathcal{E}'(\mathbb{R})$ by

$$S(\mathcal{X}) = \check{\mathcal{X}},$$

with $\check{\mathcal{X}}$ is the distribution on $\mathcal{E}'(\mathbb{R})$ given by

$$\langle\check{\mathcal{X}},\varphi\rangle = \langle\mathcal{X},\check{\varphi}\rangle,\ \varphi\in\mathcal{E}(\mathbb{R}),$$

where $\check{\varphi}$ is the function defined on $\mathbb{R}$ by

$$\check{\varphi}(x) = \varphi(-x).$$

Proposition 4.1 *The distribution $\mathcal{V}$ given by (3.1) belongs to $\mathcal{E}'(\mathbb{R})$.*

Proof.

We consider the sequence $\{\mathcal{V}_m\}_{m\in\mathbb{N}}$ of distributions in $\mathcal{E}'(\mathbb{R})$ given by

$$\mathcal{V}_m = \sum_{n=0}^{m} \frac{(-1)^n}{(2n)!}(T+2\rho S)^{(2n)}(\delta).$$

Let φ be in $\mathcal{E}(\mathbb{R})$. From Definition 1.8, we have

$$\begin{array}{rcl}\langle (T+2\rho S)(\delta),\varphi\rangle & = & -\langle \delta, T(\varphi)\rangle \\ & = & -T(\varphi)(0).\end{array}$$

Then, for all $n\in\mathbb{N}\backslash\{0\}$, we obtain

$$\langle (T+2\rho S)^{(2n)}(\delta),\varphi\rangle = T^{(2n)}(\varphi)(0). \qquad (3.2)$$

From (3.2), we have for all $m,q\in\mathbb{N}$ with $m>q$,

$$\langle \mathcal{V}_m,\varphi\rangle - \langle \mathcal{V}_q,\varphi\rangle = \sum_{n=q+1}^{m} \frac{(-1)^n}{(2n)!}T^{(2n)}(\varphi)(0). \qquad (3.3)$$

By applying the formula

$$\varphi(x)-\varphi(-x) = x\int_{-1}^{1}\varphi'(ux)du,$$

to the following expression

$$T(\varphi)(x) = \varphi'(x) + \{k\coth(x)+k'\tanh(x)\}\frac{\varphi(x)-\varphi(-x)}{2} - \rho\varphi(-x),$$

we deduce from (3.3) that for all K compact of $\mathbb{R}$, there exists $C>0$ such that

$$|\langle \mathcal{V}_m,\varphi\rangle - \langle \mathcal{V}_q,\varphi\rangle| \leq Cp_{2m,K}(\varphi)\sum_{n=q+1}^{m}\frac{1}{(2n)!} \underset{q\to+\infty}{\longrightarrow} 0,$$

where $p_{m,K}(\varphi) = \sup\limits_{\substack{0\leq n\leq m\\ x\in K}} |\varphi^{(n)}(x)|$.

Thus, $\{\mathcal{V}_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and we have

$$\langle \mathcal{V}_m,\varphi\rangle \underset{m\to+\infty}{\longrightarrow} L(\varphi).$$

We deduce that L is a distribution in $\mathcal{E}'(\mathbb{R})$ and $\mathcal{V}_m$ converges to $\mathcal{V}$ in $\mathcal{E}'(\mathbb{R})$ as m tends to infinity.

Proposition 4.2 *We consider the convolution equation*

$$\mathcal{V} *_{k,k'} \mathcal{U} = \mathcal{W},$$

with $\mathcal{V}$ given by (3.1). If $\mathcal{W}$ is given by a function $g\mathcal{A}$, $g \in \mathcal{E}(\mathbb{R})$, then $\mathcal{U}$ is given by a function $f\mathcal{A}$, $f \in \mathcal{E}(\mathbb{R})$.

Proof.

To show this result, from Theorem 2.1, it is enough to show that $\mathcal{V}$ satisfies the H-property. By using (3.2) and (1.1), we have

$$\begin{aligned}
\forall z \in \mathbb{C},\ \mathcal{H}(\mathcal{V})(z) &= \langle \mathcal{V}, G_z^{(k,k')} \rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \langle (T + 2\rho S)^{(2n)}(\delta), G_z^{(k,k')} \rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \langle \delta, T^{(2n)}(G_z^{(k,k')}) \rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n (iz)^{2n}}{(2n)!} \\
&= \cosh(z).
\end{aligned}$$

Then, the set

$$Z = \{z \in \mathbb{C}, \mathcal{H}(\mathcal{V})(z) = 0\} = \{i\frac{k\pi}{2}, k \in \mathbb{Z}\}$$

is infinite and we have

$$\lim_{\substack{|z|\to\infty \\ z \in Z}} \frac{|Imz|}{\ln |z|} = +\infty.$$

Then, the distribution $\mathcal{V}$ satisfies the i) of the H-property.
Finally, the fact that

$$\forall \lambda \in \mathbb{R},\ \cosh(\lambda) \geq 1,$$

shows that the distribution $\mathcal{V}$ satisfies the ii) of the H-property.

5 Open problems

5.1 Problem 1

Define and characterize the spaces of Jacobi-Cherednik multiplication and convolution operators on the generalized temperate distributions spaces $\mathcal{S}'_2(\mathbb{R})$.

5.2 Problem 2

Characterize hypoelliptic convolution-equations in $\mathcal{S}'_2(\mathbb{R})$ for the Jacobi-Cherednik theory on the real line.

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