On generalized Hadamard products
of certain subclass of uniformly functions
with negative coefficients

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Abstract

Using the generalized Hadamard products, we obtain some interesting characterization theorems for certain subclass of uniformly functions with negative coefficients.

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1 Introduction

Let $T(n)$ denote the class of analytic function in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z - \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0; \, n \in \mathbb{N} \setminus \{1\} = \{2, 3, \ldots\}). \quad (1)$$

For $0 \leq \alpha < 1$ and $\beta \geq 0$, let $ST(\alpha, \beta)$ and $CT(\alpha, \beta)$ be the subclasses of $T(n)$ of uniformly starlike functions of order $\alpha$ and uniformly convex functions of order $\alpha$, respectively, (see Bharati et al. [3], Goodman [6, 7] and Kanas and Srivastava [8]).

For $0 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \lambda \leq 1$, a function $f \in T(n)$ is said to be in the subclass $UL(\alpha, \beta; \lambda)$ of $T(n)$ if the following inequality holds:

$$\text{Re} \left[ \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} - \alpha \right] \geq \beta \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} - 1 \right|. \quad (2)$$

The subclass $UL(\alpha, \beta; \lambda)$ introduced and studied by Aqlan et al. [1]. We note that

(i) $UL(\alpha, \beta; 0) = ST(\alpha, \beta)$ and $UL(\alpha, \beta; 1) = CT(\alpha, \beta)$ (see [3]);

(ii) $UL(\alpha, 0; 0) = ST(\alpha)$ and $UL(\alpha, 0; 1) = CT(\alpha)$ (see [9]).

For $p_i \geq 1$ and $\sum_{i=1}^{m} \frac{1}{p_i} = 1$, the Hölder inequality is defined by (see [2]):

$$\sum_{i=2}^{\infty} \left( \prod_{j=1}^{m} a_{i,j} \right) \leq \prod_{j=1}^{m} \left( \sum_{i=2}^{\infty} a_{i,j}^{p_i} \right)^{\frac{1}{p_i}}. \quad (3)$$

Let $f_j \in T(n)$ ($j = 1, 2$) be given by

$$f_j(z) = z - \sum_{k=n}^{\infty} a_{k,j} z^k \quad (n \geq 2; \, j = 1, 2). \quad (4)$$

Then the modified Hadamard product or (convolution) $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \quad (n \geq 2). \quad (5)$$

For any real numbers $p$ and $q$, the modified generalized Hadamard product $(f_1 \Delta f_2)(p, q; z)$ defined by (see Choi and Yong [4]):

$$(f_1 \Delta f_2)(p, q; z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k \quad (n \geq 2). \quad (6)$$
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In the special case, if we take \( p = q = 1 \), then
\[
(f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z) \quad (z \in \mathbb{U}). \tag{7}
\]

In the present paper, we make use of the generalized Hadamard product to obtain some interesting characterization theorems involving the subclass \( \mathcal{UL}(\alpha, \beta; \lambda) \).

2 Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters \( 0 \leq \alpha < 1, \beta \geq 0, 0 \leq \lambda \leq 1, n \in \mathbb{N}_0 \) and \( z \in \mathbb{U} \).

In order to prove our results, we shall need the following lemma given by

Lemma 1\cite{4}. Let the function \( f \) be defined by (1). Then \( f \in \mathcal{UL}(\alpha, \beta; \lambda) \) if and only if
\[
\sum_{k=n}^{\infty} (\lambda k - \lambda + 1) \left[ (1 + \beta) k - (\alpha + \beta) \right] a_k \leq 1 - \alpha \quad (n \geq 2). \tag{8}
\]

**Theorem 1.** If the function \( f_j \ (j = 1, 2) \) defined by (4) belongs to the subclass \( \mathcal{UL}(\alpha_j, \beta; \lambda) \ (j = 1, 2) \), then
\[
(f_1 \Delta f_2) \left( \frac{1}{p}, \frac{1}{q}; z \right) \in \mathcal{UL}(\delta, \beta; \lambda), \tag{9}
\]
where \( p, q > 1 \) and \( \delta \) is given by
\[
\delta = \min_{k \geq n} \left[ 1 - \frac{(1 + \beta) (k - 1)}{\left( \lambda k - \lambda + 1 \right)^{1/p + 1/q - 1}} \frac{(1 + \beta) k - (\alpha_j + \beta)}{\left( \frac{(1 + \beta) k - (\alpha_1 + \beta)}{1 - \alpha_j} \right)^{1/p} \left( \frac{(1 + \beta) k - (\alpha_2 + \beta)}{1 - \alpha_2} \right)^{1/q} - 1} \right].
\]

**Proof.** Let \( f_j \in \mathcal{UL}(\alpha_j, \beta; \lambda) \). Then by using Lemma 1, we have
\[
\sum_{k=n}^{\infty} \frac{(\lambda k - \lambda + 1) \left[ (1 + \beta) k - (\alpha_j + \beta) \right]}{1 - \alpha_j} a_{k,j} \leq 1 \quad (j = 1, 2; \ n \geq 2). \tag{10}
\]

Moreover
\[
\left\{ \sum_{k=n}^{\infty} \left[ \frac{(\lambda k - \lambda + 1) \left[ (1 + \beta) k - (\alpha_1 + \beta) \right]}{1 - \alpha_1} \right] (a_{k,1}) \right\}^{\frac{1}{p}} \leq 1, \tag{11}
\]
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and

\[ \left\{ \sum_{k=n}^{\infty} \left[ \frac{(\lambda k - \lambda + 1) \left[(1 + \beta) k - (\alpha_2 + \beta)\right]}{1 - \alpha_2} \right] (a_{k,2})^{\frac{1}{q}} \right\}^{\frac{1}{p}} \leq 1. \quad (12) \]

Applying the Hölder inequality (3) to (11) and (12), we obtain

\[ \sum_{k=n}^{\infty} \left[ \frac{(\lambda k - \lambda + 1) \left[(1 + \beta) k - (\alpha_1 + \beta)\right]}{1 - \alpha_1} \right] \times \left[ \frac{(\lambda k - \lambda + 1) \left[(1 + \beta) k - (\alpha_2 + \beta)\right]}{1 - \alpha_2} \right]^{\frac{1}{q}} (a_{k,1})^{\frac{1}{p}} (a_{k,2})^{\frac{1}{q}} \leq 1. \quad (13) \]

Since

\[ (f_1 \Delta f_2) \left( \frac{1}{p}, \frac{1}{q}; z \right) = z - \sum_{k=n}^{\infty} (a_{k,1})^{\frac{1}{p}} (a_{k,2})^{\frac{1}{q}} z^k \quad (n \geq 2), \quad (14) \]

we see that

\[ \sum_{k=n}^{\infty} \left[ \frac{(\lambda k - \lambda + 1) \left[(1 + \beta) k - (\delta + \beta)\right]}{1 - \delta} \right] (a_{k,1})^{\frac{1}{p}} (a_{k,2})^{\frac{1}{q}} \leq 1 \quad (n \geq 2), \quad (15) \]

with

\[ \delta \leq \min_{k \geq n} \left[ 1 - \frac{(1 + \beta) (k - 1)}{(\lambda k - \lambda + 1)^{\frac{1}{p} + \frac{1}{q} - 1}} \left( \frac{(1 + \beta) k - (\alpha_1 + \beta)}{1 - \alpha_1} \right)^{\frac{1}{p}} \left( \frac{(1 + \beta) k - (\alpha_2 + \beta)}{1 - \alpha_2} \right)^{\frac{1}{q}} - 1 \right]. \]

Thus, by using Lemma 1, the proof of Theorem 1 is completed.

Putting \( \alpha_j = \alpha \) \((j = 1, 2)\) in Theorem 1, we obtain the following corollary.

**Corollary 1.** If the functions \( f_j \) \((j = 1, 2)\) defined by (4) are in the subclass \( \mathcal{UL}(\alpha; \beta; \lambda) \). Then

\[ (f_1 \Delta f_2) \left( \frac{1}{p}, \frac{1}{q}; z \right) \in \mathcal{UL}(\alpha; \beta; \lambda) \quad (p, q > 1). \quad (16) \]

**Theorem 2.** If the function \( f_j \) \((j = 1, 2, \ldots, m)\) defined by (4) belongs to the subclass \( \mathcal{UL}(\alpha_j; \beta; \lambda) \) \((j = 1, 2, \ldots, m)\), and let \( F_m(z) \) be defined by

\[ F_m(z) = z - \sum_{k=n}^{\infty} \left( \sum_{j=1}^{m} (a_{k,j})^{p} \right) z^k. \quad (17) \]

Then

\[ F_m(z) \in \mathcal{UL}(\delta; \beta; \lambda), \quad (18) \]
where
\[
\delta = 1 - \frac{m (1 + \beta) (n - 1)}{\lambda n - \lambda + 1} \left[ \frac{1 + n - (\alpha + \beta)}{1 - \alpha} \right]^{p-1} - m, \quad \alpha = \min_{1 \leq j \leq m} \{\alpha_j\},
\]
and
\[
(\lambda n - \lambda + 1)^{p-1} \left[ \frac{(1 + \beta) n - (\alpha + \beta)}{1 - \alpha} \right]^{p} \geq m [(1 + \beta) n - \beta].
\]

**Proof.** Since \( f_j \in \mathcal{UL}(\alpha_j, \beta; \lambda) \) by using Lemma 1, we have
\[
\sum_{k=n}^{\infty} \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)]}{1 - \alpha_j} a_{k,j} \leq 1 \quad (j = 1, 2, \ldots, m; n \geq 2),
\]
and
\[
\sum_{k=n}^{\infty} \left\{ \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)]}{1 - \alpha_j} \right\}^p \left( a_{k,j} \right)^p \leq \left\{ \sum_{k=n}^{\infty} \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)] a_{k,j}}{1 - \alpha_j} \right\}^p \leq 1,
\]
it follows from (20), that
\[
\sum_{k=n}^{\infty} \left( \frac{1}{m} \sum_{j=1}^{m} \left\{ \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)]}{1 - \alpha_j} \right\}^p \left( a_{k,j} \right)^p \right) \leq 1.
\]
Putting
\[
\alpha = \min_{1 \leq j \leq m} \{\alpha_j\},
\]
and by virtue of Lemma 1, we find that
\[
\sum_{k=n}^{\infty} \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\delta + \beta)]}{1 - \delta} \sum_{j=1}^{m} \left( a_{k,j} \right)^p \leq \sum_{k=n}^{\infty} \left\{ \frac{1}{m} \left[ \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)]}{1 - \alpha} \right]^p \sum_{j=1}^{m} \left( a_{k,j} \right)^p \right\}
\]
\[
\leq \sum_{k=n}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \left[ \frac{(\lambda k - \lambda + 1) [(1 + \beta) k - (\alpha_j + \beta)]}{1 - \alpha_j} \right]^p \left( a_{k,j} \right)^p \right\},
\]
if
\[
\delta \leq 1 - \frac{m (1 + \beta) (k - 1)}{\lambda k - \lambda + 1} \left[ \frac{1 + n - (\alpha + \beta)}{1 - \alpha} \right]^{p-1} - m.
\]
Now let

\[ g(k) = 1 - \frac{m (1 + \beta) (k - 1)}{(\lambda k - \lambda + 1)^{p-1} \left[ \frac{(1+\beta)k-(\alpha+\beta)}{1-\alpha} \right]^p - m} \tag{25} \]

Then \( g'(k) \geq 0 \) if \( p \geq 2 \). Hence

\[ \delta = 1 - \frac{m (1 + \beta) (n - 1)}{(\lambda n - \lambda + 1)^{p-1} \left[ \frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha} \right]^p - m}. \tag{26} \]

By the following condition

\[ (\lambda n - \lambda + 1)^{p-1} \left[ \frac{(1+\beta)n-(\alpha+\beta)}{1-\alpha} \right]^p \geq m [(1 + \beta) n - \beta] , \]

we see that \( 0 \leq \delta < 1 \). Thus the proof of Theorem 2 is completed.

**Remarks** (i) Taking \( \beta = \lambda = 0 \) in the above results we obtain the results of Choi and Kim [4, Theorem 1, Corollary 1 and Theorem 3, respectively];

(ii) Taking \( \beta = 0, \lambda = 1 \) in the above results we obtain the results of Choi and Kim [4, Theorem 2, Corollary 2 and Theorem 4, respectively];

(iii) Taking \( \lambda = 0 \) in the above results we obtain the results for the class \( \mathcal{ST}(\alpha, \beta) \);

(iv) Taking \( \lambda = 1 \) in the above results we obtain the results for the class \( \mathcal{CT}(\alpha, \beta) \).

## 3 Open Problem

The definitions, theorems and corollaries we established in this paper can be extended by using the notion of strong subordination. One can use the operator from definition ?? and the same technique to prove

**References**


