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On generalized Hadamard products of certain subclass of uniformly functions with negative coefficients

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Abstract

Using the generalized Hadamard products, we obtain some interesting characterization theorems for certain subclass of uniformly functions with negative coefficients.

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1 Introduction

Let $\mathcal{T}(n)$ denote the class of analytic function in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z - \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0; n \in \mathbb{N} \setminus \{1\} = \{2, 3, ...\}). \tag{1}$$

For $0 \leq \alpha < 1$ and $\beta \geq 0$, let $\mathcal{ST}(\alpha, \beta)$ and $\mathcal{CT}(\alpha, \beta)$ be the subclasses of $\mathcal{T}(n)$ of uniformly starlike functions of order α and uniformly convex functions of order α, respectively, (see Bharati et al. [3], Goodman [6, 7] and Kanas and Srivastava [8]).

For $0 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \lambda \leq 1$, a function $f \in \mathcal{T}(n)$ is said to be in the subclass $\mathcal{U}L(\alpha, \beta; \lambda)$ of $\mathcal{T}(n)$ if the following inequality holds:

$$\operatorname{Re}\left[\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - \alpha\right] \geq \beta \left|\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right|. \tag{2}$$

The subclass $\mathcal{U}L(\alpha, \beta; \lambda)$ introduced and studied by Aqlan et al. [1]. We note that

(i) $\mathcal{U}L(\alpha, \beta; 0) = \mathcal{ST}(\alpha, \beta)$ and $\mathcal{U}L(\alpha, \beta; 1) = \mathcal{CT}(\alpha, \beta)$ (see [3]);

(ii) $\mathcal{U}L(\alpha, 0; 0) = \mathcal{ST}(\alpha)$ and $\mathcal{U}L(\alpha, 0; 1) = \mathcal{CT}(\alpha)$ (see [9]).

For $p_i \geq 1$ and $\sum_{i=1}^{m} \frac{1}{p_i} = 1$, the Hölder inequality is defined by (see [2]):

$$\sum_{i=2}^{\infty}\left(\prod_{j=1}^{m} a_{i,j}\right) \leq \prod_{j=1}^{m}\left(\sum_{i=2}^{\infty} a_{i,j}^{p_i}\right)^{\frac{1}{p_i}}. \tag{3}$$

Let $f_j \in \mathcal{T}(n)\,(j = 1, 2)$ be given by

$$f_j(z) = z - \sum_{k=n}^{\infty} a_{k,j} z^k \quad (n \geq 2;\ j = 1, 2). \tag{4}$$

Then the modified Hadamard product or (convolution) $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \quad (n \geq 2). \tag{5}$$

For any real numbers p and q, the modified generalized Hadamard product $(f_1 \Delta f_2)(p, q; z)$ defined by (see Choi and Yong [4]):

$$(f_1 \Delta f_2)(p, q; z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k \quad (n \geq 2). \tag{6}$$

In the special case, if we take $p = q = 1$, then

$$(f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z) \quad (z \in \mathbb{U}). \tag{7}$$

In the present paper, we make use of the generalized Hadamard product to obtain some interesting characterization theorems involving the subclass $\mathcal{U}L(\alpha, \beta; \lambda)$.

2 Main results

Unless otherwise mentioned, we shell assume in the reminder of this paper that, the parameters $0 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \lambda \leq 1$, $n \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

In order to prove our results, we shall need the following lemma given by

Lemma 1[4]. Let the function f be defined by (1). Then $f \in \mathcal{U}L(\alpha, \beta; \lambda)$ if and only if

$$\sum_{k=n}^{\infty} (\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha + \beta)] a_k \leq 1 - \alpha \quad (n \geq 2). \tag{8}$$

Theorem 1. If the function f_j $(j = 1, 2)$ defined by (4) belongs to the subclass $\mathcal{U}L(\alpha_j, \beta; \lambda)$ $(j = 1, 2)$, then

$$(f_1 \Delta f_2)\left(\frac{1}{p}, \frac{1}{q}; z\right) \in \mathcal{U}L(\delta, \beta; \lambda), \tag{9}$$

where p, $q > 1$ and δ is given by

$$\delta = \min_{k \geq n} \left[1 - \frac{(1 + \beta)(k - 1)}{(\lambda k - \lambda + 1)^{\frac{1}{p} + \frac{1}{q} - 1} \left(\frac{(1+\beta)k - (\alpha_1 + \beta)}{1 - \alpha_1} \right)^{\frac{1}{p}} \left(\frac{(1+\beta)k - (\alpha_2 + \beta)}{1 - \alpha_2} \right)^{\frac{1}{q}} - 1} \right].$$

Proof. Let $f_j \in \mathcal{U}L(\alpha_j, \beta; \lambda)$. Then by using Lemma 1, we have

$$\sum_{k=n}^{\infty} \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)]}{1 - \alpha_j} a_{k,j} \leq 1 \quad (j = 1, 2;\ n \geq 2). \tag{10}$$

Moreover

$$\left\{ \sum_{k=n}^{\infty} \left[\frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_1 + \beta)]}{1 - \alpha_1} \right] (a_{k,1}) \right\}^{\frac{1}{p}} \leq 1, \tag{11}$$

and

$$\left\{\sum_{k=n}^{\infty}\left[\frac{(\lambda k-\lambda+1)\left[(1+\beta)k-(\alpha_2+\beta)\right]}{1-\alpha_2}\right](a_{k,2})\right\}^{\frac{1}{q}}\leq 1. \qquad (12)$$

Applying the Hölder inequality (3) to (11) and (12), we obtain

$$\sum_{k=n}^{\infty}\left[\frac{(\lambda k-\lambda+1)\left[(1+\beta)k-(\alpha_1+\beta)\right]}{1-\alpha_1}\right]^{\frac{1}{p}}\times$$
$$\left[\frac{(\lambda k-\lambda+1)\left[(1+\beta)k-(\alpha_2+\beta)\right]}{1-\alpha_2}\right]^{\frac{1}{q}}(a_{k,1})^{\frac{1}{p}}\ \ (a_{k,2})^{\frac{1}{q}}\leq 1. \qquad (13)$$

Since

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right)=z-\sum_{k=n}^{\infty}(a_{k,1})^{\frac{1}{p}}(a_{k,2})^{\frac{1}{q}}z^k \quad (n\geq 2), \qquad (14)$$

we see that

$$\sum_{k=n}^{\infty}\left[\frac{(\lambda k-\lambda+1)\left[(1+\beta)k-(\delta+\beta)\right]}{1-\delta}\right](a_{k,1})^{\frac{1}{p}}(a_{k,2})^{\frac{1}{q}}\leq 1 \quad (n\geq 2), \qquad (15)$$

with

$$\delta\leq\min_{k\geq n}\left[1-\frac{(1+\beta)(k-1)}{(\lambda k-\lambda+1)^{\frac{1}{p}+\frac{1}{q}-1}\left(\frac{(1+\beta)k-(\alpha_1+\beta)}{1-\alpha_1}\right)^{\frac{1}{p}}\left(\frac{(1+\beta)k-(\alpha_2+\beta)}{1-\alpha_2}\right)^{\frac{1}{q}}-1}\right].$$

Thus, by using Lemma 1, the proof of Theorem 1 is completed .

Putting $\alpha_j=\alpha\,(j=1,2)$ in Theorem 1, we obtain the following corollary.

Corollary 1. If the functions $f_j\,(j=1,2)$ defined by (4) are in the subclass $\mathcal{U}L\,(\alpha,\beta;\lambda)$. Then

$$(f_1\ \Delta f_2\ )\left(\frac{1}{p},\frac{1}{q};z\right)\in\mathcal{U}L\,(\alpha,\beta;\lambda) \quad (p,q>1)\,. \qquad (16)$$

Theorem 2. If the function $f_j\ \ (j=1,2,...,m)$ defined by (4) belongs to the subclass $\mathcal{U}L\,(\alpha_j,\beta;\lambda)$ $(j=1,2,...,m)$, and let $F_m\,(z)$ be defined by

$$F_m\,(z)=z-\sum_{k=n}^{\infty}\left(\sum_{j=1}^{m}(a_{k,j})^p\right)z^k. \qquad (17)$$

Then

$$F_m\,(z)\in\mathcal{U}L\,(\delta,\beta;\lambda)\,, \qquad (18)$$

where

$$\delta = 1 - \frac{m(1+\beta)(n-1)}{(\lambda n - \lambda + 1)^{p-1}\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p - m}, \;\; \alpha = \min_{1\le j\le m}\{\alpha_j\},$$

and

$$(\lambda n - \lambda + 1)^{p-1}\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^p \ge m\left[(1+\beta)n-\beta\right].$$

Proof. Since $f_j \in \mathcal{U}L(\alpha_j, \beta; \lambda)$ by using Lemma 1, we have

$$\sum_{k=n}^{\infty}\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\alpha_j+\beta)\right]}{1-\alpha_j}a_{k,j} \le 1 \quad (j = 1, 2, .., m; n \ge 2), \tag{19}$$

and

$$\begin{aligned}&\sum_{k=n}^{\infty}\left\{\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\alpha_j+\beta)\right]}{1-\alpha_j}\right\}^p (a_{k,j})^p\\ \le\;& \left\{\sum_{k=n}^{\infty}\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\alpha_j+\beta)\right]}{1-\alpha_j}a_{k,j}\right\}^p \le 1,\end{aligned} \tag{20}$$

it follow from (20), that

$$\sum_{k=n}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}\left\{\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\alpha_j+\beta)\right]}{1-\alpha_j}\right\}^p (a_{k,j})^p\right) \le 1. \tag{21}$$

Putting

$$\alpha = \min_{1\le j\le m}\{\alpha_j\}, \tag{22}$$

and by virtue of Lemma 1, we find that

$$\begin{aligned}&\sum_{k=n}^{\infty}\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\delta+\beta)\right]}{1-\delta}\sum_{j=1}^{m}(a_{k,j})^p\\ \le\;& \sum_{k=n}^{\infty}\left\{\frac{1}{m}\left[\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\alpha+\beta)\right]}{1-\alpha}\right]^p\sum_{j=1}^{m}(a_{k,j})^p\right\}\\ \le\;& \sum_{k=n}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left[\frac{(\lambda k - \lambda + 1)\left[(1+\beta)k-(\alpha_j+\beta)\right]}{1-\alpha_j}\right]^p (a_{k,j})^p\right\},\end{aligned} \tag{23}$$

if

$$\delta \le 1 - \frac{m(1+\beta)(k-1)}{(\lambda k - \lambda + 1)^{p-1}\left[\frac{(1+\beta)k-(\alpha+\beta)}{1-\alpha}\right]^p - m}. \tag{24}$$

Now let

$$g\left(k\right)=\ 1-\frac{m\left(1+\beta\right)\left(k-1\right)}{\left(\lambda k-\lambda+1\right)^{p-1}\left[\frac{(1+\beta)k-(\alpha+\beta)}{1-\alpha}\right]^{p}-m}. \tag{25}$$

Then $g^{'}\left(k\right)\geq 0$ if $p\geq 2$. Hence

$$\delta=\ 1-\frac{m\left(1+\beta\right)\left(n-1\right)}{\left(\lambda n-\lambda+1\right)^{p-1}\left[\frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha}\right]^{p}-m}. \tag{26}$$

By the following condition

$$\left(\lambda n-\lambda+1\right)^{p-1}\left[\frac{\left(1+\beta\right)n-\left(\alpha+\beta\right)}{1-\alpha}\right]^{p}\geq m\left[\left(1+\beta\right)n-\beta\right],$$

we see that $0\leq\delta<1$.Thus the proof of Theorem 2 is completed.

Remarks (i) Taking $\beta=\lambda=0$ in the above results we obtain the results of Choi and Kim [4, Theorem 1, Corollary 1 and Theorem 3, respectively];

(ii) Taking $\beta=0$, $\lambda=1$ in the above results we obtain the results of Choi and Kim [4, Theorem 2, Corollary 2 and Theorem 4, respectively];

(iii) Taking $\lambda=0$ in the above results we obtain the results for the class $\mathcal{ST}\left(\alpha,\beta\right)$;

(iv) Taking $\lambda=1$ in the above results we obtain the results for the class $\mathcal{CT}\left(\alpha,\beta\right)$.

3 Open Problem

The definitions, theorems and corollaries we established in this paper can be extended by using the notion of strong subordination. One can use the operator from definition **??** and the same technique to prove

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