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On fractional inequalities via Montgomery identities

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Abstract

In the present work we give several new integral inequalities via Riemann-Liouville fractional integral and Montgomery identities.

Keywords: *Riemann-Liouville fractional integral, Ostrowski inequality.*

1 Introduction

The inequality of Ostrowski [8] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ is the best possible.

For some generalizations of this classic fact see the book [5, p.468-484] by Mitrinovic, Pecaric and Fink. A simple proof of this fact can be done by using the following identity [5]:

If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with the first derivative f' integrable on $[a, b]$, then Montgomery identity holds:

$$f(t) = \frac{1}{b-a} \int_a^b f(s)ds + \int_a^b P(t, s)f'(s)ds, \tag{1}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(t,s) := \begin{cases} \frac{s-a}{b-a}, & a \le s < t \\ \\ \frac{s-b}{b-a}, & t \le s \le b. \end{cases} \tag{2}$$

Suppose now that $w : [a,b] \to [0,\infty)$ is some probability density function, i.e. it is a positive integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The following identity (given by Pečarić in [7]) is the weighted generalization of the Montgomery identity:

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt, \tag{3}$$

where the weighted Peano kernel is

$$P_w(x,t) := \begin{cases} W(t), & a \le t < x \\ \\ W(t) - 1, & x \le t \le b. \end{cases}$$

The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$ with $a \ge 0$ is defined by

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad (4) \\ J_a^0 f(x) &= f(x). \end{aligned}$$

Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, see ([1]-[4], [6], [9]-[11]) and the references cited therein.

2 Main Results

Theorem 2.1 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$ such that $f' \in L_p[a,b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $\alpha \ge 0$. Then the following inequality holds:*

$$\begin{aligned} &\left| \Gamma(\alpha+1) J_a^\alpha f(b) - (b-a)^{\alpha-1} \int_a^b f(s)ds \right| \qquad (5) \\ &\le (b-a)^{\alpha+\frac{1}{q}} \left(\frac{1}{(\alpha q+1)^{\frac{1}{q}}} + \frac{1}{(q+1)^{\frac{1}{q}}} \right) \|f'\|_p. \end{aligned}$$

Proof. We write the Riemann-Liouville fractional integral operator as follows:

$$\Gamma(\alpha) J_a^\alpha f(b) = \int_a^b (b-t)^{\alpha-1} f(t)\,dt. \tag{6}$$

Thus, using Montgomery identity in (6), we have

$$\begin{aligned}\Gamma(\alpha) J_a^\alpha f(b) &= \int_a^b (b-t)^{\alpha-1}\left[\frac{1}{b-a}\int_a^b f(s)\,ds + \int_a^b P(t,s) f'(s)\,ds\right]dt \\ &= \frac{1}{b-a}\int_a^b (b-t)^{\alpha-1}\left[\int_a^b f(s)\,ds + \int_a^t (s-a) f'(s)\,ds\right. \\ &\quad \left. + \int_t^b (s-b) f'(s)\,ds\right]dt.\end{aligned} \tag{7}$$

By an interchange of the order of integration, we get

$$\int_a^b (b-t)^{\alpha-1}\left(\int_a^b f(s)\,ds\right)dt = \frac{(b-a)^\alpha}{\alpha}\int_a^b f(s)\,ds, \tag{8}$$

$$\begin{aligned}&\int_a^b (b-t)^{\alpha-1}\left(\int_a^t (s-a) f'(s)\,ds\right)dt \\ &= \frac{b-a}{\alpha}\int_a^b (b-s)^\alpha f'(s)\,ds - \frac{1}{\alpha}\int_a^b (b-s)^{\alpha+1} f'(s)\,ds,\end{aligned} \tag{9}$$

$$\begin{aligned}&\int_a^b (b-t)^{\alpha-1}\left(\int_t^b (s-b) f'(s)\,ds\right)dt \\ &= \frac{1}{\alpha}\int_a^b (b-s)^{\alpha+1} f'(s)\,ds - \frac{(b-a)^\alpha}{\alpha}\int_a^b (b-s) f'(s)\,ds.\end{aligned} \tag{10}$$

Thus, using (8), (9) and (10) in (7) we get

$$\begin{aligned}&\Gamma(\alpha+1)\,J_a^\alpha f(b)-(b-a)^{\alpha-1}\int_a^b f(s)\,ds\\ &= \int_a^b (b-s)^\alpha f'(s)\,ds-(b-a)^{\alpha-1}\int_a^b (b-s)\,f'(s)\,ds,\ \alpha\geq 0.\end{aligned}\tag{11}$$

By taking the modulus and applying Hölder inequality, we have

$$\begin{aligned}&\left|\Gamma(\alpha+1)\,J_a^\alpha f(b)-(b-a)^{\alpha-1}\int_a^b f(s)\,ds\right|\\ &\leq \left(\int_a^b \left|f'(s)\right|^p ds\right)^{\frac{1}{p}}\left(\int_a^b (b-s)^{\alpha q}\,ds\right)^{\frac{1}{q}}\\ &\quad+(b-a)^{\alpha-1}\left(\int_a^b \left|f'(s)\right|^p ds\right)^{\frac{1}{p}}\left(\int_a^b (b-s)^{q}\,ds\right)^{\frac{1}{q}}\\ &= (b-a)^{\alpha+\frac{1}{q}}\left(\frac{1}{(\alpha q+1)^{\frac{1}{q}}}+\frac{1}{(q+1)^{\frac{1}{q}}}\right)\left\|f'\right\|_p.\end{aligned}$$

The proof is completed.

Theorem 2.2 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$ and $|f'(x)|\leq M$, for every $x\in[a,b]$ and $\alpha\geq 0$. Then the following inequality holds:*

$$\left|J_a^\alpha f(b)-\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\int_a^b f(s)\,ds\right|\leq\frac{M(\alpha+3)(b-a)^{\alpha+1}}{2\Gamma(\alpha+2)}.\tag{12}$$

Proof. By use the (11), we have

$$\begin{aligned}&\left|\Gamma(\alpha+1)\,J_a^\alpha f(b)-(b-a)^{\alpha-1}\int_a^b f(s)\,ds\right|\\ &\leq \int_a^b (b-s)^\alpha\left|f'(s)\right|ds+(b-a)^{\alpha-1}\int_a^b (b-s)\left|f'(s)\right|ds.\end{aligned}\tag{13}$$

Since $\left|f'(x)\right| \leq M$, we get the require inequality which the proof is completed.

Theorem 2.3 *Let $w : [a,b] \to [0,\infty)$ be a probability density function, i.e. $\int_a^b w(t)\,dt = 1$, and set $W(t) = \int_a^t w(x)\,dx$ for $a \leq t \leq b$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$ such that $f' \in L_p[a,b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $\alpha \geq 0$. Then the following inequality holds:*

$$\left|\Gamma(\alpha+1)\,J_a^\alpha f(b) - (b-a)^\alpha \int_a^b w(s)\,f(s)\,ds\right| \tag{14}$$

$$\leq \left\|f'\right\|_p (b-a)^\alpha \left[\left(\int_a^b |W(s)-1|^q\,ds\right)^{\frac{1}{q}} + \left(\frac{b-a}{\alpha q+1}\right)^{\frac{1}{q}}\right].$$

Proof. By using (3) in (6), we have

$$\begin{aligned}
\Gamma(\alpha)\,J_a^\alpha f(b) &= \int_a^b (b-t)^{\alpha-1}\left[\int_a^b w(s)\,f(s)\,ds + \int_a^b P_w(t,s)\,f'(s)\,ds\right]dt \\
&= \int_a^b (b-t)^{\alpha-1}\left(\int_a^b w(s)\,f(s)\,ds\right)dt \qquad (15) \\
&\quad + \int_a^b (b-t)^{\alpha-1}\left(\int_a^t W(s)\,f'(s)\,ds\right)dt \\
&\quad + \int_a^b (b-t)^{\alpha-1}\left(\int_t^b (W(s)-1)\,f'(s)\,ds\right)dt.
\end{aligned}$$

By an interchange of the order of integration, we get

$$\int_a^b (b-t)^{\alpha-1}\left(\int_a^b w(s)\,f(s)\,ds\right)dt = \frac{(b-a)^\alpha}{\alpha}\int_a^b w(s)\,f(s)\,ds, \tag{16}$$

$$\int_a^b (b-t)^{\alpha-1}\left(\int_a^t W(s)\,f'(s)\,ds\right)dt = \frac{1}{\alpha}\int_a^b (b-s)^\alpha\,W(s)\,f'(s)\,ds, \tag{17}$$

and

$$\int_a^b (b-t)^{\alpha-1}\left(\int_t^b (W(s)-1) f'(s)\,ds\right)dt \tag{18}$$
$$= \frac{1}{\alpha}\left[(b-a)^{\alpha}\int_a^b [W(s)-1] f'(s)\,ds + \int_a^b (b-s)^{\alpha} f'(s)\,ds\right].$$

Thus, using (16), (17) and (18) in (15) we get

$$\Gamma(\alpha+1) J_a^{\alpha} f(b) - (b-a)^{\alpha}\int_a^b w(s) f(s)\,ds \tag{19}$$
$$= (b-a)^{\alpha}\int_a^b [W(s)-1] f'(s)\,ds + \int_a^b (b-s)^{\alpha} f'(s)\,ds.$$

By taking the modulus and applying Hölder inequality, we have

$$\left|\Gamma(\alpha+1) J_a^{\alpha} f(b) - (b-a)^{\alpha}\int_a^b w(s) f(s)\,ds\right|$$
$$\leq (b-a)^{\alpha}\left(\int_a^b \left|f'(s)\right|^p ds\right)^{\frac{1}{p}}\left(\int_a^b |W(s)-1|^q\,ds\right)^{\frac{1}{q}}$$
$$+\left(\int_a^b \left|f'(s)\right|^p ds\right)^{\frac{1}{p}}\left(\int_a^b (b-s)^{\alpha q}\,ds\right)^{\frac{1}{q}}$$
$$= \left\|f'\right\|_p (b-a)^{\alpha}\left[\left(\int_a^b |W(s)-1|^q\,ds\right)^{\frac{1}{q}} + \left(\frac{b-a}{\alpha q+1}\right)^{\frac{1}{q}}\right]$$

which the proof is completed.

Theorem 2.4 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$ and $|f'(x)|\leq M$, for every $x\in[a,b]$ and $\alpha\geq 0$. Then the following inequality holds:*

$$\left|\Gamma(\alpha+1)\,J_a^{\alpha}f(b)-(b-a)^{\alpha}\int_a^b w(s)\,f(s)\,ds\right| \le M(b-a)^{\alpha}\left(\int_a^b |W(s)-1|\,ds-\frac{b-a}{\alpha+1}\right). \tag{20}$$

Proof. From (19), we have

$$\left|\Gamma(\alpha+1)\,J_a^{\alpha}f(b)-(b-a)^{\alpha}\int_a^b w(s)\,f(s)\,ds\right| \le (b-a)^{\alpha}\int_a^b |W(s)-1|\left|f'(s)\right|ds+\int_a^b (b-s)^{\alpha}\left|f'(s)\right|ds. \tag{21}$$

By using $\left|f'(x)\right| \le M$, the proof is completed.

3 Open Problem

In this paper, we have investigated several new integral inequalities via Riemann-Liouville fractional integral and Montgomery identities. We will continue exploring other inequalities of this type. So, there is one questions as follows:

How can be established the new versions of the inequalities (5), (12), (14) and (20) involving several differentiable functions and probability density function via other fractional integrals.

References

[1] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Inequal. in Pure and Appl. Math, 10(3), 2009, Art. 97, 6 pp.

[2] Z. Dahmani, L. Tabharit and S. Taf, *Some fractional integral inequalities*, Nonlinear Science Letters A, 2(1), 2010, p.155-160.

[3] Z. Dahmani, L. Tabharit and S. Taf, *New inequalities via Riemann-Liouville fractional integration*, J. Advance Research Sci. Comput., 2(1), 2010, p.40-45.

[4] R. Gorenflo, F. Mainardi, *Fractionalcalculus: integral and differentiable equations of fractional order*, Springer Verlag, Wien, 1997, p.223-276.

[5] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities involving functions and their integrals and derivatives,* Kluwer Academic Publishers, Dordrecht, 1991.

[6] S. G. Samko, A. A Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives Theory and Application*, Gordan and Breach Science, New York, 1993.

[7] J. Pečarić, On the Čebyšev inequality, Bul. Inst. Politehn. Timisoara, 25(39), 1980, pp.10-11.

[8] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[9] M.Z. Sarikaya and H. Ogunmez, *On new inequalities via Riemann-Liouville fractional integration*, Abstract and Applied Analysis, Volume 2012, Article ID 428983, 10 pages, doi:10.1155/2012/428983.

[10] M.Z. Sarikaya, E. Set, H. Yaldiz and N., Basak, *Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57, 2013, 2403-2407.

[11] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals,* Computers and Mathematics with Applications, 63(7), 2012, 1147-1154.