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Differential superordinations u using Ruscheweyh derivative and generalized Sălăgean operator

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Abstract

In the present paper we establish several differential superordinations regarding the operator $RD_{\lambda,\alpha}^m$ ***defined by using Ruscheweyh derivative*** $R^m f(z)$ ***and the generalized Sălăgean operator*** $D_\lambda^m f(z)$***,*** $RD_{\lambda,\alpha}^m : \mathcal{A}_n \to \mathcal{A}_n$***,*** $RD_{\lambda,\alpha}^m f(z) = (1-\alpha)R^m f(z) + \alpha D_\lambda^m f(z),\ z \in U$***, where*** $m, n \in \mathbb{N}$***,*** $\lambda, \alpha \geq 0$ ***and*** $f \in \mathcal{A}_n$, $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j,\ z \in U\}$. ***A number of interesting consequences of some of these superordination results are discussed. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.***

Keywords: *differential superordination, convex function, best subordinant, differential operator.*

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1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ and $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{ f \in \mathcal{A}_n : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$, the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is superordinate to g, written $g \prec f$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $g(z) = f(w(z))$ for all $z \in U$. If f is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p\left(z\right), zp'\left(z\right); z\right)$ are univalent in U and satisfies the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad z \in U, \tag{1}$$

then p is called a solution of the differential superordination. The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1).

An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant of (1). The best subordinant is unique up to a rotation of U.

Definition 1.1. *(Al Oboudi [9]) For $f \in \mathcal{A}_n$, $\lambda \geq 0$ and $n, m \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} D_\lambda^0 f\left(z\right) &= f\left(z\right) \\ D_\lambda^1 f\left(z\right) &= \left(1-\lambda\right) f\left(z\right) + \lambda z f'(z) = D_\lambda f\left(z\right)\ , ..., \\ D_\lambda^{m+1} f(z) &= \left(1-\lambda\right) D_\lambda^m f\left(z\right) + \lambda z \left(D_\lambda^m f\left(z\right)\right)' = D_\lambda\left(D_\lambda^m f\left(z\right)\right), \quad z \in U. \end{aligned}$$

Remark 1.2. *If $f \in \mathcal{A}_n$ and $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $D_\lambda^m f\left(z\right) = z + \sum_{j=n+1}^{\infty} \left[1 + \left(j-1\right)\lambda\right]^m a_j z^j$, $z \in U$.*

For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [15].

Definition 1.3. *(Ruscheweyh [14]) For $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} R^0 f\left(z\right) &= f\left(z\right) \\ R^1 f\left(z\right) &= z f'\left(z\right)\ , ..., \\ \left(m+1\right) R^{m+1} f\left(z\right) &= z\left(R^m f\left(z\right)\right)' + m R^m f\left(z\right), \quad z \in U. \end{aligned}$$

Remark 1.4. *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $R^m f\left(z\right) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j$, $z \in U$.*

Definition 1.5. *[3] Let $\alpha, \lambda \geq 0$, $n, m \in \mathbb{N}$. Denote by $RD_{\lambda,\alpha}^m$ the operator given by $RD_{\lambda,\alpha}^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$RD_{\lambda,\alpha}^m f(z) = (1-\alpha) R^m f(z) + \alpha D_\lambda^m f(z), \qquad z \in U.$$

Remark 1.6. *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then*
$RD_{\lambda,\alpha}^m f(z) = z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^m + (1-\alpha) C_{m+j-1}^m \right\} a_j z^j$, $z \in U$.
This operator was studied also in [6], [7], [4], [10], [11], [12].

Remark 1.7. *For $\alpha = 0$, $RD_{\lambda,0}^m f(z) = R^m f(z)$, where $z \in U$ and for $\alpha = 1$, $RD_{\lambda,1}^m f(z) = D_\lambda^m f(z)$, where $z \in U$.*

For $\lambda = 1$, we obtain $RD_{1,\alpha}^m f(z) = L_\alpha^m f(z)$ which was studied in [1], [2], [5].

For $m = 0$, $RD_{\lambda,\alpha}^0 f(z) = (1-\alpha) R^0 f(z) + \alpha D_\lambda^0 f(z) = f(z) = R^0 f(z) = D_\lambda^0 f(z)$, where $z \in U$.

Definition 1.8. *We denote by Q the set of functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.*

We will use the following lemmas.

Lemma 1.9. *(Miller and Mocanu [13, Th. 3.1.6, p. 71]) Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and $h(z) \prec p(z) + \frac{1}{\gamma} z p'(z)$, $z \in U$, then $q(z) \prec p(z)$, $z \in U$, where $q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, $z \in U$. The function q is convex and is the best subordinant.*

Lemma 1.10. *(Miller and Mocanu [13]) Let q be a convex function in U and let $h(z) = q(z) + \frac{1}{\gamma} z q'(z)$, $z \in U$, where $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and $q(z) + \frac{1}{\gamma} z q'(z) \prec p(z) + \frac{1}{\gamma} z p'(z)$, $z \in U$, then $q(z) \prec p(z)$, $z \in U$, where $q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, $z \in U$. The function q is the best subordinant.*

2 Main results

Theorem 2.1. *Let h be a convex function, $h(0) = 1$. Let $n, m \in \mathbb{N}$, $\lambda, \alpha, \delta \geq 0$, $f \in \mathcal{A}_n$ and suppose that $\left(\frac{RD_{\lambda,\alpha}^m f(z)}{z}\right)^{\delta-1} \left(RD_{\lambda,\alpha}^m f(z)\right)'$ is univalent and $\left(\frac{RD_{\lambda,\alpha}^m f(z)}{z}\right)^{\delta} \in \mathcal{H}[1, n\delta] \cap Q$. If*

$$h(z) \prec \left(\frac{RD_{\lambda,\alpha}^m f(z)}{z}\right)^{\delta-1} \left(RD_{\lambda,\alpha}^m f(z)\right)', \qquad z \in U, \tag{2}$$

then

$$q(z) \prec \left(\frac{RD_{\lambda,\alpha}^m f(z)}{z}\right)^{\delta}, \ z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. *The function* q *is convex and it is the best subordinant.*

Proof Consider
$p(z) = \left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta} = \left(\frac{z+\sum_{j=n+1}^{\infty}\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)C^n_{n+j-1}\right\}a_j z^j}{z}\right)^{\delta} = 1+p_{n\delta}z^{n\delta}+p_{n\delta+1}z^{n\delta+1}+...,\quad z\in U.$

Differentiating we obtain $\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z)\right)' = p(z)+\frac{1}{\delta}zp'(z)$, $z\in U$.

Then (2) becomes $h(z)\prec p(z)+\frac{1}{\delta}zp'(z)$, $z\in U$. By using Lemma 1.9 for $\gamma=\delta$, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)\prec\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z\in U$, where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. The function q is convex and it is the best subordinant.

Corollary 2.2. *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0\le\beta<1$. *Let* $n,m\in\mathbb{N}$, $\lambda,\alpha,\delta\ge 0$, $f\in\mathcal{A}_n$ *and suppose that* $\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z)\right)'$ *is univalent and* $\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}\in\mathcal{H}[1,n\delta]\cap Q$. *If*

$$h(z)\prec\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z)\right)',\qquad z\in U,\tag{3}$$

then $q(z)\prec\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z\in U$, *where* q *is given by* $q(z)=2\beta-1+\frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}}\int_0^z\frac{t^{\frac{\delta}{n}-1}}{1+t}dt$, $z\in U$. *The function* q *is convex and it is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.1 and considering $p(z)=\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, the differential superordination (3) becomes $h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+\frac{z}{\delta}p'(z)$, $z\in U$.

By using Lemma 1.9 for $\gamma=\delta$, we have $q(z)\prec p(z)$, i.e.,
$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z t^{\frac{\delta}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt=$
$\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z\left[(2\beta-1)\,t^{\frac{\delta}{n}-1}+2\,(1-\beta)\,\frac{t^{\frac{\delta}{n}-1}}{1+t}\right]dt=(2\beta-1)+\frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}}\int_0^z\frac{t^{\frac{\delta}{n}-1}}{1+t}dt$
$\prec\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z\in U$.

The function q is convex and it is the best subordinant.

Theorem 2.3. *Let q be convex in U and let h be defined by $h(z) = q(z) + \frac{z}{\delta}q'(z)$. If $n, m \in \mathbb{N}$, $\lambda, \alpha, \delta \geq 0$, $f \in \mathcal{A}_n$, suppose that $\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z)\right)'$ is univalent and $\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1, n\delta] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + \frac{z}{\delta}q'(z) \prec \left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z)\right)', \quad z \in U, \tag{4}$$

then

$$q(z) \prec \left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}, \; z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. The function q is the best subordinant.

Proof Following the same steps as in the proof of Theorem 2.1 and considering $p(z) = \left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, the differential superordination (4) becomes $q(z) + \frac{z}{\delta}q'(z) \prec p(z) + \frac{z}{\delta}p'(z)$, $z \in U$.

Using Lemma 1.10 for $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt \prec \left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z \in U$, and q is the best subordinant.

Remark 2.4. *For $n = 1$, $\lambda = \frac{1}{2}$, $\alpha = 2$, $\delta = 1$ we obtain the same example as in [8, Example 4.3.2, p. 136].*

Theorem 2.5. *Let h be a convex function, $h(0) = 1$. Let $\lambda, \alpha, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)}\right]$ is univalent and $z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} \in \mathcal{H}[0, n] \cap Q$. If*

$$h(z) \prec z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} +$$

$$\frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)}\right], \quad z \in U, \tag{5}$$

then

$$q(z) \prec z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}, \; z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt$. The function q is convex and it is the best subordinant.

Proof Consider $p(z) = z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}$ and we obtain $p(z) + \frac{z}{\delta}p'(z) = z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}\cdot$
$\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)}\right]$. Relation (5) becomes $h(z) \prec p(z) + \frac{z}{\delta}p'(z)$, $z \in U$.

By using Lemma 1.10 for $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt \prec z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}$, $z \in U$. The function q is convex and it is the best subordinant.

Theorem 2.6. *Let q be convex in U and let h be defined by $h(z) = q(z) + \frac{z}{\delta}q'(z)$. If $n, m \in \mathbb{N}$, $\lambda, \alpha, \delta \geq 0$, $f \in \mathcal{A}_n$, suppose that $z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)}\right]$ is univalent and $z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} \in \mathcal{H}[0,n] \cap Q$ and satisfies the differential superordination*

$$h(z) \prec z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} +$$

$$\frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)}\right], \qquad z \in U, \tag{6}$$

then

$$q(z) \prec z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}, \ z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt$. The function q is the best subordinant.

Proof Let $p(z) = z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}$, $z \in U$. Differentiating, we obtain $p(z) + \frac{z}{\delta}p'(z) = z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)}\right]$, $z \in U$, and (6) becomes $h(z) = q(z) + \frac{z}{\delta}q'(z) \prec p(z) + \frac{z}{\delta}p'(z)$, $z \in U$.

By using Lemma 1.10 for $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt \prec z\frac{RD^m_{\lambda,\alpha}f(z)}{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)^2}$, $z \in U$, and q is the best subordinant.

Theorem 2.7. *Let h be a convex function in U with $h(0) = 1$ and let $\lambda, \alpha, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$,*
$z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)''}{RD^m_{\lambda,\alpha}f(z)} - \left(\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)^2\right]$ *is univalent and*

noindent $z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} \in \mathcal{H}[0,n] \cap Q$. *If*

$$h(z) \prec z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}+$$

$$\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)''}{RD^m_{\lambda,\alpha}f(z)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)^2\right], \quad z \in U, \tag{7}$$

then

$$q(z) \prec z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}, \ z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. *The function* q *is convex and it is the best subordinant.*

Proof Let $p(z) = z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}$, $z \in U$.

Differentiating, we obtain $z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)''}{RD^m_{\lambda,\alpha}f(z)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)^2\right]$ $= p(z) + \frac{z}{\delta}p'(z)$, $z \in U$. Using the notation in (7), the differential superordination becomes $h(z) \prec p(z) + \frac{z}{\delta}p'(z)$, $z \in U$.

By using Lemma 1.9 for $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt \prec z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}$, $z \in U$. The function q is convex and it is the best subordinant.

Theorem 2.8. *Let* q *be a convex function in* U *and* $h(z) = q(z) + \frac{z}{\delta}q'(z)$. *Let* $\lambda, \alpha, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)''}{RD^m_{\lambda,\alpha}f(z)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)^2\right]$ *is univalent in* U *and* $z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)} \in \mathcal{H}[0,n] \cap Q$ *and satisfies the differential superordination*

$$h(z) \prec z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}+$$

$$\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)''}{RD^m_{\lambda,\alpha}f(z)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)^2\right], \quad z \in U, \tag{8}$$

then

$$q(z) \prec z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}, z \in U,$$

where $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. *The function* q *is the best subordinant.*

Proof Let $p(z)=z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}$. Differentiating, we obtain $p\left(z\right)+\frac{z}{\delta}p'\left(z\right)=z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)''}{RD^m_{\lambda,\alpha}f(z)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)^2\right]$, $z\in U$. Using the notation in (8), the differential superordination becomes $h(z)=q(z)+\frac{z}{\delta}q'(z)\prec p(z)+\frac{z}{\delta}p'(z)$.

By using Lemma 1.10 for $\gamma=\delta$ we have $q(z)\prec p(z)$, i.e., $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt\prec z^2\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}$, $z\in U$. The function q is the best subordinant.

Theorem 2.9. *Let* h *be a convex function,* $h(0)=1$. *Let* $n,m\in\mathbb{N}$, $\lambda,\alpha,\delta\geq 0$, $f\in\mathcal{A}_n$ *and suppose that* $1-\frac{RD^m_{\lambda,\alpha}f(z)\cdot\left(RD^m_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f(z)\right)'\right]^2}$ *is univalent and* $\frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}\in\mathcal{H}\left[1,n\right]\cap Q$. *If*

$$h(z)\prec 1-\frac{RD^m_{\lambda,\alpha}f\left(z\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)'\right]^2},\qquad z\in U,\tag{9}$$

then

$$q(z)\prec\frac{RD^m_{\lambda,\alpha}f\left(z\right)}{z\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)'},\ z\in U,$$

where q *is given by* $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, $z\in U$. *The function* q *is convex and it is the best subordinant.*

Proof Let $p(z)=\frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}$, $z\in U$. Differentiating, we obtain $1-\frac{RD^m_{\lambda,\alpha}f(z)\cdot\left(RD^m_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f(z)\right)'\right]^2}=p\left(z\right)+zp'\left(z\right)$, $z\in U$, and (9) becomes $h(z)\prec p(z)+zp'(z)$, $z\in U$.

Using Lemma 1.9 for $\gamma=1$ we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt\prec\frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}$, $z\in U$. The function q is convex and it is the best subordinant.

Corollary 2.10. *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0\leq\beta<1$. *Let* $n,m\in\mathbb{N}$, $\lambda,\alpha,\delta\geq 0$, $f\in\mathcal{A}_n$ *and suppose that* $1-\frac{RD^m_{\lambda,\alpha}f(z)\cdot\left(RD^m_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f(z)\right)'\right]^2}$ *is univalent and* $\frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}\in\mathcal{H}\left[1,n\right]\cap Q$. *If*

$$h(z)\prec 1-\frac{RD^m_{\lambda,\alpha}f\left(z\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)'\right]^2},\qquad z\in U,\tag{10}$$

then

$$q(z) \prec \frac{RD^m_{\lambda,\alpha}f\left(z\right)}{z\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)'},\ z \in U,$$

where q is given by $q(z) = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}$, $z \in U$. The function q is convex and it is the best subordinant .

Proof Following the same steps as in the proof of Theorem 2.9 and considering $p(z) = \frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}$, the differential subordination (10) becomes $h(z) = \frac{1+(2\beta-1)z}{1+z} \prec p(z) + zp'(z)$, $z \in U$.

By using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e.
$q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{1+(2\beta-1)t}{1+t}t^{\frac{1}{n}-1}dt =$
$\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\left[(2\beta-1) + \frac{2(1-\beta)}{1+t}\right]dt = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} \prec \frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}$, $z \in U$.

Theorem 2.11. *Let q be convex in U and let h be defined by $h\left(z\right) = q(z) + zq'(z)$. If $n, m \in \mathbb{N}$, $\lambda, \alpha, \delta \geq 0$, $f \in \mathcal{A}_n$, suppose that $1 - \frac{RD^m_{\lambda,\alpha}f(z)\cdot\left(RD^m_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f(z)\right)'\right]^2}$ is univalent and $\frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'} \in \mathcal{H}\left[1,n\right] \cap Q$ and satisfies the differential superordination*

$$h(z) \prec 1 - \frac{RD^m_{\lambda,\alpha}f\left(z\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)'\right]^2}, \quad z \in U, \tag{11}$$

then

$$q(z) \prec \frac{RD^m_{\lambda,\alpha}f\left(z\right)}{z\left(RD^m_{\lambda,\alpha}f\left(z\right)\right)'},\ z \in U,$$

where q is given by $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, $z \in U$. The function q is the best subordinant.

Proof Let $p(z) = \frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}$. Differentiating, we obtain
$1 - \frac{RD^m_{\lambda,\alpha}f(z)\cdot\left(RD^m_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^m_{\lambda,\alpha}f(z)\right)'\right]^2} = p\left(z\right) + zp'\left(z\right)$, $z \in U$, and (11) becomes $h(z) = q(z) + zq'(z) \prec p(z) + zp'(z)$, $z \in U$.

Using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt \prec \frac{RD^m_{\lambda,\alpha}f(z)}{z\left(RD^m_{\lambda,\alpha}f(z)\right)'}$, $z \in U$. The function q is the best subordinant.

Example 2.12. *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$. *Let* $f(z) = z+z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $\lambda = \frac{1}{2}$, $\alpha = 2$, *we obtain* $RD^1_{\frac{1}{2},2} f(z) = -R^1 f(z) + 2D^1_{\frac{1}{2}} f(z) = -zf'(z) + 2\left(\frac{1}{2}f(z) + \frac{1}{2}zf'(z)\right) = f(z) = z+z^2$, $z \in U$.

Then $\left(RD^1_{\frac{1}{2},2} f(z)\right)' = f'(z) = 1+2z$, $\frac{RD^1_{\frac{1}{2},2} f(z)}{z\left(RD^1_{\frac{1}{2},2} f(z)\right)'} = \frac{z+z^2}{z(1+2z)} = \frac{1+z}{1+2z}$, $1 - \frac{RD^1_{\frac{1}{2},2} f(z) \cdot \left(RD^1_{\frac{1}{2},2} f(z)\right)''}{\left[\left(RD^1_{\frac{1}{2},2} f(z)\right)'\right]^2} = 1 - \frac{\left(z+z^2\right)\cdot 2}{(1+2z)^2} = \frac{2z^2+2z+1}{(1+2z)^2}$. *We have* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.9 we obtain

$$\frac{1-z}{1+z} \prec \frac{2z^2+2z+1}{(1+2z)^2}, \; z \in U,$$

induce

$$-1 + \frac{2\ln(1+z)}{z} \prec \frac{1+z}{1+2z}, \; z \in U.$$

Theorem 2.13. *Let* h *be a convex function,* $h(0) = 1$ *and let* $\lambda, \alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $\left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)''$ *is univalent and* $\frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z} \in \mathcal{H}[0,n] \cap Q$. *If*

$$h(z) \prec \left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'', \; z \in U, \tag{12}$$

then

$$q(z) \prec \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}, \; z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. *The function* q *is convex and it is the best subordinant.*

Proof Let $p(z) = \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}$, $z \in U$. Differentiating, we obtain $\left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'' = p(z) + zp'(z)$, $z \in U$, and (12) becomes $h(z) \prec p(z) + zp'(z)$, $z \in U$.

Using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt \prec \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}$, $z \in U$. The function q is convex and it is the best subordinant.

Corollary 2.14. *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0 \leq \beta < 1$. *Let* $\lambda, \alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $\left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)''$ *is univalent and* $\frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z} \in \mathcal{H}[0,n] \cap Q$. *If*

$$h(z) \prec \left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'', \ z \in U, \tag{13}$$

then

$$q(z) \prec \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}, \ z \in U,$$

where q *is given by* $q(z) = (2\beta-1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}$, $z \in U$. *The function* q *is convex and it is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.13 and considering $p(z) = \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}$, the differential superordination (13) becomes $h(z) = \frac{1+(2\beta-1)z}{1+z} \prec p(z) + zp'(z)$, $z \in U$.

By using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e.,

$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1+(2\beta-1)t}{1+t} t^{\frac{1}{n}-1} dt =$
$\frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \left[(2\beta-1) + \frac{2(1-\beta)}{1+t}\right] dt = (2\beta-1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}$
$\prec \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}$, $z \in U$.

The function q is convex and it is the best subordinant.

Theorem 2.15. *Let* q *be a convex function in* U *and* h *be defined by* $h(z) = q(z) + zq'(z)$. *Let* $\lambda, \alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $\left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)''$ *is univalent and* $\frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z} \in \mathcal{H}[0,n] \cap Q$ *and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left[\left(RD^m_{\lambda,\alpha} f(z)\right)'\right]^2 + RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'', \ z \in U, \tag{14}$$

then

$$q(z) \prec \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}, \quad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. *The function* q *is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.13 and considering $p(z) = \frac{RD^m_{\lambda,\alpha} f(z) \cdot \left(RD^m_{\lambda,\alpha} f(z)\right)'}{z}$, the differential superordination (14) becomes $h(z) = q(z) + zq'(z) \prec p(z) + zp'(z)$, $z \in U$.

By using Lemma 1.10 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e., $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt \prec \frac{RD^m_{\lambda,\alpha}f(z)\cdot\left(RD^m_{\lambda,\alpha}f(z)\right)'}{z}$, $z \in U$. The

Example 2.16. *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$. *Let* $f(z) = z+z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $\lambda = \frac{1}{2}$, $\alpha = 2$, *we obtain* $RD^1_{\frac{1}{2},2}f(z) = -R^1f(z)+2D^1_{\frac{1}{2}}f(z) = -zf'(z)+2\left(\frac{1}{2}f(z)+\frac{1}{2}zf'(z)\right) = f(z) = z+z^2$, $z \in U$.

Then $\left(RD^1_{\frac{1}{2},2}f(z)\right)' = f'(z) = 1+2z$, $\frac{RD^1_{\frac{1}{2},2}f(z)\cdot\left(RD^1_{\frac{1}{2},2}f(z)\right)'}{z} = \frac{\left(z+z^2\right)(1+2z)}{z} = 2z^2+3z+1$, $\left[\left(RD^1_{\frac{1}{2},2}f(z)\right)'\right]^2 + RD^1_{\frac{1}{2},2}f(z)\cdot\left(RD^1_{\frac{1}{2},2}f(z)\right)'' = (1+2z)^2 + (z+z^2)\cdot 2 = 6z^2+6z+1$. *We have* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.13 we obtain

$$\frac{1-z}{1+z} \prec 6z^2+6z+1, \ z \in U,$$

induce

$$-1+\frac{2\ln(1+z)}{z} \prec 2z^2+3z+1, \ z \in U.$$

Theorem 2.17. *Let* h *be a convex function,* $h(0) = 1$. *Let* $\lambda, \alpha \geq 0$, $\delta \in (0,1)$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, *and suppose that* $\left(\frac{z}{RD^m_{\lambda,\alpha}f(z)}\right)^\delta \frac{RD^{m+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)} - \delta\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right)$ *is univalent and* $\frac{RD^{m+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha}f(z)}\right)^\delta \in \mathcal{H}[0, n(1-\delta)] \cap Q$. *If*

$$h(z) \prec \left(\frac{z}{RD^m_{\lambda,\alpha}f(z)}\right)^\delta \frac{RD^{m+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)} - \delta\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right), \tag{15}$$

$z \in U$, *then*

$$q(z) \prec \frac{RD^{m+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha}f(z)}\right)^\delta, \ z \in U,$$

where $q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$. *The function* q *is convex and it is the best subordinant.*

Proof Let $p(z) = \frac{RD^{m+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha}f(z)}\right)^\delta$, $z \in U$. Differentiating, we obtain $\left(\frac{z}{RD^m_{\lambda,\alpha}f(z)}\right)^\delta \frac{RD^{m+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z)\right)'}{RD^{m+1}_{\lambda,\alpha}f(z)} - \delta\frac{\left(RD^m_{\lambda,\alpha}f(z)\right)'}{RD^m_{\lambda,\alpha}f(z)}\right) = p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$, and (15) becomes $h(z) \prec p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$.

Using Lemma 1.9, we have $q(z) \prec p(z)$, $z \in U$, i.e.
$q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt \prec \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{z} \cdot \left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta}$, $z \in U$. The function q is convex and it is the best subordinant.

Theorem 2.18. *Let q be a convex function in U and $h(z) = q(z) + \frac{z}{1-\delta}q'(z)$. If $\lambda, \alpha \geq 0$, $\delta \in (0,1)$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that* $\left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta} \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{1-\delta} \left(\frac{\left(RD_{\lambda,\alpha}^{m+1}f(z)\right)'}{RD_{\lambda,\alpha}^{m+1}f(z)} - \delta\frac{\left(RD_{\lambda,\alpha}^{m}f(z)\right)'}{RD_{\lambda,\alpha}^{m}f(z)}\right)$ *is univalent and* $\frac{RD_{\lambda,\alpha}^{m+1}f(z)}{z} \cdot \left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta} \in \mathcal{H}[0, n(1-\delta)] \cap Q$ *satisfies the differential superordination*

$$h(z) \prec \left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta} \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{1-\delta} \left(\frac{\left(RD_{\lambda,\alpha}^{m+1}f(z)\right)'}{RD_{\lambda,\alpha}^{m+1}f(z)} - \delta\frac{\left(RD_{\lambda,\alpha}^{m}f(z)\right)'}{RD_{\lambda,\alpha}^{m}f(z)}\right), \tag{16}$$

$z \in U$, then

$$q(z) \prec \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{z} \cdot \left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta}, \ z \in U,$$

where $q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$. The function q is the best subordinant.

Proof Let $p(z) = \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{z} \cdot \left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta}$. Differentiating, we obtain $\left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta} \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{1-\delta} \left(\frac{\left(RD_{\lambda,\alpha}^{m+1}f(z)\right)'}{RD_{\lambda,\alpha}^{m+1}f(z)} - \delta\frac{\left(RD_{\lambda,\alpha}^{m}f(z)\right)'}{RD_{\lambda,\alpha}^{m}f(z)}\right) = p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$.

Using the notation in (16), the differential superordination becomes $h(z) = q(z) + \frac{z}{1-\delta}q'(z) \prec p(z) + \frac{1}{1-\delta}zp'(z)$. By using Lemma 1.10, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt \prec \frac{RD_{\lambda,\alpha}^{m+1}f(z)}{z} \cdot \left(\frac{z}{RD_{\lambda,\alpha}^{m}f(z)}\right)^{\delta}$, $z \in U$, and q is the best subordinant.

3 Open Problem

The definitions, theorems and corollaries we established in this paper can be extended by using the notion of strong superordination. One can use the operator from Definition 1.5 and the same technique to prove earlier results and to obtain a new set of results.

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