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Some growth properties of iterated entire functions on the basis of slowly changing functions

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Abstract

In the paper we prove some comparative growth properties of iterated entire functions using generalized L^*-order and generalized L^*-lower order.

Keywords: *Iterated entire function, Nevanlinna's characteristic function, maximum term, maximum modulus, growth, slowly changing function, generalised L^*-order (generalised L^*-lower order).*

2010 Mathematical Subject Classification: 30D30, 30D35.

1 Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers and f be entire defined in the finite complex plane $\mathbb{C}$. We use the standard notations and definitions in the theory of entire functions which are available in [14]. In the sequel the following notation is used :

$\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

The Nevanlinna's characteristic function $T(r, f)$, the maximum term $\mu(r, f)$ and the maximum modulus $M(r, f)$ of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are respectively defined as $T(r, f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta$, $\mu(r, f) = \max(|a_n| r^n)$ and $M(r, f) = \max\limits_{|z|=r} |f(z)|$.

According to Lahiri and Banerjee [6], if $f(z)$ and $g(z)$ be entire functions, then the iteration of f with respect to g is defined as follows:

$$
\begin{aligned}
f(z) &= f_1(z) \\
f(g(z)) &= f(g_1(z)) = f_2(z) \\
f(g(f(z))) &= f(g(f_1(z))) = f(g_2(z)) = f_3(z) \\
\\
\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots
\end{aligned}
$$

$$f(g(f.........(f(z) \text{ or } g(z))..........)) = f_n(z), \text{ according as } n \text{ is odd or even,}$$

and so

$$
\begin{aligned}
g(z) &= g_1(z) \\
g(f(z)) &= g(f_1(z)) = g_2(z) \\
g(f(g(z))) &= g(f(g_1(z))) = g(f_2(z)) = g_3(z) \\
\\
\ldots\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots
\end{aligned}
$$

$$g(f(g_{n-2}(z))) = g(f_{n-1}(z)) = g_n(z) \ .$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Lakshminarasimhan [5] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [7] worked on the entire functions of L-bounded index and of non uniform L-bounded index. In this paper we would like to investigate some growth properties of iterated entire functions on the basis of their Nevanlinna's characteristic functions, maximum terms and maximum moduli using generalised L^*-order and generalised L^*-lower order.

To start our paper we just recall the following definitions :

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Extending this notion, Sato [9] defined the generalised order and generalised lower order of an entire function as follows :

Definition 2. *[9]Let p be an integer ≥ 2. The generalised order $\rho_f^{[p]}$ and generalised lower order $\lambda_f^{[p]}$ of an entire function f are defined by*

$$\rho_f^{[p]} = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log r} \text{ and } \lambda_f^{[p]} = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log r}$$

respectively.

For $p = 2$, Definition 2 reduces to Definition 1.
If $\rho_f < \infty$ then f is of finite order. Also $\rho_f = 0$ means that f is of order zero. In this connection Datta and Biswas [2] gave the following definition :

Definition 3. *[2]Let f be an entire function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by*

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \text{ and } \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [10] defined it in the following way:

Definition 4. *[10]A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \text{ for } r \geq r(\varepsilon) \text{ and}$$

uniformly for $k (\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 .$$

Somasundaram and Thamizharasi [11] introduced the notions of L-order (L-lower order) for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order (L-lower order) for entire function are L^*-order (L^*-lower order). Their definitions are as follows:

Definition 5. *[11]The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

In the line of Sato [9] , Datta and Biswas [2] one can define the generalised L^*-order $\rho_f^{[p]L^*}$ and generalised L^*-lower order$\lambda_f^{[p]L^*}$ of an entire function f in the following manner :

Definition 6. *Let p be an integer ≥ 1. The generalised L^*-order $\rho_f^{[p]L^*}$ and generalised L^*-lower order $\lambda_f^{[p]L^*}$ of an entire function f are defined as*

$$\rho_f^{[p]L^*} = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[p]L^*} = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

respectively.

With the help of the inequality

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r} T(R,f) \quad \{cf.\ [3]\} \text{ for } 0 \leq r < R < \infty,$$

one can easily verify that

$$\rho_f^{[p]L^*} = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[p]L^*} = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log\left[re^{L(r)}\right]} .$$

Also for $0 \leq r < R$,

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r} \mu(R,f) \quad \{cf.\ [13]\},$$

it is easy to see that

$$\rho_f^{[p]L^*} = \limsup_{r\to\infty} \frac{\log^{[p]} \mu(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[p]L^*} = \liminf_{r\to\infty} \frac{\log^{[p]} \mu(r,f)}{\log\left[re^{L(r)}\right]} .$$

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[12] Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu(r, f\circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right) .$$

Lemma 2. *[1] If f and g are any two entire functions then for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right)\leq M(r,f\circ g)\leq M\left(M\left(r,g\right),f\right)\ .$$

Lemma 3. *[8] Let f and g be any two entire functions. Then we have*

$$T\left(r,f\circ g\right)\geq\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g\right)+O(1),f\right)\ .$$

Lemma 4. *Let f and g be any two entire functions such that $\rho_f^{[p]L^*}<\infty$ and $\rho_g^{[q]L^*}<\infty$ where p and q are any two positive integers. Then for any $\varepsilon>0$ and for all sufficiently large values of r,*

$$\begin{aligned}&\log^{\left[\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)(q-1)\right]}T\left(r,f_n\right)\\ \leq\ &\left(\rho_f^{[p]L^*}+\varepsilon\right)\left(\log M\left(r,g\right)+L\left(M\left(r,g\right)\right)\right)+O(1)\ \textit{when } n \textit{ is even}\end{aligned}$$

and

$$\begin{aligned}&\log^{\left[\frac{(n-1)}{2}\{(p-1)+(q-1)\}\right]}T\left(r,f_n\right)\\ \leq\ &\left(\rho_g^{[q]L^*}+\varepsilon\right)\left[\log M\left(r,f\right)+L\left(M\left(r,f\right)\right)\right]+O(1)\ \textit{when } n \textit{ is odd and } n\neq1\ .\end{aligned}$$

Proof. Let us consider n to be an even number.
Then in view of Lemma 2 and the inequality $T\left(r,f\right)\leq\log^{+}M\left(r,f\right)\leq\frac{R+r}{R-r}T\left(R,f\right)$ for $0\leq r<R<\infty$, we get for all sufficiently large values of r that

$$\begin{aligned}T\left(r,f_n\right)&\leq\log M\left(r,f_n\right)\\ \textit{i.e.,}\ T\left(r,f_n\right)&\leq\log M\left(M\left(r,g_{n-1}\right),f\right)\\ \textit{i.e.,}\ \log^{[p-1]}T\left(r,f_n\right)&\leq\log^{[p]}M\left(M\left(r,g_{n-1}\right),f\right)\\ \textit{i.e.,}\ \log^{[p-1]}T\left(r,f_n\right)&\leq\left(\rho_f^{[p]L^*}+\varepsilon\right)\log\left[M\left(r,g_{n-1}\right)e^{L(M(r,g_{n-1}))}\right]\\ \textit{i.e.,}\ \log^{[p-1]}T\left(r,f_n\right)&\leq\left(\rho_f^{[p]L^*}+\varepsilon\right)\left[\log M\left(r,g_{n-1}\right)+L\left(M\left(r,g_{n-1}\right)\right)\right]\\ \textit{i.e.,}\ \log^{[p-1]}T\left(r,f_n\right)&\leq\left(\rho_f^{[p]L^*}+\varepsilon\right)\left[\log M\left(M\left(r,f_{n-2}\right),g\right)+L\left(M\left(r,g_{n-1}\right)\right)\right]\end{aligned}$$

$$\begin{aligned}&\textit{i.e.,}\ \log^{[p-1]}T\left(r,f_n\right)\\ \leq\ &\log M\left(M\left(r,f_{n-2}\right),g\right)\left[\left(\rho_f^{[p]L^*}+\varepsilon\right)+\frac{\left(\rho_f^{[p]L^*}+\varepsilon\right)L\left(M\left(r,g_{n-1}\right)\right)}{\log M\left(M\left(r,f_{n-2}\right),g\right)}\right]\end{aligned}$$

$$i.e., \ \log^{[(p-1)+(q-1)]} T(r, f_n) \leq \log^{[q]} M(M(r, f_{n-2}), g) + O(1)$$

$$\begin{aligned} & i.e., \ \log^{[(p-1)+(q-1)]} T(r, f_n) \\ \leq \ & \left(\rho_g^{[q]L^*} + \varepsilon\right) [\log M(M(r, g_{n-3}), f) + L(M(r, f_{n-2}))] + O(1) \end{aligned}$$

....

....

Therefore,

$$\begin{aligned} & \log^{\left[\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)(q-1)\right]} T(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right) [\log M(r, g) + L(M(r, g))] + O(1) \text{ when } n \text{ is even.} \end{aligned}$$

Similarly,

$$\begin{aligned} & \log^{\left[\frac{(n-1)}{2}\{(p-1)+(q-1)\}\right]} T(r, f_n) \\ \leq \ & \left(\rho_g^{[q]L^*} + \varepsilon\right) [\log M(r, f) + L(M(r, f))] + O(1) \text{ when } n \text{ is odd and } n \neq 1 \ . \end{aligned}$$

This proves the lemma. □

Lemma 5. *Let f and g be any two entire functions such that $\rho_f^{[p]L^*} < \infty$ and $\rho_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon > 0$ and for all sufficiently large values of r,*

$$\begin{aligned} & \log^{\left[p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}\right]} M(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right) (\log M(r, g) + L(M(r, g))) + O(1) \ \textit{when } n \textit{ is even} \end{aligned}$$

and

$$\begin{aligned} & \log^{\left[p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)\right]} M(r, f_n) \\ \leq \ & \left(\rho_g^{[q]L^*} + \varepsilon\right) [\log M(r, f) + L(M(r, f))] + O(1) \ \textit{when } n \textit{ is odd and } n \neq 1 \ . \end{aligned}$$

We omit the proof of the lemma because it can be carried out in the line of Lemma 4 and with the help of Lemma 2.

Similarly the following lemma can be carried out in the line of Lemma 5 and with the help of the inequality $\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f)$ for $0 \leq r < R < \infty$.

Lemma 6. *Let f and g be any two entire functions such that $\rho_f^{[p]L^*} < \infty$ and $\rho_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon > 0$ and for all sufficiently large values of r,*

$$\begin{aligned} & \log^{\left[p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}\right]} \mu(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right) (\log \mu(\beta r, g) + L(\mu(r, g))) + O(1) \ \textit{when } n \textit{ is even} \end{aligned}$$

and

$$\begin{aligned}&\log^{\left[p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)\right]}\mu\left(r,f_n\right)\\ \leq\ &\left(\rho_g^{[q]L^*}+\varepsilon\right)\left[\log\mu\left(\beta r,f\right)+L\left(\mu\left(r,f\right)\right)\right]+O(1)\ \text{when } n \text{ is odd and } n\neq 1\end{aligned}$$

where $\beta > 1$.

We omit the proof of the lemma.

Lemma 7. *Let* f *and* g *be any two entire functions such that* $0<\lambda_f^{[p]L^*}<\infty$ *and* $0<\lambda_g^{[q]L^*}<\infty$ *where* p *and* q *are any two positive integers. Then for any* $\varepsilon\left(0<\varepsilon<\min\left\{\lambda_f^{[p]L^*},\lambda_g^{[q]L^*}\right\}\right)$ *and for all sufficiently large values of* r,

$$\begin{aligned}&\log^{\left[\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)(q-1)\right]}T\left(r,f_n\right)\\ \geq\ &\left(\lambda_f^{[p]L^*}-\varepsilon\right)\left(\log M\left(\frac{r}{4^{n-1}},g\right)+L\left(M\left(\frac{r}{4^{n-1}},g\right)\right)\right)+O(1)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{when } n \text{ is even}\end{aligned}$$

and

$$\begin{aligned}&\log^{\left[\frac{(n-1)}{2}\{(p-1)+(q-1)\}\right]}T\left(r,f_n\right)\\ \geq\ &\left(\lambda_g^{[q]L^*}-\varepsilon\right)\left(\log M\left(\frac{r}{4^{n-1}},f\right)+L\left(M\left(\frac{r}{4^{n-1}},f\right)\right)\right)+O(1)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{when } n \text{ is odd and } n\neq 1\ .\end{aligned}$$

Proof. We choose ε in such a way that $\varepsilon\left(0<\varepsilon<\min\left\{\lambda_f^{[p]L^*},\lambda_g^{[q]L^*}\right\}\right)$.
Also let us consider n is an even number .
Now in view of Lemma 3 and the inequality $T\left(r,f\right)\leq\log^{+}M\left(r,f\right)\leq\frac{R+r}{R+r}$ $T\left(R,f\right)$ for $0\leq r<R<\infty$ $\{cf.\ [3]\}$, we get for all sufficiently large values of r that

$$\begin{aligned}T\left(r,f_n\right)&=T\left(r,f\circ g_{n-1}\right)\\ \text{i.e., } T\left(r,f_n\right)&\geq\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right)\\ \text{i.e., } \log^{[p-1]}T\left(r,f_n\right)&\geq\log^{[p]}M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right)+O(1)\end{aligned}$$

$$\begin{aligned}\text{i.e., } \log^{[p-1]}T\left(r,f_n\right)\ \geq\ &\left(\lambda_f^{[p]L^*}-\varepsilon\right)\log\left[\left(\frac{1}{9}M\left(\frac{r}{4},g_{n-1}\right)\right)e^{L\left(\frac{1}{9}M\left(\frac{r}{4},g_{n-1}\right)\right)}\right]\\ &+O(1)\end{aligned}$$

$$i.e.,\ \log^{[p-1]} T(r,f_n) \geq \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4}, g_{n-1}\right) + L\left(\frac{1}{9}M\left(\frac{r}{4}, g_{n-1}\right)\right)\right] + O(1)$$

$$i.e.,\ \log^{[p-1]} T(r,f_n) \geq \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left[T\left(\frac{r}{4}, g_{n-1}\right) + L\left(\frac{1}{9}M\left(\frac{r}{4}, g_{n-1}\right)\right)\right] + O(1)$$

$$i.e.,\ \log^{[p-1]} T(r,f_n) \geq \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left[\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + L\left(\frac{1}{9}M\left(\frac{r}{4}, g_{n-1}\right)\right)\right] + O(1)$$

$$i.e.,\ \log^{[p-1]} T(r,f_n) \geq \log M\left(\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right), g\right) \cdot \left[\frac{1}{3}\left(\lambda_f^{[p]L^*} - \varepsilon\right) + \frac{\left(\lambda_f^{[p]L^*} - \varepsilon\right) L\left(\frac{1}{9}M\left(\frac{r}{4}, g_{n-1}\right)\right) + O(1)}{\log M\left(\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right), g\right)}\right]$$

$$i.e.,\ \log^{[(p-1)+(q-1)]} T(r,f_n) \geq \log^{[q]} M\left(\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right), g\right) + O(1)$$

$$i.e.,\ \log^{[(p-1)+(q-1)]} T(r,f_n) \geq \left(\lambda_g^{[q]L^*} - \varepsilon\right) \cdot \log\left\{\left(\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right)\right) e^{L\left(\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right)\right)}\right\} + O(1)$$

$$i.e.,\ \log^{[(p-1)+(q-1)]} T(r,f_n) \geq \left(\lambda_g^{[q]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4^2}, f_{n-2}\right) + L\left(M\left(\frac{r}{4^2}, f_{n-2}\right)\right)\right]$$

....

....

Therefore

$$\log^{\left[\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)(q-1)\right]} T(r,f_n) \geq \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4^{n-1}}, g\right) + L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right)\right] \text{ when } n \text{ is even .}$$

Similarly,

$$\log^{\left[\frac{(n-1)}{2}\{(p-1)+(q-1)\}\right]} T(r,f_n) \geq \left(\lambda_g^{[q]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4^{n-1}}, f\right) + L\left(M\left(\frac{r}{4^{n-1}}, f\right)\right)\right] \text{ when } n \text{ is odd and } n \neq 1 \text{ .}$$

Thus the lemma follows. $\square$

Lemma 8. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon \left(0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{[q]L^*}\right\}\right)$ and for all sufficiently large values of r,*

$$\begin{aligned} & \log^{\left[p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}\right]} M(r, f_n) \\ \geq & \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left(\log M\left(\frac{r}{4^{n-1}}, g\right) + L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right)\right) + O(1) \end{aligned}$$

when n is even

and

$$\begin{aligned} & \log^{\left[p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)\right]} M(r, f_n) \\ \geq & \left(\lambda_g^{[q]L^*} - \varepsilon\right)\left(\log M\left(\frac{r}{4^{n-1}}, f\right) + L\left(M\left(\frac{r}{4^{n-1}}, f\right)\right)\right) + O(1) \end{aligned}$$

when n is odd and $n \neq 1$.

We omit the proof of the lemma because it can be carried out in the line of Lemma 7 and with the help of Lemma 3.

Similarly, the following lemma can be carried out in the line of Lemma 8 and in view of Lemma 1.

Lemma 9. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon \left(0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{[q]L^*}\right\}\right)$ and for all sufficiently large values of r,*

$$\begin{aligned} & \log^{\left[p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}\right]} \mu(r, f_n) \\ \geq & \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left(\log \mu\left(\frac{r}{2^{n-1}}, g\right) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)\right) + O(1) \end{aligned}$$

when n is even

and

$$\begin{aligned} & \log^{\left[p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)\right]} \mu(r, f_n) \\ \geq & \left(\lambda_g^{[q]L^*} - \varepsilon\right)\left(\log \mu\left(\frac{r}{2^{n-1}}, f\right) + L\left(\mu\left(\frac{r}{2^{n-1}}, f\right)\right)\right) + O(1) \end{aligned}$$

when n is odd and $n \neq 1$.

The proof is omitted.

3 Theorems

In this section we present the main results of the paper.

Theorem 10. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ where $p \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then for every constant A and real number x,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{(np-2)}{2}\right]} T(r, f_n)}{\left\{\log^{[p-1]} T(r^A, f)\right\}^{1+x}} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{(np-2)}{2}\right]} T(r, f_n)}{\{\log T(r^A, g)\}^{1+x}} = \infty$$

where n is any even number.

Proof. If x is such that $1 + x \leq 0$, then the theorem is obvious. So we suppose that $1 + x > 0$.
Now in view of Lemma 7, we get for all sufficiently large values of r that

$$\log^{\left[\frac{(np-2)}{2}\right]} T(r, f_n) \geq O(1) + \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4^{n-1}}, g\right) + L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right)\right]$$

$$i.e.,\ \log^{\left[\frac{(np-2)}{2}\right]} T(r, f_n) \geq O(1) + \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left\{\left(\frac{r}{4^{n-1}}\right)e^{L\left(\frac{r}{4^{n-1}}\right)}\right\}^{\lambda_g^{L^*} - \varepsilon} + \left(\lambda_f^{[p]L^*} - \varepsilon\right)L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right) \quad (1)$$

where we choose $0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{L^*}\right\}$.
Also for all sufficiently large values of r we obtain that

$$\log^{[p-1]} T\left(r^A, f\right) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}$$

$$i.e.,\ \log^{[p-1]} T\left(r^A, f\right) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}$$

$$i.e.,\ \left\{\log^{[p-1]} T\left(r^A, f\right)\right\}^{1+x} \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)^{1+x}\left(A\log r + L\left(r^A\right)\right)^{1+x}. \quad (2)$$

Similarly, we get for all sufficiently large values of r, that

$$\log T\left(r^A, g\right) \leq \left(\rho_g^{L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}$$

$$i.e.,\ \log T\left(r^A, g\right) \leq \left(\rho_g^{L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}$$

$$i.e.,\ \left\{\log T\left(r^A, g\right)\right\}^{1+x} \leq \left(\rho_g^{L^*} + \varepsilon\right)^{1+x}\left(A\log r + L\left(r^A\right)\right)^{1+x}. \quad (3)$$

Now combining (1) and (2), it follows for all sufficiently large values of r that

$$\frac{\log^{\left[\frac{(np-2)}{2}\right]} T(r, f_n)}{\left\{\log^{[p-1]} T(r^A, f)\right\}^{1+x}}$$

$$\geq \frac{O(1) + \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left\{\left(\frac{r}{4^{n-1}}\right) e^{L\left(\frac{r}{4^{n-1}}\right)}\right\}^{\lambda_g^{L^*} - \varepsilon} + \left(\lambda_f^{[p]L^*} - \varepsilon\right) L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right)}{\left(\rho_f^{[p]L^*} + \varepsilon\right)^{1+x} \left(A \log r + L\left(r^A\right)\right)^{1+x}}. \tag{4}$$

Thus the first part of the theorem follows from (4).
Again from (1) and (3) we get for all sufficiently large values of r that

$$\frac{\log^{\left[\frac{(np-2)}{2}\right]} T(r, f_n)}{\left\{\log T(r^A, g)\right\}^{1+x}}$$

$$\geq \frac{O(1) + \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left\{\left(\frac{r}{4^{n-1}}\right) e^{L\left(\frac{r}{4^{n-1}}\right)}\right\}^{\lambda_g^{L^*} - \varepsilon} + \left(\lambda_f^{[p]L^*} - \varepsilon\right) L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right)}{\left(\rho_g^{L^*} + \varepsilon\right)^{1+x} \left(A \log r + L\left(r^A\right)\right)^{1+x}}$$

$$i.e., \lim_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)\right]} T(r, f_n)}{\left\{\log T(r^A, g)\right\}^{1+x}} = \infty .$$

Thus the second part of the theorem is established. □

In the line of Theorem 10, we may state the following theorem without its proof :

Theorem 11. *Let f and g be any two entire functions such that $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ where $p \geq 1$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$. Then for every constant A and real number x,*

$$(i) \lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}\right]} T(r, f_n)}{\left\{\log T(r^A, f)\right\}^{1+x}} = \infty$$

and

$$(ii) \lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}\right]} T(r, f_n)}{\left\{\log^{[q-1]} T(r^A, g)\right\}^{1+x}} = \infty$$

when n is odd and $n \neq 1$.

The proof is omitted.

The following two theorems can be carried out in the line of Theorem 10 and Theorem 11 and with the help of Lemma 5 .

Theorem 12. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ where $p \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then for every constant A and real number x,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} M(r, f_n)}{\left\{\log^{[p]} M(r^A, f)\right\}^{1+x}} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} M(r, f_n)}{\left\{\log^{[2]} M(r^A, g)\right\}^{1+x}} = \infty$$

where n is any even number.

Theorem 13. *Let f and g be any two entire functions such that $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ where $p \geq 1$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$. Then for every constant A and real number x,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{nq-q+2}{2}\right]} M(r, f_n)}{\left\{\log^{[2]} M(r^A, f)\right\}^{1+x}} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{nq-q+2}{2}\right]} M(r, f_n)}{\left\{\log^{[q]} M(r^A, g)\right\}^{1+x}} = \infty$$

when n is odd and $n \neq 1$.

Replacing maximum modulus by maximum term in Theorem 12 and Theorem 13, we respectively get Theorem 14 and Theorem 15.

Theorem 14. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ where $p \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then for every constant A and real number x,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} \mu(r, f_n)}{\left\{\log^{[p]} \mu(r^A, f)\right\}^{1+x}} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} \mu(r, f_n)}{\left\{\log^{[2]} \mu(r^A, g)\right\}^{1+x}} = \infty,$$

where n is any even number.

Theorem 15. *Let f and g be any two entire functions such that $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ where $p \geq 1$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$. Then for every constant A and real number x,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{nq-q+2}{2}\right]} \mu\left(r, f_n\right)}{\left\{\log^{[2]} \mu\left(r^A, f\right)\right\}^{1+x}} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{nq-q+2}{2}\right]} \mu\left(r, f_n\right)}{\left\{\log^{[q]} \mu\left(r^A, g\right)\right\}^{1+x}} = \infty$$

when n is odd and $n \neq 1$.

The proofs of the above two theorems are omitted as those can be carried out with the help of Lemma 9 and in the line of Theorem 12 and Theorem 13 .

Theorem 16. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where $p \geq 1$. Then for any two positive integers α and β and for any even n,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^\alpha\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^\beta\right), f\right) + L\left(\exp\left(\exp\left(r^\alpha\right)\right)\right)} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^\alpha\right)\right), f_n\right)}{\log T\left(\exp\left(r^\beta\right), g\right) + L\left(\exp\left(\exp\left(r^\alpha\right)\right)\right)} = \infty$$

where $K\left(r, \alpha; L\right) = \begin{cases} 0 \ \textit{if}\ r^\beta = o\left\{L\left(\exp\left(\exp\left(r^\alpha\right)\right)\right)\right\} \ \textit{as}\ r \to \infty \\ L\left(\exp\left(\exp\left(r^\alpha\right)\right)\right) \ \textit{otherwise} . \end{cases}$

Proof. Taking $x = 0$ and $A = 1$ in the first part of Theorem 10, we obtain for $K > 1$ and for all sufficiently large values of r that

$$\begin{aligned} \log^{\left[\frac{(np-2)}{2}\right]} T\left(r, f_n\right) &> K \log^{[p-1]} T\left(r, f\right) \\ i.e.,\ \log^{\left[\frac{(np-4)}{2}\right]} T\left(r, f_n\right) &> \left\{\log^{[p-2]} T\left(r, f\right)\right\}^K \\ i.e.,\ \log^{\left[\frac{(np-4)}{2}\right]} T\left(r, f_n\right) &> \log^{[p-2]} T\left(r, f\right)\ . \end{aligned} \tag{5}$$

Therefore from (5) we get for all sufficiently large values of r that

$$\log^{\left[\frac{(np-2)}{2}\right]} T\left(\exp\left(\exp\left(r^\alpha\right)\right), f_n\right) > \log^{[p-1]} T\left(\exp\left(\exp\left(r^\alpha\right)\right), f\right)$$

$$\begin{aligned} & i.e., \;\; \log^{\left[\frac{(np-2)}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) \\ > & \;\; \left(\lambda_f^{[p]L^*} - \varepsilon\right). \log\left\{\exp\left(\exp\left(r^{\alpha}\right)\right). \exp L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \end{aligned}$$

$$\begin{aligned} & i.e., \;\; \log^{\left[\frac{(np-2)}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) \\ > & \;\; \left(\lambda_f^{[p]L^*} - \varepsilon\right). \left\{\left(\exp\left(r^{\alpha}\right)\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \end{aligned}$$

$$\begin{aligned} & i.e., \;\; \log^{\left[\frac{(np-2)}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) \\ > & \;\; \left(\lambda_f^{[p]L^*} - \varepsilon\right). \left\{\left(\exp\left(r^{\alpha}\right)\right)\left(1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\left(\exp\left(r^{\alpha}\right)\right)}\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) \;\; > \;\; & O(1) + \log\exp\left(r^{\alpha}\right) \\ & + \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\left(\exp\left(r^{\alpha}\right)\right)}\right\} \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) > O(1) + r^{\alpha} & \\ + \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\left(\exp\left(r^{\alpha}\right)\right)}\right\} & \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) > O(1) + r^{\alpha} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) & \\ - \log\left[\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}\right] & \\ + \log\left[1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\exp\left(\mu r^{\alpha}\right)}\right] & \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) > O(1) + r^{\alpha} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) & \\ + \log\left[\frac{1}{\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}}\right. & \\ \left. + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}.\exp\left(r^{\alpha}\right)}\right] & \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) > O(1) + r^{(\alpha-\beta)}.r^{\beta} & \\ + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \;\; . & \end{aligned} \tag{6}$$

Again we have for all sufficiently large values of r that

$$\begin{aligned}
&\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) \le \left(\rho_f^{[p]L^*} + \varepsilon\right) \log\left\{\exp\left(r^{\beta}\right) e^{L\left(\exp\left(r^{\beta}\right)\right)}\right\} \\
i.e.,\ &\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) \le \left(\rho_f^{[p]L^*} + \varepsilon\right) \left\{\log\exp\left(r^{\beta}\right) + L\left(\exp\left(r^{\beta}\right)\right)\right\} \\
i.e.,\ &\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) \le \left(\rho_f^{[p]L^*} + \varepsilon\right) \left\{r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right\}
\end{aligned}$$

$$i.e.,\ \frac{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) - \left(\rho_f^{[p]L^*} + \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right)}{\left(\rho_f^{[p]L^*} + \varepsilon\right)} \le r^{\beta}\ . \tag{7}$$

Now from (6) and (7), it follows for all sufficiently large values of r that

$$\begin{aligned}
&\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right) \\
&\ge O(1) + \left(\frac{r^{(\alpha-\beta)}}{\rho_f^{[p]L^*} + \varepsilon}\right) \left[\log^{[p-1]} \mu\left(\exp\left(r^{\beta}\right), f\right) - \left(\rho_f^{[p]L^*} + \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right)\right] \\
&\qquad + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)
\end{aligned} \tag{8}$$

$$\begin{aligned}
i.e.,\ \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)} &\ge \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) + O(1)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)} \\
&+ \frac{r^{(\alpha-\beta)}}{\rho_f^{[p]L^*} + \varepsilon} \left\{1 - \frac{\left(\rho_f^{[p]L^*} + \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)}\right\}\ .
\end{aligned} \tag{9}$$

Again from (8) we get for all sufficiently large values of r that

$$\begin{aligned}
&\frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \\
&\ge \frac{O(1) - r^{(\alpha-\beta)} L\left(\exp\left(r^{\beta}\right)\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \\
&+ \frac{\left(\frac{r^{(\alpha-\beta)}}{\rho_f^{[p]L^*} + \varepsilon}\right) \log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \\
&\qquad + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}
\end{aligned}$$

$$i.e., \quad \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \geq \frac{\frac{O(1) - r^{(\alpha-\beta)} L\left(\exp\left(r^{\beta}\right)\right)}{L(\exp(\exp(r^{\alpha})))}}{\frac{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)}{L(\exp(\exp(r^{\alpha})))} + 1}$$

$$+ \frac{\left(\frac{r^{(\alpha-\beta)}}{\rho_f^{[p]L^*} + \varepsilon}\right)}{1 + \frac{L(\exp(\exp(r^{\alpha})))}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)}} + \frac{1}{1 + \frac{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)}{L(\exp(\exp(r^{\alpha})))}} . \tag{10}$$

Case I. If $r^{\beta} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}$, then it follows from (9) that

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)} = \infty .$$

Case II. If $r^{\beta} \neq o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}$, then two sub cases may arise:
Sub case (a). If $L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) = o\left\{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)\right\}$, then we get from (10) that

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty .$$

Sub case (b). If $L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \sim \log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)$ then

$$\lim_{r\to\infty} \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right)} = 1$$

and we obtain from (10) that

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty .$$

Combining Case I and Case II, we obtain that

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p-1]} T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty ,$$

where $K(r, \alpha; L) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \text{ as } r \to \infty \\ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \text{ otherwise .} \end{cases}$

This proves the first part of the theorem.
Now, with the help of the second part of the Theorem 10 and in the line of the first part of Theorem 16, one may easily prove the seond part of theorem.16.

□

Theorem 17. *Let f and g be any two entire functions with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$ where $p \geq 1$. Then for any two positive integers α and β and for any odd n except 1,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log T\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} T\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[q-1]} T\left(\exp\left(r^{\beta}\right), g\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty$$

where $K(r,\alpha;L) = \begin{cases} 0 \ \textit{if}\ r^{\beta} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \ \textit{as}\ r\to\infty \\ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \ \textit{otherwise}\ . \end{cases}$

The proof of the theorem is omitted because it can be carried out with the help of Theorem 11 and in the line of Theorem 16.

Theorem 18. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where $p \geq 1$. Then for any two positive integers α and β and for any even n,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}+1\right]} M\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[p]} M\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}+1\right]} M\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[2]} M\left(\exp\left(r^{\beta}\right), g\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty$$

where $K(r,\alpha;L) = \begin{cases} 0 \ \textit{if}\ r^{\beta} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \ \textit{as}\ r\to\infty \\ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \ \textit{otherwise}\ . \end{cases}$

Theorem 19. *Let f and g be any two entire functions with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$ where $p \geq 1$. Then for any two positive integers α and β and for any odd n except 1,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{nq-q+4}{2}\right]} M\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[2]} M\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{\left[\frac{nq-q+4}{2}\right]} M\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f_n\right)}{\log^{[q]} M\left(\exp\left(r^{\beta}\right), g\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty$$

where $K(r,\alpha;L) = \begin{cases} 0 \ \textit{if}\ r^{\beta} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \ \textit{as}\ r\to\infty \\ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \ \textit{otherwise}\ . \end{cases}$

Remark 1. *In view of Theorem 14 and Theorem 15 , the results analogous to Theorem 18 and Theorem 19 can also be derived in terms of maximum terms of iterated entire functions.*

Theorem 20. *Let f and g be any two entire functions such that $0 < \rho_g^{L^*} < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ where $p \geq 1$. Then for any even number n and for any $\beta > 1$,*

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np-2}{2}\right]} T(r, f_n)}{\log^{[p-1]} T(r, f) \cdot K(r, g; L)} = 0 \ ,$$

where $K(r, g; L) = \begin{cases} 1 \text{ if } L\left(\exp\left(\frac{\beta+1}{\beta-1} T(\beta r, g)\right)\right) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \text{ as } r \to \infty \\ \qquad\qquad\qquad\qquad \text{and for some } \alpha < \lambda_f^{[p]L^*} \\ L\left(\exp\left(\frac{\beta+1}{\beta-1} T(\beta r, g)\right)\right) \text{ otherwise.} \end{cases}$

Proof. In view of Lemma 4 and taking $R = \beta r$ in the inequality $T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f)$ $\{cf.\ [3]\}$, we have for all sufficiently large values of r that

$$\begin{aligned} & i.e., \ \log^{\left[\frac{np-2}{2}\right]} T(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M(r, g))\right) + O(1) \end{aligned}$$

$$\begin{aligned} & i.e., \ \log^{\left[\frac{np-2}{2}\right]} T(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L\left(\exp\left(\frac{\beta+1}{\beta-1} T(\beta r, g)\right)\right)\right) \\ & + O(1) \end{aligned} \tag{11}$$

Also we obtain for all sufficiently large values of r that

$$\begin{aligned} \log^{[p-1]} T(r, f) &\geq \left(\lambda_f^{[p]L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ \log^{[p-1]} T(r, f) &\geq \left(\lambda_f^{[p]L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ \log^{[p-1]} T(r, f) &\geq \left[re^{L(r)}\right]^{\left(\lambda_f^{[p]L^*}-\varepsilon\right)} \ . \end{aligned} \tag{12}$$

Now from (11) and (12), we get for all sufficiently large values of r that

$$\begin{aligned} & \frac{\log^{\left[\frac{np-2}{2}\right]} T(r, f_n)}{\log^{[p-1]} T(r, f)} \\ \leq \ & \frac{\left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L\left(\exp\left(\frac{\beta+1}{\beta-1} T(\beta r, g)\right)\right)\right) + O(1)}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[p]L^*}-\varepsilon\right)}} \ . \end{aligned} \tag{13}$$

Since $\rho_g^{L^*} < \lambda_f^{[p]L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho_g^{L^*} + \varepsilon < \lambda_f^{[p]L^*} - \varepsilon\ . \tag{14}$$

Case I. Let $L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some $\alpha < \lambda_f^{[p]L^*}$.
As $\alpha < \lambda_f^{[p]L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\alpha < \lambda_f^{[p]L^*} - \varepsilon\ . \tag{15}$$

Since $L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r \to \infty$, we get on using (15) that

$$\frac{L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right)}{r^{\alpha}e^{\alpha L(r)}} \to 0 \text{ as } r \to \infty$$

$$i.e.,\ \frac{L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right)}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[p]L^*}-\varepsilon\right)}} \to 0 \text{ as } r \to \infty\ . \tag{16}$$

Now in view of (13), (14) and (16) we obtain that

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np-2}{2}\right]} T\left(r, f_n\right)}{\log^{[p-1]} T\left(r, f\right)} = 0\ . \tag{17}$$

Case II. If $L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right) \neq o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some $\alpha < \lambda_f^{[p]L^*}$ then from (13) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{\left[\frac{np-2}{2}\right]} T\left(r, f_n\right)}{\log^{[p-1]} T\left(r, f\right).L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right)} \leq \frac{\left(\rho_f^{[p]L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)}}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[p]L^*}-\varepsilon\right)} L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right)} + \frac{\left(\rho_f^{[p]L^*} + \varepsilon\right)}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[p]L^*}-\varepsilon\right)}}\ . \tag{18}$$

Now using (14) it follows from (18) that

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np-2}{2}\right]} T\left(r, f_n\right)}{\log^{[p-1]} T\left(r, f\right).L\left(\exp\left(\frac{\beta+1}{\beta-1}T\left(\beta r, g\right)\right)\right)} = 0\ . \tag{19}$$

Combining (17) and (19) we obtain that

$$\lim_{r\to\infty}\frac{\log^{\left[\frac{np-2}{2}\right]}T(r,f_n)}{\log^{[p-1]}T(r,f).L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,g)\right)\right)}=0\ ,$$

$$\text{where } K(r,g;L)=\begin{cases}1 \text{ if } L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,g)\right)\right)=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad\text{and for some } \alpha<\lambda_f^{[p]L^*}\\ L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,g)\right)\right) \text{ otherwise.}\end{cases}$$

Thus the theorem is established. □

The following theorem can be carried out in the line of Theorem 20 and therefore its proof is omitted :

Theorem 21. *Let f and g be any two entire functions with $0<\rho_g^{L^*}<\rho_f^{[p]L^*}<\infty$ where $p\geq 1$. Then for any even number n and for any $\beta>1$,*

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{np-2}{2}\right]}T(r,f_n)}{\log^{[p-1]}T(r,f)\cdot K(r,g;L)}=0\ ,$$

$$\textit{where } K(r,g;L)=\begin{cases}1 \textit{ if } L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,g)\right)\right)=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \textit{ as } r\to\infty\\ \qquad\qquad\textit{and for some } \alpha<\rho_f^{[p]L^*}\\ L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,g)\right)\right) \textit{ otherwise.}\end{cases}$$

Theorem 22. *Let f and g be any two entire functions such that $0<\rho_f^{L^*}<\lambda_g^{[q]L^*}\leq\rho_g^{[q]L^*}<\infty$ where $q\geq 1$. Then for any odd number n $(\neq 1)$ and for any $\beta>1$,*

$$\lim_{r\to\infty}\frac{\log^{\left[\frac{(n-1)q}{2}\right]}T(r,f_n)}{\log^{[q-1]}T(r,g)\cdot K(r,f;L)}=0\ ,$$

$$\textit{where } K(r,f;L)=\begin{cases}1 \textit{ if } L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,f)\right)\right)=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \textit{ as } r\to\infty\\ \qquad\qquad\textit{and for some } \alpha<\lambda_g^{[q]L^*}\\ L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,f)\right)\right) \textit{ otherwise.}\end{cases}$$

Theorem 23. *Let f and g be any two entire functions with $0<\rho_f^{L^*}<\rho_g^{[q]L^*}<\infty$ where $q\geq 1$. Then for any odd number n $(\neq 1)$ and for any $\beta>1$,*

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{(n-1)q}{2}\right]}T(r,f_n)}{\log^{[q-1]}T(r,g)\cdot K(r,g;L)}=0\ ,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \ \textit{if } L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,f)\right)\right) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \ \textit{as } r\to\infty \\ \qquad\qquad\qquad\qquad \textit{and for some } \alpha < \rho_g^{[q]L^*} \\ L\left(\exp\left(\frac{\beta+1}{\beta-1}T(\beta r,f)\right)\right) \ \textit{otherwise.} \end{cases}$$

We omit the proof of Theorem 22 and Theorem 23 as those can be carried out in the line of Theorem 20 and Theorem 21 and with the help of second part of Lemma 4.

The following four theorems can be carried out in the line of Theorem 20, Theorem 21, Theorem 22 and Theorem 23 respectively and with the help of Lemma 5. Therefore their proofs are omitted.

Theorem 24. *Let f and g be any two entire functions such that $0 < \rho_g^{L^*} < \lambda_f^{[p]L^*} \le \rho_f^{[p]L^*} < \infty$ where $p \ge 1$. Then for any even number n,*

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} M(r,f_n)}{\log^{[p]} M(r,f)\cdot K(r,g;L)} = 0 \ ,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \ \textit{if } L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \ \textit{as } r\to\infty \\ \qquad\qquad \textit{and for some } \alpha < \lambda_f^{[p]L^*} \\ L(M(r,g)) \ \textit{otherwise.} \end{cases}$$

Theorem 25. *Let f and g be any two entire functions with $0 < \rho_g^{L^*} < \rho_f^{[p]L^*} < \infty$ where $p \ge 1$. Then for any even number n,*

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} M(r,f_n)}{\log^{[p]} M(r,f)\cdot K(r,g;L)} = 0 \ ,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \ \textit{if } L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \ \textit{as } r\to\infty \\ \qquad\qquad \textit{and for some } \alpha < \rho_f^{[p]L^*} \\ L(M(r,g)) \ \textit{otherwise.} \end{cases}$$

Theorem 26. *Let f and g be any two entire functions such that $0 < \rho_f^{L^*} < \lambda_g^{[q]L^*} \le \rho_g^{[q]L^*} < \infty$ where $q \ge 1$. Then for any odd number n $(\neq 1)$,*

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q+2}{2}\right]} M(r,f_n)}{\log^{[q]} M(r,g)\cdot K(r,f;L)} = 0 \ ,$$

$$\text{where } K(r,f;L) = \begin{cases} 1 \ \textit{if } L(M(r,f)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \ \textit{as } r\to\infty \\ \qquad\qquad \textit{and for some } \alpha < \lambda_g^{[q]L^*} \\ L(M(r,f)) \ \textit{otherwise.} \end{cases}$$

Theorem 27. *Let f and g be any two entire functions with $0<\rho_f^{L^*}<\rho_g^{[q]L^*}<\infty$ where $q\geq 1$. Then for any odd number $n\ (\neq 1)$,*

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{(n-1)q}{2}\right]}M(r,f_n)}{\log^{[q]}M(r,g)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(M(r,f))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad \text{and for some } \alpha<\rho_g^{[q]L^*}\\ L(M(r,f)) \text{ otherwise.}\end{cases}$

Replacing maximum modulus by maximum term in Theorem 24, Theorem 25, Theorem 26 and Theorem 27 we respectively get Theorem 28, Theorem 29, Theorem 30 and Theorem 31 :

Theorem 28. *Let f and g be any two entire functions such that $0<\rho_g^{L^*}<\lambda_f^{[p]L^*}\leq\rho_f^{[p]L^*}<\infty$ where $p\geq 1$. Then for any even number n and for any $\beta>1$,*

$$\lim_{r\to\infty}\frac{\log^{\left[\frac{np}{2}\right]}\mu(r,f_n)}{\log^{[p]}\mu(r,f)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(\mu(\beta r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad \text{and for some } \alpha<\lambda_f^{[p]L^*}\\ L(\mu(\beta r,g)) \text{ otherwise.}\end{cases}$

Theorem 29. *Let f and g be any two entire functions with $0<\rho_g^{L^*}<\rho_f^{[p]L^*}<\infty$ where $p\geq 1$. Then for any even number n and for any $\beta>1$,*

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{np}{2}\right]}\mu(r,f_n)}{\log^{[p]}\mu(r,f)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(\mu(\beta r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad \text{and for some } \alpha<\rho_f^{[p]L^*}\\ L(\mu(\beta r,g)) \text{ otherwise.}\end{cases}$

Theorem 30. *Let f and g be any two entire functions such that $0<\rho_f^{L^*}<\lambda_g^{[q]L^*}\leq\rho_g^{[q]L^*}<\infty$ where $q\geq 1$. Then for any odd number $n\ (\neq 1)$ and for any $\beta>1$,*

$$\lim_{r\to\infty}\frac{\log^{\left[\frac{(n-1)q}{2}\right]}\mu(r,f_n)}{\log^{[q]}\mu(r,g)\cdot K(r,f;L)}=0\ ,$$

where $K(r,f;L)=\begin{cases}1 \text{ if } L(\mu(\beta r,f))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad \text{and for some } \alpha<\lambda_g^{[q]L^*}\\ L(\mu(\beta r,f)) \text{ otherwise.}\end{cases}$

Theorem 31. *Let f and g be any two entire functions with $0 < \rho_f^{L^*} < \rho_g^{[q]L^*} < \infty$ where $q \geq 1$. Then for any odd number n $(\neq 1)$ and for any $\beta > 1$,*

$$\liminf_{r \to \infty} \frac{\log^{\left[\frac{(n-1)q}{2}\right]} \mu(r, f_n)}{\log^{[q]} \mu(r, g) \cdot K(r, g; L)} = 0 ,$$

where $K(r, g; L) = \begin{cases} 1 \text{ if } L(\mu(\beta r, f)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \text{ as } r \to \infty \\ \qquad \text{and for some } \alpha < \rho_g^{[q]L^*} \\ L(\mu(\beta r, f)) \text{ otherwise.} \end{cases}$

We omit the proof of Theorem 28, Theorem 29, Theorem 30 and Theorem 31 as those can be carried out in the line of Theorem 24, Theorem 25, Theorem 26 and Theorem 27 and with the help of Lemma 6.

Theorem 32. *Let f and g be any two entire functions with $\rho_f^{[p]L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where p is any positive integer. Then for any even n and $\beta > 1$,*
(a) if $L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right) = o\{\log T(r, g)\}$ then

$$\limsup_{r \to \infty} \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r, g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} \leq \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}$$

and (b) if $\log T(r, g) = o\left\{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)\right\}$ then

$$\lim_{r \to \infty} \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r, g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} = 0 .$$

Proof. Taking $R = \beta r$ in the inequality $T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f)$ $\{cf.$ [3]$\}$ and also using $\log\left\{1 + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}\right\} \sim \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}$, we get from Lemma 4 for all sufficiently large values of r that

$$\log^{\left[\frac{np-2}{2}\right]} T(r, f_n) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right) \log M(r, g) \left[1 + \frac{O(1) + L(M(r, g))}{\log M(r, g)}\right]$$

$$\begin{aligned} & i.e., \ \log^{\left[\frac{np-2}{2}\right]} T(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right) \log M(r, g) \left[1 + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}\right] \end{aligned}$$

$$i.e., \ \log^{\left[\frac{np}{2}\right]} T(r, f_n) \leq \log\left(\rho_f^{[p]L^*} + \varepsilon\right) + \log^{[2]} M(r, g) + \log\left\{1 + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}\right\}$$

$$i.e., \ \log^{\left[\frac{np}{2}\right]} T(r, f_n) \leq \log\left(\rho_f^{[p]L^*} + \varepsilon\right) + \left(\rho_g^{L^*} + \varepsilon\right) \log\left\{re^{L(r)}\right\} + \log\left\{1 + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}\right\}$$

$$i.e., \ \log^{\left[\frac{np}{2}\right]} T(r, f_n) \leq \log\left(\rho_f^{[p]L^*} + \varepsilon\right) + \left(\rho_g^{L^*} + \varepsilon\right) \log\left\{re^{L(r)}\right\} + \log\left\{1 + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}\right\}$$

$$i.e., \ \log^{\left[\frac{np}{2}\right]} T(r, f_n) \leq O(1) + \left(\rho_g^{L^*} + \varepsilon\right)\{\log r + L(r)\} + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}. \tag{20}$$

Again from the definition of L^*-lower order, we get for all sufficiently large values of r that

$$\begin{aligned} \log T(r, g) &\geq \left(\lambda_g^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ \log T(r, g) &\geq \left(\lambda_g^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \ \log T(r, g) &\geq \left(\lambda_g^{L^*} - \varepsilon\right) [\log r + L(r)] \\ i.e., \ \log r + L(r) &\leq \frac{\log T(r, g)}{\left(\lambda_g^{L^*} - \varepsilon\right)} . \end{aligned} \tag{21}$$

Hence from (20) and (21), it follows for all sufficiently large values of r that

$$\begin{aligned} & \log^{\left[\frac{np}{2}\right]} T(r, f_n) \\ \leq \ & O(1) + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right) \cdot \log T(r, g) + \frac{O(1) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)}{\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)} \end{aligned}$$

$$i.e, \ \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r,g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} \leq \frac{\frac{O(1)}{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right)T(\beta r,g)\right)\right)}}{\frac{\log T(r,g)}{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right)T(\beta r,g)\right)\right)} + 1}$$

$$+ \frac{\left(\frac{\rho_g^{L^*}+\varepsilon}{\lambda_g^{L^*}-\varepsilon}\right)}{1 + \frac{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right)T(\beta r,g)\right)\right)}{\log T(r,g)}} + \frac{1}{\left[1 + \frac{\log T(r,g)}{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right)T(\beta r,g)\right)\right)}\right] \left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)}. \quad (22)$$

Since $L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right) = o\{\log T(r,g)\}$ as $r \to \infty$ and $\varepsilon (> 0)$ is arbitrary, we obtain from (22) that

$$\limsup_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r,g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} \leq \frac{\rho_g^{L^*}}{\lambda_g^{L^*}} . \quad (23)$$

Again if $\log T(r,g) = o\left\{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)\right\}$ then from (22) we get that

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r,g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} = 0 . \quad (24)$$

Thus the theorem follows from (23) and (24). □

Corollary 33. *Let f and g be any two entire functions with $\rho_f^{[p]L^*} < \infty$, $0 < \rho_g^{L^*} < \infty$ where $p \geq 1$. Then for any even n and $\beta > 1$,*
(a) if $L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right) = o\{\log T(r,g)\}$ then

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r,g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} \leq 1$$

and (b) if $\log T(r,g) = o\left\{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np}{2}\right]} T(r, f_n)}{\log T(r,g) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, g)\right)\right)} = 0 .$$

We omit the proof of Corollary 33 because it can be carried out in the line of Theorem 32.

Theorem 34. *Let f and g be any two entire functions with $\rho_f^{[p]L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where p is any positive integer. Then for any even n,*

(a) *if* $L(M(r,g)) = o\left\{\log^{[2]} M(r,g)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{\left[\frac{np+2}{2}\right]} M(r,f_n)}{\log^{[2]} M(r,g) + L(M(r,g))} \le \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}$$

and (b) *if* $\log^{[2]} M(r,g) = o\{L(M(r,g))\}$ *then*

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np+2}{2}\right]} M(r,f_n)}{\log^{[2]} M(r,g) + L(M(r,g))} = 0 \ .$$

Corollary 35. *Let f and g be any two entire functions with $\rho_f^{[p]L^*} < \infty$, $0 < \rho_g^{L^*} < \infty$ where $p \ge 1$. Then for any even n,*
(a) *if* $L(M(r,g)) = o\left\{\log^{[2]} M(r,g)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np+2}{2}\right]} M(r,f_n)}{\log^{[2]} M(r,g) + L(M(r,g))} \le 1$$

and (b) *if* $\log^{[2]} M(r,g) = o\{L(M(r,g))\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{np+2}{2}\right]} M(r,f_n)}{\log^{[2]} M(r,g) + L(M(r,g))} = 0 \ .$$

Using $\log\left\{1 + \frac{O(1)+L(M(r,g))}{\log M(r,g)}\right\} \sim \frac{O(1)+L(M(r,g))}{\log M(r,g)}$ and with the help of Lemma 5, Theorem 34 and Corollary 35 can be carried out in the line of Theorem 32 and Corollary 33 respectively. Hence their proofs are omitted.

Analogusly using $\log\left\{1 + \frac{O(1)+L(\mu(\beta r,g))}{\log^{[2]}\mu(r,g)}\right\} \sim \frac{O(1)+L(\mu(\beta r,g))}{\log^{[2]}\mu(r,g)}$ for any $\beta > 1$ and with the help of Lemma 6, the following theorem and corollary can be carried out in the line of Theorem 34 and Corollary 35 respectively.

Theorem 36. *Let f and g be any two entire functions with $\rho_f^{[p]L^*} < \infty$, $0 < \lambda_g^{L^*} \le \rho_g^{L^*} < \infty$ where p is any positive integer. Then for any even n and $\beta > 1$,*
(a) *if* $L(\mu(\beta r,g)) = o\left\{\log^{[2]} \mu(r,g)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{\left[\frac{np+2}{2}\right]} \mu(r,f_n)}{\log^{[2]} \mu(r,g) + L(\mu(\beta r,g))} \le \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}$$

and (b) *if* $\log^{[2]} \mu(r,g) = o\{L(\mu(\beta r,g))\}$ *then*

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{np+2}{2}\right]} \mu(r,f_n)}{\log^{[2]} \mu(r,g) + L(\mu(\beta r,g))} = 0 \ .$$

Corollary 37. *Let f and g be any two entire functions with $\rho_f^{[p]L^*} < \infty$, $0 < \rho_g^{L^*} < \infty$ where $p \geq 1$. Then for any even n and $\beta > 1$,*
(a) if $L(\mu(\beta r, g)) = o\left\{\log^{[2]} \mu(r, g)\right\}$ then

$$\liminf_{r \to \infty} \frac{\log^{\left[\frac{np}{2}\right]} \mu(r, f_n)}{\log^{[2]} \mu(r, g) + L(\mu(\beta r, g))} \leq 1$$

and (b) if $\log^{[2]} \mu(r, g) = o\{L(\mu(\beta r, g))\}$ then

$$\liminf_{r \to \infty} \frac{\log^{\left[\frac{np}{2}\right]} \mu(r, f_n)}{\log^{[2]} \mu(r, g) + L(\mu(\beta r, g))} = 0 \ .$$

Theorem 38. *Let f and g be any two entire functions with $\rho_g^{[q]L^*} < \infty$, $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ where q is any positive integer. Then for any odd $n \neq 1$ and $\beta > 1$,*
(a) if $L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right) = o\{\log T(r, f)\}$ then

$$\limsup_{r \to \infty} \frac{\log^{\left[\frac{q(n-2)}{2}\right]} T(r, f_n)}{\log T(r, f) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right)} \leq \frac{\rho_f^{L^*}}{\lambda_f^{L^*}}$$

and (b) if $\log T(r, f) = o\left\{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right)\right\}$ then

$$\lim_{r \to \infty} \frac{\log^{\left[\frac{q(n-2)}{2}\right]} T(r, f_n)}{\log T(r, f) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right)} = 0 \ .$$

Corollary 39. *Let f and g be any two entire functions with $\rho_g^{[q]L^*} < \infty$, $0 < \rho_f^{L^*} < \infty$ where $q \geq 1$. Then for any odd $n \neq 1$ and $\beta > 1$,*
(a) if $L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right) = o\{\log T(r, f)\}$ then

$$\liminf_{r \to \infty} \frac{\log^{\left[\frac{q(n-2)}{2}\right]} T(r, f_n)}{\log T(r, f) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right)} \leq 1$$

and (b) if $\log T(r, f) = o\left\{L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right)\right\}$ then

$$\liminf_{r \to \infty} \frac{\log^{\left[\frac{q(n-2)}{2}\right]} T(r, f_n)}{\log T(r, f) + L\left(\exp\left(\left(\frac{\beta+1}{\beta-1}\right) T(\beta r, f)\right)\right)} = 0 \ .$$

Theorem 40. *Let f and g be any two entire functions with $\rho_g^{[q]L^*} < \infty$, $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ where q is any positive integer. Then for any odd $n \neq 1$,*
(a) if $L(M(r,f)) = o\left\{\log^{[2]} M(r,f)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} M(r,f_n)}{\log^{[2]} M(r,f) + L(M(r,f))} \leq \frac{\rho_f^{L^*}}{\lambda_f^{L^*}}$$

and (b) if $\log^{[2]} M(r,f) = o\{L(M(r,f))\}$ then

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} M(r,f_n)}{\log^{[2]} M(r,f) + L(M(r,f))} = 0\ .$$

Corollary 41. *Let f and g be any two entire functions with $\rho_g^{[q]L^*} < \infty$, $0 < \rho_f^{L^*} < \infty$ where $q \geq 1$. Then for any odd $n \neq 1$,*
(a) if $L(M(r,f)) = o\left\{\log^{[2]} M(r,f)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} M(r,f_n)}{\log^{[2]} M(r,f) + L(M(r,f))} \leq 1$$

and (b) if $\log^{[2]} M(r,f) = o\{L(M(r,f))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} M(r,f_n)}{\log^{[2]} M(r,f) + L(M(r,f))} = 0\ .$$

Theorem 42. *Let f and g be any two entire functions with $\rho_g^{[q]L^*} < \infty$, $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ where q is any positive integer. Then for any odd $n \neq 1$ and $\beta > 1$,*
(a) if $L(\mu(\beta r,f)) = o\left\{\log^{[2]} \mu(r,f)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} \mu(r,f_n)}{\log^{[2]} \mu(r,f) + L(\mu(\beta r,f))} \leq \frac{\rho_f^{L^*}}{\lambda_f^{L^*}}$$

and (b) if $\log^{[2]} \mu(r,f) = o\{L(\mu(\beta r,f))\}$ then

$$\lim_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} \mu(r,f_n)}{\log^{[2]} \mu(r,f) + L(\mu(\beta r,f))} = 0\ .$$

Corollary 43. *Let f and g be any two entire functions with $\rho_g^{[q]L^*} < \infty$, $0 < \rho_f^{L^*} < \infty$ where $q \geq 1$. Then for any odd $n \neq 1$ and $\beta > 1$,*
(a) if $L(\mu(\beta r,f)) = o\left\{\log^{[2]} \mu(r,f)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} \mu(r,f_n)}{\log^{[2]} \mu(r,f) + L(\mu(\beta r,f))} \leq 1$$

and (b) *if* $\log^{[2]} \mu(r,f) = o\{L(\mu(\beta r, f))\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{(n-1)q}{2}+1\right]} \mu(r,f_n)}{\log^{[2]} \mu(r,f) + L(\mu(\beta r, f))} = 0 \ .$$

We omit the proof of Theorem 38, Theorem 40, Theorem 42, Corollary 39, Corollary 41 and Corollary 43 because those can be carried out in view of the second part of Lemma 4, Lemma 5 and Lemma 6 respectively.

Remark 2. *The equality sign in Theorem 32 and Corollary 33 cannot be removed as we see in the following example.*

Example 1. *Let* $f = g = \exp z$, $p = 2$, $\beta = 2$ *and* $L(r) = \frac{1}{m}\exp\left(\frac{1}{r}\right)$ *where* m *is any positive real number.*
Then

$$\rho_f^{L^*} = \lambda_g^{L^*} = \rho_g^{L^*} = 1.$$

Now for any even n,

$$f_n = \exp^{[n]} z.$$

Now

$$\begin{aligned} T(r,f_n) &\leq \log M(r,f_n) = \exp^{[n-1]} r \\ i.e.,\ \log^{[n]} T(r,f_n) &\leq \log^{[n]}\left(\exp^{[n-1]} r\right) \\ i.e.,\ \log^{[n]} T(r,f_n) &\leq \log r \\ and\ 3T(2r,g) &\geq \log M(r,g) = r \\ i.e.,\ \log T(2r,g) &\geq \log r + O(1) \\ i.e.,\ \log T(r,g) &\geq \log r + O(1)\ . \end{aligned}$$

Also

$$\begin{aligned} \log^{[n]} T(r,f_n) &\geq \log r + O(1), \\ and\ \log T(r,g) &\leq \log r\ . \end{aligned}$$

So

$$L(\exp(3T(2r,g))) \geqslant L(M(r,g)) = L(\exp r) = \frac{1}{m}\exp\left(\frac{1}{\exp r}\right)$$

and

$$L(\exp(3T(2r,g))) \leq L\left(M\left(r^\delta, g\right)\right) = L\left(\exp r^\delta\right) = \frac{1}{m}\exp\left(\frac{1}{\exp r^\delta}\right) \quad for\ any\ \delta > 1.$$

Hence

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\log^{[n]}T\left(r,f_n\right)}{\log T\left(r,g\right)+L\left(\exp\left(3T\left(2r,g\right)\right)\right)}\\
\leq\;&\limsup_{r\to\infty}\frac{\log r}{\log r+O\left(1\right)+\frac{1}{m}\exp\left(\frac{1}{\exp r}\right)}=1
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{r\to\infty}\frac{\log^{[n]}T\left(r,f_n\right)}{\log T\left(r,g\right)+L\left(\exp\left(3T\left(2r,g\right)\right)\right)}\\
\geq\;&\liminf_{r\to\infty}\frac{\log r+O(1)}{\log r+\frac{1}{m}\exp\left(\frac{1}{\exp r^{\delta}}\right)}=1\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\liminf_{r\to\infty}\frac{\log^{[n]}T\left(r,f_n\right)}{\log T\left(r,g\right)+L\left(\exp\left(3T\left(2r,g\right)\right)\right)}\\
=\;&\limsup_{r\to\infty}\frac{\log^{[n]}T\left(r,f_n\right)}{\log T\left(r,g\right)+L\left(\exp\left(3T\left(2r,g\right)\right)\right)}\\
=\;&1\ .
\end{aligned}$$

Remark 3. *Considering $f=g=\exp z$, $p=2$ and $L\left(r\right)=\frac{1}{m}\exp\left(\frac{1}{r}\right)$ where m is any positive real number, one can easily verify that the equality sign in Theorem 34, Theorem 36, Corollary 35 and Corollary 37 cannot be removed.*

Remark 4. *Considering $f=g=\exp z$, $q=2$ and $L\left(r\right)=\frac{1}{m}\exp\left(\frac{1}{r}\right)$ where m is any positive real number, one can easily verify that the equality sign in Theorem 38, Theorem 40, Theorem 42, Corollary 39, Corollary 41 and Corollary 43 cannot be removed.*

4 Open Problem

Extending the notion of classical growth indicators *order* and *lower order*, Juneja et al. [4] defined the (p,q)-*th order* and (p,q)-*th lower order* of an entire function f as follows:

$$\rho_f\left(p,q\right)=\limsup_{r\to\infty}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r}\text{ and }\lambda_f\left(p,q\right)=\liminf_{r\to\infty}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r},$$

where p,q are positive integers with $p\geq q$.

In the line Juneja et al. [4], one can define the (p,q,m)-*th L-order* and (p,q,m)-*th L-lower order* of an entire function f in the following way :

$$\begin{aligned} \rho_f^L(p,q,m) &= \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[r \exp^{[m]} L(r)\right]} \text{ and} \\ \lambda_f^{[p]L}(p,q,m) &= \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[r \exp^{[m]} L(r)\right]} \end{aligned}$$

where p, q and m are any three positive integers with $p \geq q$. Also using the inequalities $T(r,f) \leq \log^+ M(r,f) \leq 3T(2r,f)$ $\{cf.$ [3]$\}$ and $\mu(r,f) \leq M(r,f) \leq 2\mu(2r,f)$ $\{cf.$ [13]$\}$ respectively, the above definition can also be reformulated in terms of *maximum terms* and *Nevanlinna's characteristic functions* of entire functions. Now the following natural question may arrise for the workers of this branch :
Can the results which we have established in this paper be modified by the treatment of the notions of (p,q,m)-*th L-order* and (p,q,m)-*th L-lower order*?

References

[1] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical Essays dedicated to A. J. Macintyre,Ohio University Press, (1970), 75-92.

[2] S.K. Datta and T. Biswas, *On the definition of a meromorphic function of order zero*, International Mathematical Forum, 4 (37), (2009), 1851-1861.

[3] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, (1964).

[4] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math., 282 (1976), 53-67.

[5] T.V. Lakshminarasimhan, *A note on entire functions of bounded index*, J. Indian Math. Soc., 38, (1974), 43-49.

[6] B. K. Lahiri and D. Banerjee, *Relative fix points of entire functions*, J. India Acad. Math., 19 (1), (1997), 87-97.

[7] I. Lahiri and N.R. Bhattacharjee, *Functions of L-bounded index and of non-uniform L-bounded index*, Indian J. Math., 40 (1), (1998), 43-57.

[8] K. Niino and C. C. Tang, *Some growth relationships on factors of two composite entire functions*, Factorization Theory of Meromorphic Functions and Related Topocs, Marcel Dekker Inc. (New York and Basel), (1982), 95-99.

[9] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 69, (1963), 411-414.

[10] S.K. Singh and G.P. Barker, *Slowly changing functions and their applications*, Indian J. Math., 19 (1), (1977), 1-6.

[11] D. Somasundaram and R. Thamizharasi, *A note on the entire functions of L-bounded index and L-type*, Indian J. Pure Appl. Math., 19 (3), (March 1988), 284-293.

[12] A. P. Singh, *On maximum term of composition of entire functions*, Proc. Nat. Acad. Sci. India, 59(A) (Part I), (1989), 103-115.

[13] A. P. Singh and M. S. Baloria, *On maximum modulus and maximum term of composition of entire functions*, Indian J. Pure Appl. Math., 22 (12), (1991), 1019-1026.

[14] G. Valiron, *Lectures on the general theory of integral functions*, Chelsea Publishing Company, (1949).