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The Jacobi-Cherednik operator on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

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Abstract

In this paper we study the differential-difference Jacobi-Cherednik operator defined by

$$T^{(k,k')}f(\theta) := f'(\theta) + (k\cot\theta - k'\tan\theta)\,(f(\theta) - f(-\theta)) - i(k+k')f(-\theta),$$

$$f \in C^1\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right),$$

where $k > 0$ ***and*** $k' \geq 0$***, and the operator which intertwines*** $T^{(k,k')}$ ***and the derivative operator*** $\frac{d}{d\theta}$***. Estimates for the eigenfunctions of the operator*** $T^{(k,k')}$ ***are also given.***

Keywords: *Jacobi-Cherednik operator, Laplace formula, Estimates for the Jacobi-Cherednik kernel, Intertwining operator, Intertwining dual operator.*

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1 Introduction

For a crystallographic root system R in $\mathbb{R}^d$, a fixed positive subsystem R_+ and a nonnegative multiplicity function k defined on R, the Cherednik operator ([1], [9], [10]) in the direction $\xi \in \mathbb{R}^d$ is defined, for $f \in C^1(\mathbb{R}^d)$, by

$$T_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-\langle \alpha, x \rangle}} (f(x) - f(\sigma_\alpha(x)) - \langle \rho, \xi \rangle f(x),$$

where $\langle ., . \rangle$ is the usual scalar product, σ_α is the orthogonal reflexion in the hyperplane orthogonal to α, $\rho := \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha$ and the function k is invariant by the finite reflection group W generated by the reflections σ_α $(\alpha \in R)$.

Thanks to these operators, G.J. Heckmann and E.M. Opdam developped a theory generalizing harmonic analysis on symmetric spaces ([5], [6], [9]). Important results have yet been obtained in this direction ([10]) but despite to recent interesting results ([11]), applications remain restricted, in particular for lack of precise information on the eigenfunctions of these operators T_ξ. One of the main obstacle is that no Laplace formula with a positive kernel is known for the eigenfunctions (or the so called Opdam-Cherednik kernel) equivalently no positive operator intertwining T_ξ and the derivative operator ∂_ξ, is known for the moment. As a contribution towards this fondamental question and via the study of a more general differential-difference operator, L. Gallardo and K. Trimèche gave in [4] a complete solution for the case $d = 1$.

In the present paper, we give a solution for the case of a bounded interval. More precisely, we consider the differential-difference operator, which we will call the Jacobi-Cherednik operator, defined for $f \in C^1\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$, by

$$T^{(k,k')} f(\theta) := f'(\theta) + (k \cot(\theta) - k' \tan(\theta)) (f(\theta) - f(-\theta)) - i(k+k') f(-\theta), \quad (1)$$

for $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$ and $T^{(k,k')} f(0) := (2k+1) f'(0) - i(k+k') f(0)$ [1], where $k > 0$ and $k' \geq 0$ are two parameters satisfying the following condition :

$$(C): \text{ either } k' = 0 < k, \text{ or } 0 < k' \leq k. \quad (2)$$

[1]Note that $T^{(k,k')} f$ is a continuous function on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

For every $\lambda\in\mathbb{C}$, let us denote by $G_\lambda^{(k,k')}$ the unique solution of the eigenvalue problem

$$\left\{\begin{array}{rcl} T^{(k,k')}f(\theta) &=& i\lambda f(\theta),\\ f(0) &=& 1. \end{array}\right. \tag{3}$$

Noting that $G_\lambda^{(k,k')}$ can be expressed in terms of Jacobi functions and using results obtained by T.H. Koornwinder ([7]), we show that there exists a continuous kernel $K^{(k,k')}(\theta,\phi)$ $\left(\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\},\ -|\theta|\le\phi\le|\theta|\right)$ which we call Laplace kernel, such that for all $\lambda\in\mathbb{C}$ and $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$,

$$G_\lambda^{(k,k')}(\theta)=\int_{-|\theta|}^{|\theta|}K^{(k,k')}(\theta,\phi)e^{i\lambda\phi}d\phi. \tag{4}$$

We can deduce precise estimates on the function $G_\lambda^{(k,k')}$ which allows the Jacobi-Cherednik kernel $(\lambda,\theta)\longmapsto G_\lambda^{(k,k')}(\theta)$ to be considered as a good kernel for the Fourier-Opdam transform.
We then study the associated intertwining operator defined by

$$V^{(k,k')}f(\theta):=\int_{-|\theta|}^{|\theta|}K^{(k,k')}(\theta,\phi)f(\phi)\,d\phi,$$

for $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$ and $V^{(k,k')}f(0):=f(0)$. It intertwins the operators $T^{(k,k')}$ and $\frac{d}{d\theta}$ on the space of C^∞ functions, that is

$$T^{(k,k')}V^{(k,k')}f=V^{(k,k')}\frac{d}{d\theta}f, \tag{5}$$

for all C^∞ function f. Moreover, we show that is a topological automorphism of the space $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ (of complex valued C^∞ functions on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ carrying the topology of uniform convergence on compact sets of all derivatives) and we determine explicitely the inverse automorphism $\left(V^{(k,k')}\right)^{-1}$. We also study the dual operator ${}^tV^{(k,k')}$ defined, on the space of all continuous function $g:\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\longrightarrow\mathbb{C}$ with compact support, by

$$\forall\phi\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[,\quad {}^tV^{(k,k')}g(\phi):=\int_{|\theta|>|\phi|}K^{(k,k')}(\theta,\phi)g(\theta)(\sin(|\theta|))^{2k}(\cos\theta)^{2k'}\,d\theta.$$

We show that it is a topological automorphism of the space $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ of complex valued C^∞ functions with compact support and it satisfies the following unusual intertwining relation :

$$\forall g\in\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right),\quad {}^tV^{(k,k')}\left(T^{(k,k')}+2i(k+k')S\right)g=\frac{d}{d\theta}\,{}^tV^{(k,k')}g,$$

where S is the operator defined on $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ by $Sg(\theta) := g(-\theta)$.

In a forthcoming paper we study the Fourier-Opdam transform associated to the Jacobi-Cherednik kernel, we prove a Fourier inversion formula, a Plancherel theorem and a Paley-Wiener theorem and we show how the intertwining operator $V^{(k,k')}$ can be used to define generalized translation operators and a convolution structure naturally associated to the Jacobi-Cherednik operators.

In the sequel, we always suppose that the parameters k and k' satisfy the condition (C) given by (2) and we denote by $\rho := k + k' > 0$.

2 The Jacobi-Cherednik kernel

Proposition 2.1. *For every $\lambda \in \mathbb{C}$, the eigenfunction equation* (3) *has a unique solution of the form*

$$\begin{aligned} G_\lambda^{(k,k')}(\theta) &= R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)) - \frac{i}{\lambda-\rho}\frac{d}{d\theta}\left[R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta))\right] \qquad (6) \\ &= R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)) + i\frac{\lambda+\rho}{2(2k+1)}\sin(2\theta)R_{\frac{\lambda-(\rho+2)}{2}}^{(k+\frac{1}{2},k'+\frac{1}{2})}(\cos(2\theta)), (7) \end{aligned}$$

$$\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[,$$

where $R_{\frac{\lambda-\rho}{2}}^{(\alpha,\beta)}$ is the Jacobi function of index (α,β) given by

$$R_{\frac{\lambda-\rho}{2}}^{(\alpha,\beta)}(\cos(2\theta)) := {}_2F_1\left(\frac{\rho-\lambda}{2},\frac{\rho+\lambda}{2};\alpha+1;(\sin\theta)^2\right),$$

where ${}_2F_1$ is the Gaussian hypergeometric function (see [7], p.147, formula 2.3).

Proof. In order to simplify notations let us denote by

$$T := T^{(k,k')}, \quad \varphi(\theta) := R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)), \quad G := \varphi - \frac{i}{\lambda-\rho}\varphi'.$$

In view of [7], the function φ satisfies the differential equation

$$\varphi''(\theta) + 2\left(k\cot\theta - k'\tan\theta\right)\varphi'(\theta) = -(\lambda^2-\rho^2)\varphi, \quad \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}.$$

Using the fact that φ is even and φ' is odd on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, we immediately deduce that $TG = i\lambda G$. As $G(0) = 1$, this proves that G satisfies the eigenfunction equation (3). In order to see that it is the unique solution, it remains to show that if $h \in C^1\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ is a solution of

$$\begin{cases} Th(\theta) &= 0, \\ h(0) &= 0, \end{cases} \tag{8}$$

then $h = 0$. Let us denote by $h_e(\theta) := \frac{1}{2}(h(\theta) + h(-\theta))$ and $h_o(\theta) := \frac{1}{2}(h(\theta) - h(-\theta))$ respectively the even and odd parts of h. Taking into account that the function $q(\theta) := k\cot\theta - k'\tan\theta$ is odd on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$, the function h satisfies (8) if and only if it satisfies the two following conditions:

$$\begin{cases} h_o'(\theta) + 2q(\theta)h_o(\theta) - i\rho h_e(-\theta) &= 0, \\ h(0) &= 0 \end{cases} \tag{9}$$

and

$$\begin{cases} h_e'(\theta) - i\rho h_o(-\theta) &= 0, \\ h(0) &= 0. \end{cases} \tag{10}$$

From the equation (10) we deduce that $h_e \in C^2\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $h_e''(\theta) = -i\rho h_o'(\theta)$ and by introducing this value in the equation (9), we see that h_e satisfies the following second order differential equation :

$$\begin{cases} h_e''(\theta) + 2q(\theta)h_e'(\theta) - \rho^2 h_e(\theta) = 0, \\ h_e(0) = 0, \quad h_e'(0) = 0, \end{cases}$$

which admits a unique solution $h_e = 0$. Then, by (10), $h_o = 0$ and so $h = 0$. The second expression (7) follows from the formula giving the derivative of φ. □

Examples 2.2. *For all* $\lambda \in \mathbb{C}$ *and* $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, *we have*

1. *If* $k = 1$ *and* $k' = 0$, *then* $R_{\frac{\lambda-1}{2}}^{(\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)) = \dfrac{\sin(\lambda\theta)}{\lambda\sin\theta}$ *and*

$$G_\lambda^{(1,0)}(\theta) = \frac{\sin(\lambda\theta)}{\lambda\sin\theta} + \frac{i}{\lambda-1}\left(\frac{\sin(\lambda\theta)\cos\theta}{\lambda(\sin\theta)^2} - \frac{\cos(\lambda\theta)}{\sin\theta}\right).$$

2. *If* $k = k' = 1$, *then* $R_{\frac{\lambda-2}{2}}^{(\frac{1}{2},\frac{1}{2})}(\cos(2\theta)) = \dfrac{2\sin(\lambda\theta)}{\lambda\sin(2\theta)}$ *and*

$$G_\lambda^{(1,1)}(\theta) = 2\left[\frac{\sin(\lambda\theta)}{\lambda\sin(2\theta)} + \frac{i}{\lambda-2}\left(\frac{2\sin(\lambda\theta)\cos(2\theta)}{\lambda(\sin(2\theta))^2} - \frac{\cos(\lambda\theta)}{\sin(2\theta)}\right)\right].$$

As an interesting direct consequence of Proposition 2.1, we now present a functional relation between the eigenfunction $G_\lambda^{(k,k')}$ and the Jacobi function $R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta))$ which can be compared to the relation between the functions G_λ and F_λ of E.M. Opdam in [9], p.89.

Corollary 2.3. *For all $\lambda \in \mathbb{C}$ and $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, we have*

$$(\lambda-\rho)G_\lambda^{(k,k')}(\theta) = p\left(\lambda, T^{(k,k')}\right) R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)),$$

where $p(\lambda, X) := -iX + \lambda I$.

Proof. Formula (1) applied to $\varphi(\theta) = R_{\frac{\lambda-\rho}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta))$ implies $\varphi'(\theta) = T^{(k,k')}\varphi(\theta) + i\rho\varphi(\theta)$. From (6), we immediately deduce that $(\lambda-\rho)G_\lambda^{(k,k')}(\theta) = -iT^{(k,k')}\varphi(\theta) + \lambda\varphi(\theta)$, which is the announced result. □

3 Laplace representation formula for the Jacobi-Cherednik kernel

From the explicit expression of the eigenfunctions given by Proposition 2.1, and using the integral representation of the Jacobi functions obtained by T.H. Koornwinder in [7], we will obtain a Laplace integral representation of the function $G_\lambda^{(k,k')}$ which we will call the Jacobi-Cherednik kernel. We first recall the result of T.H. Koornwinder.

Theorem 3.1. *For all $\lambda \in \mathbb{C}$, $\theta \in \left]0, \frac{\pi}{2}\right[$, $\alpha > -\frac{1}{2}$ and $\beta \in \mathbb{R}$, the Jacobi function $R_{\frac{\lambda-\rho}{2}}^{(\alpha,\beta)}$ has the following integral representation [7] :*

$$R_{\frac{\lambda-\rho}{2}}^{(\alpha,\beta)}(\cos(2\theta)) = \int_0^\theta K_0^{(\alpha,\beta)}(\theta,\phi)\cos(\lambda\phi)\,d\phi, \tag{11}$$

where

$$\begin{aligned} K_0^{(\alpha,\beta)}(\theta,\phi) &= \frac{2^{-\alpha+\frac{3}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}(\sin\theta)^{-2\alpha}(\cos\theta)^{-(\alpha+\beta)}(\cos(2\phi)-\cos(2\theta))^{\alpha-\frac{1}{2}} \\ &\times \ {}_2F_1\left(\alpha+\beta, \alpha-\beta; \alpha+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{2\cos\theta}\right)\mathbf{1}_{]-\theta,\theta[}(\phi). \end{aligned} \tag{12}$$

This kernel can also be written in the following forms [8] :

1. *If* $-\frac{1}{2}<\beta<\alpha$,

$$\begin{aligned} K_0^{(\alpha,\beta)}(\theta,\phi) &= \frac{2^{\alpha-2\beta+\frac{3}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(\sin\theta)^{-2\alpha}(\cos\theta)^{-2\beta} \\ &\times \left(\int_{|\phi|}^{\theta}\sin t(\cos(2t)-\cos(2\theta))^{\beta-1/2}(\cos\phi-\cos t)^{\alpha-\beta-1}\,dt\right) \\ &\times \mathbf{1}_{]-\theta,\theta[}(\phi). \end{aligned} \tag{13}$$

2. *If* $-\frac{1}{2}<\beta=\alpha$,

$$K_0^{(\alpha,\beta)}(\theta,\phi)=\frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}(\sin(2\theta))^{-2\alpha}(\cos(2\phi)-\cos(2\theta))^{\alpha-1/2}\mathbf{1}_{]-\theta,\theta[}(\phi). \tag{14}$$

3. *If* $-\frac{1}{2}=\beta<\alpha$,

$$K_0^{(\alpha,\beta)}(\theta,\phi)=\frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}(\sin\theta)^{-2\alpha}(\cos\phi-\cos\theta)^{\alpha-\frac{1}{2}}\mathbf{1}_{]-\theta,\theta[}(\phi). \tag{15}$$

Remark 3.2. *The function* $\theta\longmapsto R^{(\alpha,\beta)}_{\frac{\lambda-\rho}{2}}(\cos(2\theta))$ *is even on* $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$. *The kernel* $K_0^{(\alpha,\beta)}(\theta,\phi)$ *is even in the variable* ϕ *but we can also extend it in an even function in the variable* θ *by defining* $K_0^{(\alpha,\beta)}(\theta,\phi):=K_0^{(\alpha,\beta)}(|\theta|,\phi)$ *if* $\theta\in\left]-\frac{\pi}{2},0\right[$*; in fact this is equivalent, in view of* (12)*, to define* $(\sin\theta)^{-2\alpha}$ *by* $(\sin\theta)^{-2\alpha}:=((\sin\theta)^2)^{-\alpha}=(\sin(|\theta|))^{-2\alpha}$ *if* $\theta\in\left]-\frac{\pi}{2},0\right[$. *In the sequel, we always consider that* $K_0^{(\alpha,\beta)}(\theta,\phi)$ *is so defined for all* $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$ *and* $-|\theta|<\phi<|\theta|$ *and is even in both variables* θ *and* ϕ. *Formulas* (12), (15), (14) *and* (27) *are then valid with* θ *replaced by* $|\theta|$ *and we can write*

$$\forall\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\},\quad R^{(\alpha,\beta)}_{\frac{\lambda-\rho}{2}}(\cos(2\theta))=\frac{1}{2}\int_{-|\theta|}^{|\theta|}K_0^{(\alpha,\beta)}(\theta,\phi)e^{i\lambda\phi}\,d\phi. \tag{16}$$

In his paper [7], p.150, T.H. Koornwinder also gives a Laplace representation formula for the derivative $\frac{d}{d\theta}\left[R^{(\alpha,\beta)}_{\frac{\lambda-\rho}{2}}(\cos(2\theta))\right]$ but it is not adequate for our purpose. We will derive here a crucial integral formula adapted to our problem.

Theorem 3.3. *For all* $\lambda \in \mathbb{C}$, $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$, $\alpha, \beta \in \mathbb{R}$; $-\frac{1}{2} \leq \beta \leq \alpha$ *and* $\alpha > -\frac{1}{2}$, *we have*

$$-\frac{1}{\lambda-\rho}\frac{d}{d\theta}\left[R^{(\alpha,\beta)}_{\frac{\lambda-\rho}{2}}(\cos(2\theta))\right] = \frac{\operatorname{sgn}(\theta)}{2A_{\alpha,\beta}(|\theta|)}\int_{-|\theta|}^{|\theta|}\left(\rho\Psi_\theta(|\phi|) + i\operatorname{sgn}(\phi)\frac{\partial\Psi_\theta}{\partial v}(|\phi|)\right)e^{i\lambda\phi}d\phi, \quad (17)$$

where

$$\Psi_\theta(v) = \int_v^{|\theta|} K_0^{(\alpha,\beta)}(t,v)A_{\alpha,\beta}(t)\,dt, \quad 0 \leq v < |\theta|, \quad (18)$$

$\operatorname{sgn}(\theta)$ *denotes the sign of* θ *and*

$$\forall t \in \left]0, \frac{\pi}{2}\right[, \quad A_{\alpha,\beta}(t) = 2^{2(\alpha+\beta+1)}(\sin t)^{2\alpha+1}(\cos t)^{2\beta+1}. \quad (19)$$

Proof. Let $\lambda \in \mathbb{C}$, $\theta \in \left]0, \frac{\pi}{2}\right[$, $\alpha, \beta \in \mathbb{R}$; $-\frac{1}{2} \leq \beta \leq \alpha$, $\alpha > -\frac{1}{2}$ and $\varphi(\theta) := R^{(\alpha,\beta)}_{\frac{\lambda-\rho}{2}}(\cos(2\theta))$. Let us consider the Jacobi operator

$$\Delta_{\alpha,\beta} = \frac{1}{A_{\alpha,\beta}(\theta)}\frac{d}{d\theta}\left(A_{\alpha,\beta}(\theta)\frac{d}{d\theta}\right).$$

By (11) and the equality $\Delta_{\alpha,\beta}\varphi(\theta) = -(\lambda^2-\rho^2)\varphi(\theta)$, we have

$$\begin{aligned}\varphi'(\theta) &= \frac{1}{A_{\alpha,\beta}(\theta)}\int_0^\theta \Delta_{\alpha,\beta}\varphi(t)A_{\alpha,\beta}(t)\,dt = -\frac{\lambda^2-\rho^2}{A_{\alpha,\beta}(\theta)}\int_0^\theta \varphi(t)A_{\alpha,\beta}(t)\,dt \\ &= -\frac{\lambda^2-\rho^2}{A_{\alpha,\beta}(\theta)}\int_0^\theta\left(\int_0^t K_0^{(\alpha,\beta)}(t,\phi)\cos(\lambda\phi)\,d\phi\right)A_{\alpha,\beta}(t)\,dt.\end{aligned}$$

By Fubini-Tonelli's theorem, we can write

$$\varphi'(\theta) = -\frac{\lambda^2-\rho^2}{A_{\alpha,\beta}(\theta)}\int_0^\theta\left(\int_\phi^\theta K_0^{(\alpha,\beta)}(t,\phi)A_{\alpha,\beta}(t)\,dt\right)\cos(\lambda\phi)d\phi.$$

Therefore

$$-\frac{1}{\lambda-\rho}\varphi'(\theta) = \frac{\lambda+\rho}{A_{\alpha,\beta}(\theta)}\int_0^\theta\left(\int_\phi^\theta K_0^{(\alpha,\beta)}(t,\phi)A_{\alpha,\beta}(t)\,dt\right)\cos(\lambda\phi)\,d\phi.$$

Integration by parts gives

$$\begin{aligned}-\frac{1}{\lambda-\rho}\varphi'(\theta) &= \frac{\rho}{A_{\alpha,\beta}(\theta)}\int_0^\theta\left(\int_\phi^\theta K_0^{(\alpha,\beta)}(t,\phi)A_{\alpha,\beta}(t)\,dt\right)\cos(\lambda\phi)\,d\phi \\ &\quad -\frac{1}{A_{\alpha,\beta}(\theta)}\int_0^\theta\frac{\partial}{\partial\phi}\left(\int_\phi^\theta K_0^{(\alpha,\beta)}(t,\phi)A_{\alpha,\beta}(t)\,dt\right)\sin(\lambda\phi)d\phi.\end{aligned}$$

Since the fonction φ' is odd on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, then for all $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$,

$$\begin{aligned}
-\frac{1}{\lambda-\rho}\varphi'(\theta) &= -\frac{\operatorname{sgn}(\theta)}{\lambda-\rho}\varphi'(|\theta|)\\
&= \frac{\rho\operatorname{sgn}(\theta)}{A_{\alpha,\beta}(|\theta|)}\int_0^{|\theta|}\left(\int_\phi^{|\theta|}K_0^{(\alpha,\beta)}(t,\phi)A_{\alpha,\beta}(t)\,dt\right)\cos(\lambda\phi)d\phi\\
&\quad-\frac{\operatorname{sgn}(\theta)}{A_{\alpha,\beta}(|\theta|)}\int_0^{|\theta|}\frac{\partial}{\partial\phi}\left(\int_\phi^{|\theta|}K_0^{(\alpha,\beta)}(t,\phi)A_{\alpha,\beta}(t)\,dt\right)\sin(\lambda\phi)\,d\phi.
\end{aligned}$$

Finally, if we denote by Ψ_θ the function $v\longmapsto\displaystyle\int_v^{|\theta|}K_0^{(\alpha,\beta)}(t,v)A_{\alpha,\beta}(t)\,dt$, $0\le v<|\theta|$, we verify immediately that

$$-\frac{1}{\lambda-\rho}\varphi'(\theta)$$

$$=\frac{\operatorname{sgn}(\theta)}{2A_{\alpha,\beta}(|\theta|)}\left(\int_{-|\theta|}^{|\theta|}\rho\Psi_\theta(|\phi|)\cos(\lambda\phi)\,d\phi-\int_{-|\theta|}^{|\theta|}\operatorname{sgn}(\phi)\frac{\partial\Psi_\theta}{\partial v}(|\phi|)\sin(\lambda\phi)\,d\phi\right)$$

and this finishes the proof. □

We can now derive a formal expression for the kernel $K^{(k,k')}(\theta,\phi)$ announced in formula (4) in the introduction.

Corollary 3.4. *For all $\lambda\in\mathbb{C}$ and $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$, we have*

$$G_\lambda^{(k,k')}(\theta)=\int_{-|\theta|}^{|\theta|}K^{(k,k')}(\theta,\phi)e^{i\lambda\phi}\,d\phi, \tag{20}$$

where

$$\begin{aligned}
K^{(k,k')}(\theta,\phi) := \frac{1}{2}\Bigg[&K_0^{(k-\frac12,k'-\frac12)}(\theta,\phi)\\
&-\frac{\operatorname{sgn}(\theta)\operatorname{sgn}(\phi)}{A_{k-\frac12,k'-\frac12}(|\theta|)}\frac{\partial}{\partial v}\left(\int_v^{|\theta|}K_0^{(k-\frac12,k'-\frac12)}(t,v)A_{k-\frac12,k'-\frac12}(t)\,dt\right)(|\phi|)\\
&+i\frac{\rho\operatorname{sgn}(\theta)}{A_{k-\frac12,k'-\frac12}(|\theta|)}\int_{|\phi|}^{|\theta|}K_0^{(k-\frac12,k'-\frac12)}(t,\phi)A_{k-\frac12,k'-\frac12}(t)\,dt\Bigg]\mathbf{1}_{]-|\theta|,|\theta|[}(\phi),
\end{aligned}\tag{21}$$

$K_0^{(k-\frac12,k'-\frac12)}$ is as in Theorem 3.1 and $A_{k-\frac12,k'-\frac12}$ is given by (19).

Proof. This follows immediately from (6), (16) and Theorem 3.3. □

4 Explicit form of Laplace kernel

In this section we give an explicit expression of the function $K^{(k,k')}(\theta,\phi)$ defined by (21) which will be called Laplace kernel.

4.1 The case $k' = 0 < k$.

Theorem 4.1. *For all* $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$ *and* $-|\theta| < \phi < |\theta|$, *we have*

1. $K_0^{(k-\frac{1}{2},-\frac{1}{2})}(\theta,\phi) = \dfrac{2^k\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k)}(\sin(|\theta|))^{-2k+1}(\cos\phi - \cos\theta)^{k-1} > 0.$

2.

$$K^{(k,0)}(\theta,\phi) = K_0^{(k-\frac{1}{2},-\frac{1}{2})}(\theta,\phi)\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\theta}e^{i\frac{\theta-\phi}{2}}. \tag{22}$$

3.

$$0 < \left|K^{(k,0)}(\theta,\phi)\right| = e^{-i\frac{\theta-\phi}{2}}K^{(k,0)}(\theta,\phi) < K_0^{(k-\frac{1}{2},-\frac{1}{2})}(\theta,\phi). \tag{23}$$

Proof. Let $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$ and $-|\theta| < \phi < |\theta|$. We denote by $K_0(\theta,\phi) := K_0^{(k-\frac{1}{2},-\frac{1}{2})}(\theta,\phi)$, $K(\theta,\phi) := K^{(k,0)}(\theta,\phi)$ and $A(t) := A_{k-\frac{1}{2},-\frac{1}{2}}(t)$, $0 < t < \frac{\pi}{2}$ to simplify notations. By (15),

$$K_0(\theta,\phi) = C_1(\sin(|\theta|))^{-2k+1}(\cos\phi - \cos\theta)^{k-1}, \quad C_1 := \frac{2^k\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k)}.$$

As $\rho = k$ and $A(t) = 2^{2k}(\sin t)^{2k}$, $\quad 0 < t < \frac{\pi}{2}$, we have

$$\int_v^{|\theta|} K_0(t,v)A(t)dt = C_1\frac{2^{2k}}{k}(\cos v - \cos\theta)^k$$

and

$$\frac{\partial}{\partial v}\int_v^{|\theta|} K_0(t,v)A(t)dt = -C_1 2^{2k}\sin v(\cos v - \cos\theta)^{k-1}, \quad 0 \le v < \frac{\pi}{2}.$$

Then by (21), we get

$$\begin{aligned}
K(\theta,\phi) &= \frac{C_1}{2}(\sin(|\theta|))^{-2k+1}(\cos\phi-\cos\theta)^{k-1}\\
&\times\left[1+\frac{\mathrm{sgn}(\theta)\mathrm{sgn}(\phi)\sin(|\phi|)}{\sin(|\theta|)}+i\frac{\mathrm{sgn}(\theta)(\cos\phi-\cos\theta)}{\sin(|\theta|)}\right]\\
&= \frac{C_1}{2}\mathrm{sgn}(\theta)(\sin(|\theta|))^{-2k}(\cos\phi-\cos\theta)^{k-1}\\
&\times[\sin\theta+\sin\phi+i(\cos\phi-\cos\theta)]\\
&= i\frac{C_1}{2}\mathrm{sgn}(\theta)(\sin(|\theta|))^{-2k}(\cos\phi-\cos\theta)^{k-1}(e^{-i\phi}-e^{i\theta})\\
&= K_0(\theta,\phi)\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\theta}e^{i\frac{\theta-\phi}{2}}.
\end{aligned}$$

As $0<\dfrac{\sin\left(\frac{\theta+\phi}{2}\right)}{\sin\theta}<1$ and $K_0(\theta,\phi)>0$, then

$$0<|K(\theta,\phi)|=e^{-i\frac{\theta-\phi}{2}}K(\theta,\phi)<K_0(\theta,\phi).$$

□

4.2 The case $0<k=k'$.

Theorem 4.2. *For all* $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$ *and* $-|\theta|<\phi<|\theta|$, *we have*

1.

$$\begin{aligned}
K_0^{(k-\frac12,k-\frac12)}(\theta,\phi) &= \frac{2^{k+1}\Gamma(k+\frac12)}{\sqrt{\pi}\,\Gamma(k)}(\sin(2|\theta|))^{-2k+1}(\cos(2\phi)-\cos(2\theta))^{k-1}\\
&> 0.
\end{aligned}$$

2.

$$K^{(k,k)}(\theta,\phi)=K_0^{(k-\frac12,k-\frac12)}(\theta,\phi)\frac{\sin(\theta+\phi)}{\sin(2\theta)}e^{i(\theta-\phi)}. \tag{24}$$

3.

$$0<\left|K^{(k,k)}(\theta,\phi)\right|=e^{-i(\theta-\phi)}K^{(k,k)}(\theta,\phi)<m(\theta)K_0^{(k-\frac12,k-\frac12)}(\theta,\phi), \tag{25}$$

where

$$m(\theta):=\begin{cases}1 & \text{if}\quad 0<|\theta|\le\frac{\pi}{4},\\ \dfrac{1}{\sin(2|\theta|)} & \text{if}\quad \frac{\pi}{4}<|\theta|<\frac{\pi}{2}.\end{cases} \tag{26}$$

Proof. Let $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$ and $-|\theta| < \phi < |\theta|$. We denote by $K_0(\theta,\phi) := K_0^{(k-\frac{1}{2},k-\frac{1}{2})}(\theta,\phi)$, $K(\theta,\phi) := K^{(k,k)}(\theta,\phi)$ and $A(t) := A_{k-\frac{1}{2},k-\frac{1}{2}}(t)$, $0 < t < \frac{\pi}{2}$ to simplify notations. By (14),

$$K_0(\theta,\phi) = C_2(\sin(2|\theta|))^{-2k+1}(\cos(2\phi) - \cos(2\theta))^{k-1}, \quad C_2 := \frac{2^{k+1}\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k)}.$$

As $\rho = 2k$ and $A(t) = 2^{2k}(\sin(2t))^{2k}, \quad 0 < t < \frac{\pi}{2}$, we have

$$\int_v^{|\theta|} K_0(t,v)A(t)dt = C_2\frac{2^{2k-1}}{k}(\cos(2v) - \cos(2\theta))^k$$

and

$$\frac{\partial}{\partial v}\int_v^{|\theta|} K_0(t,v)A(t)dt = -C_2 2^{2k}\sin(2v)(\cos(2v) - \cos(2\theta))^{k-1}, \quad 0 \le v < \frac{\pi}{2}.$$

Then by (21), we get

$$\begin{aligned}
K(\theta,\phi) &= \frac{C_2}{2}(\sin(2|\theta|))^{-2k+1}(\cos(2\phi) - \cos(2\theta))^{k-1} \\
&\times \left[1 + \frac{\operatorname{sgn}(\theta)\operatorname{sgn}(\phi)\sin(2|\phi|)}{\sin(2|\theta|)} + i\frac{\operatorname{sgn}(\theta)(\cos(2\phi) - \cos(2\theta))}{\sin(2|\theta|)}\right] \\
&= \frac{C_2}{2}\operatorname{sgn}(\theta)(\sin(2|\theta|))^{-2k}(\cos(2\phi) - \cos(2\theta))^{k-1} \\
&\times [\sin(2\theta) + \sin(2\phi) + i(\cos(2\phi) - \cos(2\theta))] \\
&= i\frac{C_2}{2}\operatorname{sgn}(\theta)(\sin(2|\theta|))^{-2k}(\cos(2\phi) - \cos(2\theta))^{k-1}(e^{-2i\phi} - e^{2i\theta}) \\
&= K_0(\theta,\phi)\frac{\sin(\theta+\phi)}{\sin(2\theta)}e^{i(\theta-\phi)}.
\end{aligned}$$

If $0 < |\theta| \le \frac{\pi}{4}$, then $0 < \frac{\sin(\theta+\phi)}{\sin(2\theta)} < 1$. If $\frac{\pi}{4} < |\theta| < \frac{\pi}{2}$, then $0 < \frac{\sin(\theta+\phi)}{\sin(2\theta)} < \frac{1}{\sin(2|\theta|)}$. As $K_0(\theta,\phi) > 0$, then

$$0 < |K(\theta,\phi)| = e^{-i(\theta-\phi)}K(\theta,\phi) < m(\theta)K_0(\theta,\phi),$$

where $m(\theta)$ is given by (26). □

4.3 The case $0 < k' < k$.

Theorem 4.3. *For all* $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}$ *and* $-|\theta| < \phi < |\theta|$, *we have*

1. $$K_0^{(k-\frac{1}{2},k'-\frac{1}{2})}(\theta,\phi) = \frac{2^k\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k-k')\Gamma(k')}(\sin(|\theta|))^{-2k+1}(\cos\theta)^{-k'}$$
$$\times(\cos\phi-\cos\theta)^{k-1}\int_0^1 s^{k'-1}(1-s)^{k-k'-1}\left(1+\frac{\cos\phi-\cos\theta}{2\cos\theta}s\right)^{k'-1} ds > 0. \tag{27}$$

2. $$K^{(k,k')}(\theta,\phi) = \frac{2^{k-2}\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k-k')\Gamma(k'+1)}\mathrm{sgn}(\theta)(\sin(|\theta|))^{-2k}(\cos\theta)^{-(k'+1)}$$
$$\begin{aligned}\times &\quad (\cos\phi-\cos\theta)^{k-1}\int_0^1 s^{k'-1}(1-s)^{k-k'-1}\left(1+\frac{\cos\phi-\cos\theta}{2\cos\theta}s\right)^{k'-1}\\ \times &\quad \Big[k'\sin(2\theta)+\sin\phi\, s[(k+k')(\cos\phi-\cos\theta)s+2k\cos\theta]\\ + &\quad i(k+k')(\cos\phi-\cos\theta)s[(\cos\phi-\cos\theta)s+2\cos\theta]\Big]\, ds. \end{aligned} \tag{28}$$

3. $$\left|K^{(k,k')}(\theta,\phi)\right| \le \frac{(1+\sqrt{2})(k+k')}{4k'\cos\theta}K_0^{(k-\frac{1}{2},k'-\frac{1}{2})}(\theta,\phi). \tag{29}$$

Proof. Let $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}$ and $-|\theta| < \phi < |\theta|$. We denote by $K_0(\theta,\phi) := K_0^{(k-\frac{1}{2},k'-\frac{1}{2})}(\theta,\phi)$, $K(\theta,\phi) := K^{(k,k')}(\theta,\phi)$ and $A(t) := A_{k-\frac{1}{2},k'-\frac{1}{2}}(t)$, $0 < t < \frac{\pi}{2}$ to simplify notations. By (27),

$$K_0(\theta,\phi) = C_3(\sin(|\theta|))^{-2k+1}(\cos\theta)^{-2k'+1}I_{k,k'}(\theta,|\phi|),$$

where $C_3 := \dfrac{2^{k-2k'+2}\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k-k')\Gamma(k')}$ and $\forall v \in [0,|\theta|[$,

$$I_{k,k'}(\theta,v) := \int_v^{|\theta|}\sin w(\cos(2w)-\cos(2\theta))^{k'-1}(\cos v-\cos w)^{k-k'-1}\,dw.$$

The equality $\cos(2w)-\cos(2\theta) = 2((\cos w)^2-(\cos\theta)^2)$ and the change of variables $\cos w = \cos\theta + s(\cos v-\cos\theta)$, $\quad 0 < s < 1$ give

$$\begin{aligned} I_{k,k'}(\theta,v) &= 2^{2(k'-1)}(\cos\theta)^{k'-1}(\cos v-\cos\theta)^{k-1}\\ &\quad\times\int_0^1 s^{k'-1}(1-s)^{k-k'-1}(1+Z_v s)^{k'-1}\,ds,\end{aligned}$$

where

$$0 < Z_v = \frac{\cos v - \cos\theta}{2\cos\theta}, \quad 0 \le v < |\theta|.$$

We now compute, for $v \in [0, |\theta|[$, the integral $\int_v^{|\theta|} K_0(t,v)A(t)\,dt$ which appears in the second and third terms in the right hand side of formula (21). As $\rho = k + k'$, $A(t) = 2^{2(k+k')}(\sin t)^{2k}(\cos t)^{2k'}$, $\quad 0 < t < \frac{\pi}{2}$ and by using Fubini's theorem, we have

$$\begin{aligned}
&\int_v^{|\theta|} K_0(t,v)A(t)\,dt = C_3 2^{2(k+k')-1}\int_v^{|\theta|}\sin(2t) I_{k,k'}(t,v)\,dt\\
&= C_3 2^{2(k+k')-1}\int_v^{|\theta|}\sin(2t)\\
&\times\left(\int_v^t \sin w(\cos(2w)-\cos(2t))^{k'-1}(\cos v-\cos w)^{k-k'-1}\,dw\right)\,dt\\
&= C_3 2^{2(k+k')-1}\int_v^{|\theta|}\left(\int_w^{|\theta|}\sin(2t)(\cos(2w)-\cos(2t))^{k'-1}\,dt\right)\\
&\times(\cos v-\cos w)^{k-k'-1}\sin w\,dw\\
&= C_3 2^{2(k+k')-1}\int_v^{|\theta|}\left[\frac{(\cos(2w)-\cos(2t))^{k'}}{2k'}\right]_{t=w}^{t=|\theta|}\\
&\times(\cos v-\cos w)^{k-k'-1}\sin w\,dw\\
&= \frac{C_3 2^{2(k+k'-1)}}{k'}\int_v^{|\theta|}(\cos(2w)-\cos(2\theta))^{k'}(\cos v-\cos w)^{k-k'-1}\sin w\,dw\\
&= \frac{C_3 2^{2(k+k'-1)}}{k'} I_{k+1,k'+1}(\theta,v)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial v} I_{k+1,k'+1}(\theta,v) = -k\sin v(\cos v-\cos\theta)^{-1} I_{k+1,k'+1}(\theta,v)\\
&-2^{2k'-1}k'(\cos\theta)^{k'-1}(\cos v-\cos\theta)^k\sin v\\
&\times\int_0^1 s^{k'+1}(1-s)^{k-k'-1}(1+Z_v s)^{k'-1}\,ds\\
&= -2^{2k'}(\cos\theta)^{k'}(\cos v-\cos\theta)^{k-1}\sin v\\
&\times\int_0^1 s^{k'}(1-s)^{k-k'-1}(1+Z_v s)^{k'-1}[k+(k+k')Z_v s]\,ds.
\end{aligned}$$

From (21), we get

$$\begin{aligned} K(\theta,\phi) &= C_3 2^{2(k'-1)}(\sin(|\theta|))^{-2k+1}(\cos\theta)^{-k'}(\cos\phi-\cos\theta)^{k-1}\frac{\operatorname{sgn}(\theta)}{2k'\sin(|\theta|)} \\ &\times \int_0^1 s^{k'-1}(1-s)^{k-k'-1}(1+Zs)^{k'-1} \\ &\times \Big[k'\operatorname{sgn}(\theta)\sin(|\theta|)+\operatorname{sgn}(\phi)\sin(|\phi|)s[k+(k+k')Zs] \\ &+ i(k+k')(\cos\phi-\cos\theta)s(1+Zs)\Big]\, ds, \end{aligned}$$

with $Z=\dfrac{\cos\phi-\cos\theta}{2\cos\theta}$. Hence, we get (28). For all $s\in]0,1[$, we have

$$\begin{aligned} &\Big| k'\operatorname{sgn}(\theta)\sin(2|\theta|)+\operatorname{sgn}(\phi)\sin(|\phi|)\, s[2k\cos\theta+(k+k')(\cos\phi-\cos\theta)s] \\ &+i(k+k')(\cos\phi-\cos\theta)s[2\cos\theta+(\cos\phi-\cos\theta)s]\Big| \\ &\le \Big[k'\sin(2|\theta|)+\sin(|\phi|)[2k\cos\theta+(k+k')(\cos\phi-\cos\theta)] \\ &+(k+k')(\cos\phi-\cos\theta)[2\cos\theta+(\cos\phi-\cos\theta)]\Big] \\ &= \sin(|\phi|)[(k+k')(\cos\phi-\sin(|\phi|))+(k-k')\cos\theta] \\ &+\sin(|\theta|)[(k+k')\sin(|\theta|)+2k'\cos\theta] \\ &\le \sin(|\theta|)[(k+k')+(k-k')\cos\theta]+\sin(|\theta|)[(k+k')\sin(|\theta|)+2k'\cos\theta] \\ &= (k+k')\sin(|\theta|)(1+\cos\theta+\sin(|\theta|)) \\ &= (k+k')\sin(|\theta|)\left(1+\sqrt{2}\cos\left(|\theta|-\frac{\pi}{4}\right)\right) \\ &\le (1+\sqrt{2})(k+k')\sin(|\theta|). \end{aligned}$$

Then, we get (29). □

5 Estimates for the Jacobi-Cherednik kernel

In this section we give some properties of the Jacobi-Cherednik kernel $G_\lambda^{(k,k')}(\theta)$ in the following cases :

5.1 The case $k'=0<k$.

Proposition 5.1.

$$\forall\lambda\in\mathbb{C},\ \forall\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[,\quad \left|G_\lambda^{(k,0)}(\theta)\right|\le 2e^{|\Im\lambda||\theta|}R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)).$$

Proof. This follows immediately from formulas (4), (23), (16) and the fact that

$$G_\lambda^{(k,0)}(0) = 1 = R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(1).$$

□

Lemma 5.2. *Let* $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\smallsetminus \{0\}$ *and*

$$I_k(\theta) := \int_0^{|\theta|} (\cos\phi - \cos\theta)^{k-1}\, d\phi. \tag{30}$$

Then we have

1.

$$I_k(\theta) = \frac{\sqrt{\pi}\,\Gamma(k)}{2^k\Gamma(k+\frac{1}{2})}(\sin(|\theta|))^{2k-1}\,{}_2F_1\left(k,k;k+\frac{1}{2};\left(\sin\left(\frac{|\theta|}{2}\right)\right)^2\right). \tag{31}$$

2.

$$I_k(\theta) = 2^{k-1}(\sin(|\theta|))^{2k-1}\int_0^{+\infty}\frac{t^{k-1}}{(1+2t\cos\theta+t^2)^k}\,dt. \tag{32}$$

3.

$$\frac{\sqrt{\pi}\,\Gamma(k)}{2^k\Gamma(k+\frac{1}{2})}(\sin(|\theta|))^{2k-1} \le I_k(\theta) \le \frac{\sqrt{\pi}\,\Gamma(\frac{k}{2})}{2\Gamma\left(\frac{k+1}{2}\right)}(\sin(|\theta|))^{2k-1}. \tag{33}$$

Proof. We get (31) in view of [3], p.383, 999, and [2], p.64, formula (23). We deduce (32) from [3], p.383, 1002 and 938. As

$$\begin{aligned} B(k,k) = \int_0^{+\infty}\frac{t^{k-1}}{(1+t)^{2k}}dt \ &\le \int_0^{+\infty}\frac{t^{k-1}}{(1+2t\cos\theta+t^2)^k}\,dt \\ &\le \int_0^{+\infty}\frac{t^{k-1}}{(1+t^2)^k}\,dt = \frac{1}{2}B\left(\frac{k}{2},\frac{k}{2}\right), \end{aligned}$$

(see [3], p.948), $\Gamma(2k) = \dfrac{2^{2k-1}}{\sqrt{\pi}}\Gamma(k)\Gamma\left(k+\dfrac{1}{2}\right)$ (see [3], p.938) and by using (32), then we obtain (33). □

Theorem 5.3.

$$\forall\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[, \quad 1 \le R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)) \le \frac{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}.$$

Proof. Let $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}$. We have

$$\begin{aligned} R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)) &= \int_0^{|\theta|} K_0^{(k-\frac{1}{2},-\frac{1}{2})}(\theta,\phi)\,d\phi \\ &= \frac{2^k\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(k)}(\sin(|\theta|))^{-2k+1}I_k(\theta), \end{aligned} \tag{34}$$

where $I_k(\theta)$ is given by (30). To finish the proof we use (33) and the equality

$$\frac{2^{k-1}\Gamma(k+\frac{1}{2})\Gamma\left(\frac{k}{2}\right)}{\Gamma(k)\Gamma\left(\frac{k+1}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}.$$

□

Corollary 5.4.

$$\forall \lambda \in \mathbb{C},\ \forall \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[, \quad \left|G_\lambda^{(k,0)}(\theta)\right| \le \frac{2\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2} e^{|\Im\lambda||\theta|}.$$

Proof. Proposition 5.1 and Theorem 5.3 give the result. □

Remarks 5.5.

1. *From* (4), (23) *and* (16) *we get*

$$\forall \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}, \quad 0 < e^{-i\frac{\theta}{2}} G_{\frac{1}{2}}^{(k,0)}(\theta) < 2R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)).$$

2. $\forall \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[,\ R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)) = {}_2F_1\left(k,k;k+\frac{1}{2};\left(\sin\left(\frac{|\theta|}{2}\right)\right)^2\right).$

5.2 The case $0 < k = k'$.

Proposition 5.6.

$$\forall \lambda \in \mathbb{C},\ \forall \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[, \quad \left|G_\lambda^{(k,k)}(\theta)\right| \le 2m(\theta)e^{|\Im\lambda||\theta|}R_{-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(\cos(2\theta)),$$

where $m(\theta)$ *is given by* (26).

Proof. This follows immediately from formulas (4), (25), (16) and the fact that

$$G_\lambda^{(k,k)}(0) = 1 = R_{-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(1).$$

□

Theorem 5.7. *For all* $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *we have*

$$(\cos\theta)^{1-\min(2k,k+1)} \leq R_{-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(\cos(2\theta)) \leq \frac{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}(\cos\theta)^{1-\max(2k,k+1)}.$$

Proof. Let $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$. We have

$$\begin{aligned} R_{-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(\cos(2\theta)) &= \frac{2^k\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k)}(\sin(|\theta|))^{-2k+1}(\cos\theta)^{-2k+1} \\ &\times \int_0^{|\theta|}(\cos\phi - \cos\theta)^{k-1}\left(\frac{\cos\phi+\cos\theta}{2}\right)^{k-1} d\phi. \end{aligned}$$

By using $(\cos\theta)^{\max(k-1,0)} \leq \left(\frac{\cos\phi+\cos\theta}{2}\right)^{k-1} \leq (\cos\theta)^{\min(k-1,0)}$ and (34) we get

$$\begin{aligned} (\cos\theta)^{1-\min(2k,k+1)}R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)) &\leq R_{-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(\cos(2\theta)) \\ &\leq (\cos\theta)^{1-\max(2k,k+1)}R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)). \end{aligned}$$

Now Theorem 5.3 gives the result. □

Corollary 5.8. *For all* $\lambda \in \mathbb{C}$ *and* $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *we have*

$$\left|G_\lambda^{(k,k)}(\theta)\right| \leq \frac{2\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{(\Gamma(\frac{k+1}{2}))^2}m(\theta)(\cos\theta)^{1-\max(2k,k+1)}\,e^{|\Im\lambda||\theta|},$$

where $m(\theta)$ *is given by* (26).

Proof. Proposition 5.6 and Theorem 5.7 give the result. □

Remark 5.9. *From* (4), (25) *and* (16) *we get*

$$\forall\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}, \quad 0 < e^{-i\theta}G_1^{(k,k)}(\theta) < 2m(\theta)R_{-k}^{(k-\frac{1}{2},k-\frac{1}{2})}(\cos(2\theta)),$$

where $m(\theta)$ *is given by* (26).

5.3 The case $0 < k' < k$.

Proposition 5.10. *For all* $\lambda \in \mathbb{C}$ *and* $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *we have*

$$\left|G_\lambda^{(k,k')}(\theta)\right| \leq \frac{(1+\sqrt{2})(k+k')}{2k'\cos\theta}e^{|\Im\lambda||\theta|}R_{-\frac{k+k'}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)).$$

Proof. This follows immediately from formulas (4), (29), (16) and the fact that

$$G_\lambda^{(k,k')}(0) = 1 = R_{-\frac{k+k'}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(1).$$

□

Theorem 5.11. *For all* $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$*, we have*

$$(\cos\theta)^{1-\min(2k',k'+1)} \le R_{-\frac{k+k'}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)) \le \frac{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}(\cos\theta)^{1-\max(2k',k'+1)}.$$

Proof. Let $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}$, $\phi \in]-|\theta|,|\theta|[$, $Z = \frac{\cos\phi - \cos\theta}{2\cos\theta}$ and $s \in]0,1[$. We have

$$1 < 1 + Zs < 1 + Z = \frac{\cos\phi + \cos\theta}{2\cos\theta} \le \frac{1+\cos\theta}{2\cos\theta} < \frac{1}{\cos\theta}.$$

So, $(\cos\theta)^{-\min(k'-1,0)} \le (1+Zs)^{k'-1} \le (\cos\theta)^{-\max(k'-1,0)}$. By using (27), we get

$$(\cos\theta)^{1-\min(2k',k'+1)} R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta))$$
$$\le R_{-\frac{k+k'}{2}}^{(k-\frac{1}{2},k'-\frac{1}{2})}(\cos(2\theta)) \le (\cos\theta)^{1-\max(2k',k'+1)} R_{-\frac{k}{2}}^{(k-\frac{1}{2},-\frac{1}{2})}(\cos(2\theta)).$$

Now Theorem 5.3 gives the result. □

Corollary 5.12. *For all* $\lambda \in \mathbb{C}$ *and* $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$*, we have*

$$\left|G_\lambda^{(k,k')}(\theta)\right| \le \frac{(1+\sqrt{2})\sqrt{\pi}(k+k')\Gamma(k+\frac{1}{2})}{2k'\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}(\cos\theta)^{-\max(2k',k'+1)}\, e^{|\Im\lambda||\theta|}.$$

Proof. Proposition 5.10 and Theorem 5.11 give the result. □

Remark 5.13. *In view of Theorem* 5.7*, we remark that Theorem* 5.11 *is also valid for* $k = k'$*.*

6 The Jacobi-Cherednik intertwining operator

Definition 6.1. *For every continuous function* $f : \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\to \mathbb{C}$*, we define*

$$V^{(k,k')}f(\theta) := \begin{cases} \int_{-|\theta|}^{|\theta|} K^{(k,k')}(\theta,\phi)f(\phi)\,d\phi & \text{if} \quad \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}, \\ f(0) & \text{if} \quad \theta = 0. \end{cases} \tag{35}$$

For functions $e_\lambda : \theta \longmapsto e^{\lambda\theta}, \quad \lambda \in \mathbb{C}$, we have $V^{(k,k')}e_\lambda = G^{(k,k')}_{-i\lambda}$ and by (3) we remark that

$$T^{(k,k')}V^{(k,k')}e_\lambda = V^{(k,k')}\frac{d}{d\theta}e_\lambda. \tag{36}$$

This is the intertwining relation (5) for functions $f = e_\lambda$. The aim of this section is to extend this relation to functions $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and to show that V is an automorphism of $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. The main tool is the general form (21) of the Laplace kernel $K^{(k,k')}(\theta,\phi)$ in terms of the kernel $K_0(\theta,\phi)$ as given by Theorem 3.1. This kernel K_0 is the Laplace kernel associated to the Jacobi functions $R^{(k-\frac{1}{2},k'-\frac{1}{2})}_{\frac{\lambda-\rho}{2}}(\cos(2\theta))$ (see (11)). By a result of K. Trimèche ([12]), we know that the operator $\mathcal{K}_0$ associated with the kernel K_0 and defined on $\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ (the space of even functions $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$) by

$$\mathcal{K}_0 f(\theta) := \begin{cases} \displaystyle\int_0^{|\theta|} K_0(\theta,\phi)f(\phi)\,d\phi & \text{if} \quad \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}, \\ f(0) & \text{if} \quad \theta = 0 \end{cases} \tag{37}$$

is the unique topological automorphism of $\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ that intertwines the second derivative $\frac{d^2}{d\theta^2}$ and the extended Jacobi operator $\Delta - \rho^2$, where

$$\Delta := \frac{1}{A(|\theta|)}\frac{d}{d\theta}\left(A(|\theta|)\frac{d}{d\theta}\right), \quad \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}, \tag{38}$$

and $A(|\theta|) := A_{k-1/2,k'-1/2}(|\theta|)$. In other words, for all $f \in \mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, we have

$$\begin{cases} \Delta\mathcal{K}_0 f &= \mathcal{K}_0\left(\frac{d^2}{d\theta^2}+\rho^2\right)f, \\ \mathcal{K}_0 f(0) &= f(0). \end{cases} \tag{39}$$

In order to simplify notations, we will simply write $K(\theta,\phi) := K^{(k,k')}(\theta,\phi)$, $V := V^{(k,k')}$ and $T := T^{(k,k')}$. We also use the notations introduced in the above sections. In addition we will consider the space $\mathcal{E}_o\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ of odd C^∞ complex valued functions on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and we suppose that the spaces $\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $\mathcal{E}_o\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ carry the induced topology of $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. For every function $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ we will denote by f_e and f_o respectively, its even and odd parts, that is

$$\forall\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[, \quad f_e(\theta) := \frac{1}{2}(f(\theta)+f(-\theta)), \quad f_o(\theta) := \frac{1}{2}(f(\theta)-f(-\theta)).$$

Theorem 6.2. *For all* $f\in\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$*, we have*

$$Vf=\mathcal{K}_0 f_e+\frac{d}{d\theta}\mathcal{K}_0\tilde{I}\left(i\rho I f_e+f_o\right), \tag{40}$$

where I *is the primitive operator, which vanishes at* 0*, defined by*

$$If(\phi):=\int_0^{\phi}f(t)\,dt$$

and $\tilde{I}$ *is the operator given by*

$$\tilde{I}f(\theta):=\int_0^{\theta}f(\phi)\cos(\rho(\theta-\phi))\,d\phi.$$

Proof. i) Let $f\in\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus\{0\}$. By (21), we get

$$\begin{aligned}Vf(\theta)&=\int_0^{|\theta|}\left(K(\theta,\phi)+K(\theta,-\phi)\right)f(\phi)\,d\phi\\&=\int_0^{|\theta|}K_0(\theta,\phi)f(\phi)\,d\phi+i\rho\frac{\operatorname{sgn}(\theta)}{A(|\theta|)}\int_0^{|\theta|}\int_{\phi}^{|\theta|}K_0(t,\phi)A(t)\,dtf(\phi)\,d\phi,\end{aligned} \tag{41}$$

and applying Fubini's theorem to the second integral of (41), we see that

$$\begin{aligned}Vf(\theta)&=\mathcal{K}_0f(\theta)+i\rho\frac{\operatorname{sgn}(\theta)}{A(|\theta|)}\int_0^{|\theta|}\left(\int_0^t K_0(t,\phi)f(\phi)\,d\phi A(t)\,dt\right)\\&=\mathcal{K}_0f(\theta)+i\rho\frac{\operatorname{sgn}(\theta)}{A(|\theta|)}\int_0^{|\theta|}\mathcal{K}_0f(t)A(t)\,dt\\&=\mathcal{K}_0f(\theta)+i\rho\frac{\operatorname{sgn}(\theta)}{A(|\theta|)}\int_0^{|\theta|}\mathcal{K}_0\left(\frac{d^2}{d\theta^2}+\rho^2\right)\tilde{I}(If)(t)A(t)\,dt\\&=\mathcal{K}_0f(\theta)+i\rho\frac{\operatorname{sgn}(\theta)}{A(|\theta|)}\int_0^{|\theta|}\Delta\mathcal{K}_0\tilde{I}(If)(t)A(t)\,dt,\end{aligned} \tag{42}$$

where we have used that the unique solution of $\left(\frac{d^2}{d\theta^2}+\rho^2\right)u=f$, $u(0)=0,\ u'(0)=0$ is $u=\tilde{I}(If)$ and the intertwining relation (39) to deduce (42). But for $g\in\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, it is easy to see that for all $\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$,

$$\frac{\operatorname{sgn}(\theta)}{A(|\theta|)}\int_0^{|\theta|}\Delta g(t)A(t)\,dt=\frac{d}{d\theta}g(\theta). \tag{43}$$

By applying this formula to the function $g = \mathcal{K}_0 \tilde{I}(If)$, the result of the theorem follows from (42).
ii) Let $f \in \mathcal{E}_o\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ and $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$. We have

$$\begin{aligned} Vf(\theta) &= \int_0^{|\theta|} (K(\theta,\phi) - K(\theta,-\phi)) f(\phi)\, d\phi \\ &= -\frac{\operatorname{sgn}(\theta)}{A(|\theta|)} \int_0^{|\theta|} \frac{\partial}{\partial v}\left(\int_v^{|\theta|} K_0(t,v)A(t)\,dt\right)(\phi) f(\phi)\,d\phi \\ &= \frac{\operatorname{sgn}(\theta)}{A(|\theta|)} \int_0^{|\theta|} \left(\int_\phi^{|\theta|} K_0(t,\phi)A(t)\,dt\right) f'(\phi)\,d\phi, \end{aligned} \tag{44}$$

by a trivial integration by parts. But if we interchange integrations in (44) we obtain

$$Vf(\theta) = \frac{\operatorname{sgn}(\theta)}{A(|\theta|)} \int_0^{|\theta|} (\mathcal{K}_0 f')(t) A(t)\,dt. \tag{45}$$

But if in (45) we note that $f' = \left(\frac{d^2}{d\theta^2} + \rho^2\right) \tilde{I} f$, we replace $\mathcal{K}_0\left(\frac{d^2}{d\theta^2} + \rho^2\right) \tilde{I} f$ by $\Delta \mathcal{K}_0 \tilde{I} f$ and we apply once again the formula (43) to the function $g = \mathcal{K}_0 \tilde{I} f$, the announced result follows immediately from (45). □

Now consider the application $\Psi_\rho : \mathcal{E}\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right) \to \mathcal{E}\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ defined for $f = f_e + f_o$ by

$$\Psi_\rho f = f_e + i\rho I f_e + f_o. \tag{46}$$

We clearly have $(\Psi_\rho f)_e = f_e$, $(\Psi_\rho f)_o = i\rho I f_e + f_o$ and (40) can then be written

$$Vf = \mathcal{K}_0((\Psi_\rho f)_e) + \frac{d}{d\theta}\mathcal{K}_0 \tilde{I}((\Psi_\rho f)_o) = \Phi(\Psi_\rho f) = \Phi \circ \Psi_\rho f, \tag{47}$$

where $\Phi : \mathcal{E}\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right) \to \mathcal{E}\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ is the operator defined for all $f = f_e + f_o \in \mathcal{E}\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ by

$$\Phi(f) = \mathcal{K}_0(f_e) + \frac{d}{d\theta}\mathcal{K}_0 \tilde{I}(f_o). \tag{48}$$

We precise the structure of V in the following lemma :

Lemma 6.3. *The operators* Ψ_ρ *and* Φ *defined by formulas* (46) *et* (48) *are automorphisms of* $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ *and their inverses are given by*
$\Psi_\rho^{-1}=\Psi_{-\rho}$ *and*

$$\Phi^{-1}(f)=\mathcal{K}_0^{-1}(f_e)+\tilde{I}^{-1}\mathcal{K}_0^{-1}I(f_o),\quad f=f_e+f_o\in\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right),\tag{49}$$

where $\tilde{I}^{-1}$ *is the inverse operator of* $\tilde{I}$ *given by*

$$\tilde{I}^{-1}(g)=I\circ\left(\frac{d^2}{d\theta^2}+\rho^2\right)(g).$$

Proof. For $f\in\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, we immediately see that $\Psi_{-\rho}(\Psi_\rho f)=f$, so Ψ_ρ is a bijection and $\Psi_\rho^{-1}=\Psi_{-\rho}$. The continuity of Ψ_ρ follows from the continuity on $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ of the applications $f\longmapsto f_e$, $f\longmapsto f_o$ and $f\longmapsto If_e$. The assertion concerning Φ follows from the fact that $f_e\longmapsto\mathcal{K}_0(f_e)$ is an automorphism of $\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $f_o\longmapsto\frac{d}{d\theta}\mathcal{K}_0\tilde{I}(f_o)$ is an automorphism of $\mathcal{E}_o\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, the formula giving Φ^{-1} is clear. □

We can now summarize the above discussion in the form of the following theorem :

Theorem 6.4. *The intertwining operator* V *is a topological automorphism of the space* $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ *and the inverse operator* V^{-1} *is given by*

$$V^{-1}f=\mathcal{K}_0^{-1}f_e-i\rho I\mathcal{K}_0^{-1}f_e+\tilde{I}^{-1}\mathcal{K}_0^{-1}If_o,\quad f\in\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right).\tag{50}$$

Proof. By Lemma 6.3 and formula (47), V is an automorphism. The form of $V^{-1}=\Psi^{-1}\circ\Phi^{-1}$ follows also from Lemma 6.3 and (46). □

As a consequence of the isomorphism Theorem 6.4, we can now prove the intertwining property announced in (5) and in the beginning of this section.

Theorem 6.5. *The operator* V *is the unique continuous operator on* $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ *satisfying*

$$\forall f\in\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right),\quad TVf=V\frac{d}{d\theta}f\quad\text{and}\quad Vf(0)=f(0).\tag{51}$$

Proof. Let us denote by $\mathcal{P}$ the linear subspace of $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ generated by the functions $e_\lambda : \theta \longmapsto e^{\lambda\theta}$ when $\lambda \in \mathbb{C}$. By (36) and the linearity of V and T, for every $p \in \mathcal{P}$, we have

$$TVp = V\frac{d}{d\theta}p. \tag{52}$$

But $\mathcal{P}$ is dense[2] in $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. So if $(p_n)_n$ is a sequence in $\mathcal{P}$ converging to $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, we have $V\frac{d}{d\theta}p_n \to V\frac{d}{d\theta}f$ and $Vp_n \to Vf$ $(n \to +\infty)$ in $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ as V is continuous. By Lemma 6.6 below, we deduce that $TVp_n \to TVf$ uniformly on compact sets. From (52) with $p = p_n$ and passing to the limit as $n \to +\infty$, we then obtain (51). Now if W is another continuous operator on $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ satisfying (51), for all $\lambda \in \mathbb{C}$, by unicity in the eigenfunction equation (3), we must have $W(e_\lambda) = G_{-i\lambda} = V(e_\lambda)$, so $W(p) = V(p)$ for all $p \in \mathcal{P}$ and $V = W$ as $\mathcal{P}$ is dense in $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. The only thing that remains to be proved is the following lemma :

Lemma 6.6. *If $(f_n)_n$ is a sequence of functions of $C^1\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $f \in C^1\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ are such that $f_n \to f$ and $f_n' \to f'$ uniformly on compact sets, then $Tf_n \to Tf$ uniformly on compact sets.*

Proof. Fix $A \in \left]0,\frac{\pi}{2}\right[$ and for g a continuous function on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, write $||g||_A = \sup_{|\theta|\le A} |g(\theta)|$. If $f \in C^1\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ by a Taylor-Lagrange expansion of order 1 at point 0, we clearly see

$$\sup_{|\theta|\le A}\left|\frac{f(\theta)-f(-\theta)}{\theta}\right| \le 2||f'||_A. \tag{53}$$

If we write

$$Tf(\theta) = f'(\theta) + 2i\left(k\frac{\theta}{1-e^{-2i\theta}} + k'\frac{\theta}{1+e^{-2i\theta}}\right)\frac{f(\theta)-f(-\theta)}{\theta} - i(k+k')f(\theta)$$

and let

$$\sup_{|\theta|\le A}\left|2i\left(k\frac{\theta}{1-e^{-2i\theta}} + k'\frac{\theta}{1+e^{-2i\theta}}\right)\right| = M < +\infty,$$

[2]This known result is an easy consequence of the Hahn-Banach theorem using the fact that the dual $\mathcal{E}'\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ of $\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ is the space of the distributions on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ with compact support.

we immediately see that

$$||Tf_n - Tf||_A \leq ||f_n' - f'||_A + 2M||f_n' - f'||_A + (k+k')||f_n - f||_A$$

and the result follows clearly. □

□

7 The dual of the Jacobi-Cherednik intertwining operator

We denote by $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ (resp. $\mathcal{D}_e(]-\frac{\pi}{2},\frac{\pi}{2}[)$ and $\mathcal{D}_o\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$) the space of complex valued $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ with compact support (resp. the subspaces $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right) \cap \mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right) \cap \mathcal{E}_o\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$).

Definition 7.1. *For all continuous function* $g : \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\to \mathbb{C}$ *with compact support we define the dual* ${}^tV$ *of the operator* V *by*

$$\forall \phi \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[, \quad {}^tVg(\phi) := \int_{|\theta|>|\phi|} K(\theta,\phi)g(\theta)A(|\theta|)\,d\theta. \tag{54}$$

Proposition 7.2. *The operator* ${}^tV$ *is the unique operator satisfying* $\forall f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, $\forall g \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf(\theta)g(\theta)A(|\theta|)\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi)\,{}^tVg(\phi)\,d\phi. \tag{55}$$

Proof. The relation (55) follows immediately from Fubini's theorem by interchanging integrations in the right hand side of (55) and the unicity of the function ${}^tVg$ satisfying (55) for all $f \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, is clear. □

Theorem 7.3. *For all* $g \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, *we have*

$${}^tVg = {}^t\mathcal{K}_0(g_e - i\rho J(g_o)) + \frac{d}{d\theta}\,{}^t\mathcal{K}_0(J(g_o)), \tag{56}$$

where ${}^t\mathcal{K}_0$ *is the dual of the operator* $\mathcal{K}_0$ *considered in* (37) *and* J *is the primitive operator which vanishes at* $-\frac{\pi}{2}$ *i.e.* $Jg_o(\theta) := \int_{-\frac{\pi}{2}}^{\theta} g_o(t)dt$.

Proof. i) Suppose $g = g_e \in \mathcal{D}_e\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$. For $f = f_e \in \mathcal{E}_e\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$, we have

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf_e(\theta)g_e(\theta)A(|\theta|)\,d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{K}_0(f_e)(\theta)g_e(\theta)A(|\theta|)\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \mathcal{K}_0(f_e)(\theta)g_e(\theta)A(|\theta|)\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} f_e(\phi)\ {}^t\mathcal{K}_0 g_e(\phi)\,d\phi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_e(\phi)\,{}^t\mathcal{K}_0 g_e(\phi)\,d\phi.\end{aligned}$$

If $f \in \mathcal{E}_o\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$, we have $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf(\theta)g(\theta)A(|\theta|)d\theta = 0$ as the function under the integral sign is odd.
ii) Suppose $g = g_o \in \mathcal{D}_o\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$. For $f = f_e \in \mathcal{E}_e\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$, by (43), we have

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf_e(\theta)g_o(\theta)A(|\theta|)\,d\theta &= i\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\theta}\mathcal{K}_0\tilde{I}(If_e)(\theta)g_o(\theta)A(|\theta|)\,d\theta \\ &= i\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sgn}(\theta)\left(\int_0^{|\theta|} \Delta\mathcal{K}_0\tilde{I}(If_e)(t)A(t)\,dt\right)g_o(\theta)\,d\theta \\ &= 2i\rho\int_0^{\frac{\pi}{2}}\left(\int_0^{\theta} \Delta\mathcal{K}_0\tilde{I}(If_e)(t)A(t)dt\right)g_o(\theta)\,d\theta.\end{aligned}$$

By integrating by parts and using the fact that $\Delta\mathcal{K}_0 = \mathcal{K}_0\left(\frac{d^2}{dt^2} + \rho^2\right)$ (see (39)) and $\left(\frac{d^2}{dt^2} + \rho^2\right)\tilde{I}(If_e) = f_e$, we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf_e(\theta)g_o(\theta)A(|\theta|)\,d\theta &= -2i\rho\int_0^{\frac{\pi}{2}} \mathcal{K}_0 f_e(t)Jg_o(t)A(t)\,dt \\ &= -2i\rho\int_0^{\frac{\pi}{2}} f_e(\phi)\ {}^t\mathcal{K}_0(Jg_o)(\phi)\,d\phi \\ &= -i\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_e(\phi)\,{}^t\mathcal{K}_0(Jg_o)(\phi)\,d\phi.\end{aligned}$$

For $f=f_o\in\mathcal{E}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ by (43), we have

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf_o(\theta)g_o(\theta)A(|\theta|)\,d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\theta}\mathcal{K}_0\tilde{I}(f_o)(\theta)g_o(\theta)A(|\theta|)\,d\theta\\ &= 2\int_0^{\frac{\pi}{2}} \frac{d}{d\theta}\mathcal{K}_0\tilde{I}(f_o)(\theta)g_o(\theta)A(|\theta|)\,d\theta\\ &= 2\int_0^{\frac{\pi}{2}}\left(\int_0^{\theta}\Delta\mathcal{K}_0\tilde{I}(f_o)(t)A(t)\,dt\right)g_o(\theta)\,d\theta.\end{aligned}$$

By integrating by parts and using the fact that $\Delta\mathcal{K}_0=\mathcal{K}_0\left(\frac{d^2}{dt^2}+\rho^2\right)$ (see (39)) and $\left(\frac{d^2}{dt^2}+\rho^2\right)\tilde{I}f_o=f_o'$, we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf_o(\theta)g_o(\theta)A(|\theta|)\,d\theta &= -2\int_0^{\frac{\pi}{2}}\mathcal{K}_0(f_o')(t)Jg_o(t)A(t)\,dt\\ &= -2\int_0^{\frac{\pi}{2}} f_o'(\phi)\ {}^t\mathcal{K}_0(Jg_o)(\phi)\,d\phi\\ &= 2\int_0^{\frac{\pi}{2}} f_o(\phi)\frac{d}{d\theta}({}^t\mathcal{K}_0Jg_o)(\phi)\,d\phi\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_o(\phi)\frac{d}{d\theta}({}^t\mathcal{K}_0Jg_o)(\phi)\,d\phi.\end{aligned}$$

iii) Finally for $f=f_e+f_o\in\mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $g=g_e+g_o\in\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, using i) and ii) and the fact that

$$Vf(\theta)g(\theta)=Vf_e(\theta)g_e(\theta)+Vf_o(\theta)g_e(\theta)+Vf_e(\theta)g_o(\theta)+Vf_o(\theta)g_o(\theta),$$

we get $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf(\theta)g(\theta)A(|\theta|)\,d\theta$

$$\begin{aligned}&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi)\left({}^t\mathcal{K}_0g_e(\phi)-i\rho{}^t\mathcal{K}_0Jg_o(\phi)+\frac{d}{d\theta}({}^t\mathcal{K}_0Jg_o)(\phi)\right)\\ &=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi){}^tVg(\phi)\,d\phi.\end{aligned}$$

This relation being true for all $f\in\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, the result of the theorem follows. □

Remark 7.4. *By [12], we know that the operator* ${}^t\mathcal{K}_0$ *is an automorphism of the space* $\mathcal{D}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. *As a consequence of the preceding theorem we deduce that* ${}^tV$ *is a linear and continuous operator of* $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ *into itself. We will see in the next result that it is indeed an automorphism of* $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$.

Theorem 7.5. *The operator* ${}^tV$ *is an automorphism of* $\mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. *The inverse operator is given by*

$$\forall g \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right), \quad {}^tV^{-1}g = {}^t\mathcal{K}_0^{-1}(g_e - i\rho J g_o) + \frac{d}{d\theta}\,{}^t\mathcal{K}_0^{-1}(Jg_o). \tag{57}$$

Proof. The result follows from the easily verified relation

$$\forall f \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right), \quad {}^tV^{-1}\circ {}^tVf = {}^tV\circ {}^tV^{-1}f \tag{58}$$

and from the fact that ${}^t\mathcal{K}_0$ is an automorphism of $\mathcal{D}_e\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. □

Theorem 7.6. *For all* $g \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, *we have*

$${}^tV(T+2i\rho S)g = \frac{d}{d\theta}\,{}^tVg, \tag{59}$$

where S *is the operator defined by* $Sg(\theta) := g(-\theta)$.

For the proof we need the following lemma :

Lemma 7.7. *For all* $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ *and* $g \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, *we have*

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Tf(\theta)g(\theta)A(|\theta|)d\theta &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)Tg(\theta)A(|\theta|)d\theta \\ &\quad - 2i\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)g(-\theta)A(|\theta|)d\theta. \end{aligned} \tag{60}$$

Proof. It follows from (1) that we have

$$Tf(\theta) = \frac{d}{d\theta}f(\theta) + \frac{1}{2}\mathrm{sgn}(\theta)\frac{A'(|\theta|)}{A(|\theta|)}\left(f(\theta)-f(-\theta)\right) - i\rho f(-\theta), \quad \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[. \tag{61}$$

Then we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Tf(\theta)g(\theta)A(|\theta|)\,d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f'(\theta)g(\theta)A(|\theta|)\,d\theta \\ &+ \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sgn}(\theta)A'(|\theta|)(f(\theta)-f(-\theta))g(\theta)\,d\theta \\ &- i\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(-\theta)g(\theta)A(|\theta|)\,d\theta. \qquad (62)\end{aligned}$$

Integrating by parts on $\left]-\frac{\pi}{2},0\right[$ and $\left]0,\frac{\pi}{2}\right[$ the first integral in the right hand side above, we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f'(\theta)g(\theta)A(|\theta|)\,d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\left(\frac{dg}{d\theta}(\theta)+\operatorname{sgn}(\theta)\frac{A'(|\theta|)}{A(|\theta|)}g(\theta)\right)A(|\theta|)\,d\theta. \qquad (63)$$

By changing variable θ into $-\theta$ in the integrals $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sgn}(\theta)A'(|\theta|)(f(-\theta))g(\theta)\,d\theta$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(-\theta)g(\theta)A(|\theta|)\,d\theta$ the result of the lemma follows easily from (62). □

Proof of the Theorem 7.6. Let $f \in \mathcal{E}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ and $g \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$. Repeated integration by parts, the definition of ${}^tV$, the intertwining relation and Lemma 7.7 give

$$\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi)\frac{d}{d\theta}\,{}^tVg(\phi)\,d\phi = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\theta}f(\phi)^tVg(\phi)\,d\phi \\ &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V\frac{d}{d\theta}f(\theta)g(\theta)A(|\theta|)\,d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} TVf(\theta)g(\theta)A(|\theta|)\,d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf(\theta)Tg(\theta)A(|\theta|)\,d\theta + 2i\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Vf(\theta)g(-\theta)A(|\theta|)\,d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi)^tV(Tg(\phi)+2i\rho g(-\phi))\,d\phi.\end{aligned}$$

This relation being true for all $f \in \mathcal{D}\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, the result of the theorem follows. □

8 Open problems

8.1 Problem 1

Find a positive constant C, which does not depend on θ and ϕ, such that

$$\left|K^{(k,k')}(\theta,\phi)\right| \leq C K_0^{(k-\frac{1}{2},k'-\frac{1}{2})}(\theta,\phi),$$

where $\theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}$ and $-|\theta| < \phi < |\theta|$.

8.2 Problem 2

Write a relationship between $K^{(k,k')}$ and $K_0^{(k-\frac{1}{2},k'-\frac{1}{2})}$, like (22) and (24), in the case $0 < k' < k$.

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