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Real Paley-Wiener theorems for the multivariable Bessel transform

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Abstract

Using the harmonic analysis assoicated with the Bessel operator $\mathcal{L}_\alpha$ on $\Omega_n = (0, +\infty)^n$. We establish real Paley-Wiener theorems and we give a necessary and sufficient condition on a function f in order to have a multivariable Bessel transform vanishing on neigberhood of the origin, on symmetric body and a on polynomial domain.

Keywords: *Bessel operator on $(0, +\infty)^n$, multivariable Bessel transform Real Paley-Wiener theorems.*

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1 Introduction

The classical Paley-Wiener Theorem characterizes the images $D(\mathbb{R})$ (the space of C^∞-functions on $\mathbb{R}$ with compact support) under the classical Fourier transform as rapidly decreasing functions having an holomorphic extension to $\mathbb{C}$ of exponential type. H.H.Bang [2] characterizes the function $f \in C^\infty(\mathbb{R})$ whose Fourier transform has compact support by a L^p grouwth condition for $1 \leq p \leq +\infty$. More precisely, he proves that for $1 \leq p \leq +\infty$ and $f \in C^\infty(\mathbb{R})$ such that its derivative $f^{(n)}$ of order n, belongs to the Lebesgue space $L^p(\mathbb{R})$, for all $n \in \mathbb{N}$, the limit $d_f = \lim_{n\to+\infty} \|f^{(n)}\|_{L^p(\mathbb{R})}^{1/p}$ always exists and $d_f = \sup\{|\lambda| \lambda \in \text{supp } \mathcal{F}(f)\}$.

This result, called real Paley-Wiener theorem, has been establish for many other integral transforms, see [3, 7, 8].

In this paper we consider the Bessel operator $\ell_{\alpha_i}, i = 1, ..., n$, on $\Omega_1 = (0, +\infty)$

$$\ell_{\alpha_i} = \frac{d^2}{dx_i^2} + \frac{2\alpha_i + 1}{x_i}\frac{d}{dx_i} \ ,$$

where $\alpha_i \in (-\frac{1}{2}, +\infty)$.

We denote by $\mathcal{L}_\alpha, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in (-1/2, +\infty)^n$, the operators $\mathcal{L}_\alpha = \ell_{\alpha_1} \otimes \ell_{\alpha_2} ... \otimes \ell_{\alpha_n}$ and $\Delta_\alpha = \sum_{i=1}^{n} \ell_{\alpha_i}$.

We studied a Multivariable Bessel transform $\mathcal{F}_B$ on Ω_n defined for a regular function f by

$$\forall \ \lambda \in \mathbb{R}^n \ , \mathcal{F}_B f(\lambda) = \int_{\Omega_n} f(x)\Lambda_\alpha(\lambda, x) d\mu_\alpha(x)$$

where $\Lambda_\alpha(\lambda, x)$ represents the Bessel kernel on $\mathbb{C}^n \times \mathbb{R}^n$ and $d\mu_\alpha$ the measure given by

$$d\mu_\alpha(x) = \prod_{i=1}^{n} \frac{|x_i|^{2\alpha_i+1}}{2^{\alpha_i}\Gamma(\alpha_i + 1)} dx_i.$$

The object of this paper is to prove real Paley-Wiener theorem on the Schwartz space $S_*(\mathbb{R}^n)$ (the space of C^∞ function on $\mathbb{R}^n$ even with respect to each variable) and on $L^2_\alpha(\Omega_n)$. Next we consider the Paley-Wiener spaces $PW^2_\alpha(\mathbb{R}^n)$ associated with the Bessel operators $\mathcal{L}_\alpha$ satisfying

$$f \in \mathcal{E}_*(\mathbb{R}^n) : \ \forall \ n \in \mathbb{N}, \ \Delta^n_\alpha f \in L^2_\alpha(\Omega_n)$$

and

$$R_f^{\Delta_\alpha} = \lim_{n\to+\infty} \|\Delta^n_\alpha f\|_{\alpha,2}^{1/2n} < +\infty\}$$

$$PW_\alpha(\mathbb{R}^n) = \{f \in \mathcal{E}_*(\mathbb{R}^n) \ : \forall \ n, m \in \mathbb{N}, (1+\|x\|)^m \Delta^n_\alpha f \in L^2_\alpha(\Omega_n) \text{ and } \ R_f^{\Delta_\alpha} < +\infty\}$$

where $\mathcal{E}_*(\mathbb{R}^n)$ is the space of C^∞-function on $\mathbb{R}^n$, even with respect to each variable, and $\|.\|_{\alpha,2}$ the norm of the space $L^2_\alpha(\Omega_n)$. We establish that $\mathcal{F}_B$ is a bijection from $PW^2_\alpha(\mathbb{R}^n)$ onto $L^2_{\alpha,c}(\Omega_n)$ (the space of functions in $L^2_\alpha(\Omega_n)$ with compact support), and from $PW_\alpha(\mathbb{R}^n)$ into $D_*(\mathbb{R}^n)$ (the space of C^∞ functions on $\mathbb{R}^n$, with compact support even with respect to each variable).

Next, we characterize the space $L^2_\alpha(K)$ where K is respectively a symmetric body (a polynomial domain) by their Multivariable Bessel transform with support in $\mathbb{R}^n$. These results are the real Paley-Wiener theorem for square integrable functions with respect to the measure $d\mu_\alpha(x)$. We shall prove that these real Paley-Wiener theorem can also be stated as follows :

- $\mathcal{F}_B(f)$ of $f \in S_*(\mathbb{R}^n)$ vanish outside a polynomial domain

$$\Gamma_P = \{x \in \mathbb{R}^n : |P(x_1^2, x_2^2, ..., x_n^2)| \leq 1\}.$$

with P a non constant polynomial, if and only if

$$\lim_{n \to +\infty} \sup \|P^n(-\mathcal{L}_\alpha)f\|_{\alpha,p}^{1/n} \leq 1$$

with $\|.\|_{\alpha,p}$ is the norm of the space $L^p_\alpha(\Omega_n)$ of p^{th} integrable functions on Ω_n with respect to the measure $d\mu_\alpha(x)$.

- A function f in $\mathcal{E}_*(\mathbb{R}^n)$ is the Multivariable Bessel transform of a square integrable function vanishing outside a symmetric body if and only if $\mathcal{L}^\beta_\alpha f$ belongs to $L^2_\alpha(\Omega_n)$ for all multi-indice $\beta = (\beta_1, ..., \beta_n)$ and

$$\sup_{a \in K^*} \|(a^2\mathcal{L}_\alpha)^k f\|_{\alpha,2} \leq M \ ,$$

where M is positive constant independent of k and $\mathcal{L}^k_\alpha = \ell^k_{\alpha_1} \otimes ... \otimes \ell^k_{\alpha_n}$, $k \in \mathbb{N}$.

- For all function f in $\mathcal{E}_*(\mathbb{R}^n)$ such that for all $k \in \mathbb{N}, \mathcal{L}^k_\alpha f$ belongs to $L^2_\alpha(\Omega_n)$ then

$$\lim_{k \to +\infty} \|\mathcal{L}^k_\alpha f\|_{\alpha,2}^{1/2k} = \sigma_f \ ,$$

where $\sigma_f = \sup\{|\lambda_1...\lambda_n|, \lambda \in \text{supp } \mathcal{F}_B(f)\}$.

This paper is arranged as follows

- In the first, second, third and fourth section we recall the main result on the harmonic analysis associated with the Multivariable Bessel transform $\mathcal{F}_B$.

- In the fifth section we give for $\mathcal{F}_B$ the analogue of the classical Paley-Wiener theorem using complex method and next we present the Paley-Wiener real theorems.

- In the sixth section we give a Multivariable Bessel transform of functions vanishing on a disc.

- The seventh section is devoted to study the function such that the support of their Multivariable Bessel transform are compact and to establish the real Paley-Wiener theorem for $\mathcal{F}_B$ on the Schawrtz space $S_*(\mathbb{R}^n)$.

In the last section we study the functions such that their Multivariable Bessel transform satisfies the symmetric body property, and we give a real Paley-Wiener type theorems which characterize these functions.

2 The operator $\mathcal{L}_\alpha$

We consider the operator $\mathcal{L}_\alpha$ on $\Omega_n = (0, +\infty)^n$ defined by :

$$\mathcal{L}_\alpha = \ell_{\alpha_1} \otimes ... \otimes \ell_{\alpha_n}.$$

Where $\alpha = (\alpha_1, ..., \alpha_n) \in (-1/2, +\infty)^n$ and ℓ_{α_i} it the Bessel operator on $\Omega_1 = (0, +\infty)$ given by

$$\ell_{\alpha_i} = \frac{d^2}{dx_i^2} + \frac{2\alpha_i + 1}{x_i} \frac{d}{dx_i}, \quad i = 1, ..., n.$$

For $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, we denote by $\mathcal{L}_\alpha^\beta$ the operator $\mathcal{L}_\alpha^\beta = \ell_{\alpha_1}^{\beta_1} \otimes ... \otimes \ell_{\alpha_n}^{\beta_n}$. Let j_{α_i} be the normalized Bessel function defined for $\lambda_i \in \mathbb{C}$ and $x_i \in \mathbb{R}$ by

$$j_{\alpha_i}(\lambda_i x_i) = \Gamma(\alpha_i + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n (\lambda_i x_i)^{2n}}{2^{2n} n! \Gamma(n + \alpha_i + 1)}.$$

The Bessel kernel $\Lambda_\alpha(\lambda, x)$ defined by

$$\Lambda_\alpha(\lambda, x) = \prod_{i=1}^n j_{\alpha_i}(\lambda_i x_i).$$

Where $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$, $x = (x_1, ..., x_n) \in \Omega_n$, is a solution of the equation

$$\begin{cases} \mathcal{L}_\alpha^\beta u(x) = (-1)^{|\beta|} \lambda^{2\beta} u(x) \\ u(0) = 1, \ \frac{\partial}{\partial x_i} u(x) = 0, \ \ i = 1, ..., n. \end{cases}$$

From the properties of the function j_α (see [6]), we deduce that the function Λ_α satisfies the following properties

i) For all $\lambda \in \mathbb{C}^n$, the function $(x_1, ..., x_n) \mapsto \Lambda_\alpha(\lambda, (x_1, ..., x_n))$ is of class $\mathcal{C}^\infty$ on $\mathbb{R}^n$ and even with respect to each variable.

ii) For all $x \in \mathbb{R}^n$, the function $(\lambda_1, ..., \lambda_n) \mapsto \Lambda_\alpha((\lambda_1, ..., \lambda_n), x)$ is entire on $\mathbb{C}^n$ and even with respect to each variable.

iii) For all $\lambda \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$, the function Λ_α admits the following integral representation

$$\Lambda_\alpha(\lambda, x) = \prod_{i=1}^n \frac{2\Gamma(\alpha_i + 1)}{\sqrt{\pi}\Gamma(\alpha_i + 1/2)} \int_0^1 (1 - t^2)^{\alpha_i - 1/2} \cos(\lambda_i x_i t) dt$$

iv) For all $\lambda \in \mathbb{C}^n, x \in \mathbb{R}^n$ we have

$$|\Lambda_\alpha(\lambda, x)| \leq e^{\|Im(\lambda)\| \|x\|}. \tag{2.1}$$

In particular for all $\lambda \in \mathbb{R}^n$ we have $|\Lambda_\alpha(\lambda, x)| \leq 1$.

v) For all $\nu \in \mathbb{N}^n, \ x \in \mathbb{R}^n, \ \lambda \in \mathbb{C}^n$ we have

$$|D_\lambda^\nu \Lambda(x, \lambda)| \leq \|x\|^{|\nu|} \exp(\|x\| \|Re\, \lambda\|) \tag{2.2}$$

3 Notations

We denote by

- $\mathcal{C}_*(\mathbb{R}^n)$ (resp $\mathcal{C}_{*,c}(\mathbb{R}^n)$) the space of continuous functions on $\mathbb{R}^n$ (resp with compact support), even with respect to each variable.
- $\mathcal{E}_*(\mathbb{R}^n)$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$, even with respect to each variable.
- $S_*(\mathbb{R}^n)$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$, rapidly decreasing together with their derivatives which are even with respect to each variable.
- $D_*(\mathbb{R}^n)$ the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$, with compact support and even with respect to each variable.
- $\mathcal{H}_*(\mathbb{C}^n)$ the space of functions on $\mathbb{C}^n$ even with respect to each variable, entire, slowly increasing, and of exponential type.

We consider also the following spaces.

- $\mathcal{E}'_*(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$ with compact support. It is the topological dual of $\mathcal{E}_*(\mathbb{R}^n)$.
- $S'_*(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. It is the topological dual of $S_*(\mathbb{R}^n)$.

4 Multivariable Bessel transform on Ω_n

.

In this section we define the Multivariable Bessel transform on Ω_n and we recall some basic results of this transform.

Notations We denote by

- $\mathcal{C}(\Omega_n)$ (resp $\mathcal{C}_c(\Omega_n)$) the space of continuous functions on Ω_n (resp with compact support).
- μ_α the measure defined on Ω_n by

$$d\mu_\alpha(x) = \prod_{i=1}^{n} \frac{x_i^{2\alpha_i+1}}{2^{\alpha_i}\Gamma(\alpha_i+1)} \quad dx_1....dx_n.$$

- $L^r_\alpha(\Omega_n), 1 \leq r \leq +\infty$, the space of measurable functions f on Ω_n, such that

$$\|f\|_{\alpha,r} = \left(\int_{\Omega_n} |f(x)|^r d\mu_\alpha(x)\right)^{1/r} < +\infty, \;\; 1 \leq r < +\infty.$$

$$\|f\|_{\alpha,\infty} = ess \sup_{x\in\Omega_n} |f(x)| < +\infty, \quad r = +\infty.$$

Definition 4.1. The Multivariable Bessel transform $\mathcal{F}_B$ is defined on $L^1_\alpha(\Omega_n)$ by

$$\forall\, \lambda \in \mathbb{R}^n, \;\; \mathcal{F}_B(f)(\lambda) = \int_{\Omega_n} f(x)\Lambda_\alpha(\lambda, x)d\mu_\alpha(x). \tag{4.1}$$

Proposition 4.1. i) For all f in $L^1_\alpha(\Omega_n)$, the function $\mathcal{F}_B(f)$ is continuous on $\mathbb{R}^n$, goes to zero at infinity and we have

$$\|\mathcal{F}_B(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1} \tag{4.2}$$

ii) For all f in $D_*(\mathbb{R}^n)$, we have

$$\forall\, \lambda \in \mathbb{R}^n,\;\; \mathcal{F}_B(\mathcal{L}^\beta_\alpha f)(x) = (-1)^{|\beta|}\lambda^{2\beta}\mathcal{F}_B(f)(\lambda) \tag{4.3}$$

$$\forall\, \lambda \in \mathbb{R}^n,\;\; \mathcal{L}^\beta_\alpha(\mathcal{F}_B(f))(\lambda) = (-1)^{|\beta|}\mathcal{F}_B(x^{2\beta}f)(\lambda). \tag{4.4}$$

Proposition 4.2. Let f be in $D_*(\mathbb{R}^n)$, then we have the inversion formula

$$\forall\, x \in \Omega_n,\; f(x) = \int_{\Omega_n} \mathcal{F}_B(f)(\lambda)\Lambda_\alpha(\lambda, x)d\mu_\alpha(\lambda). \tag{4.5}$$

Proposition 4.3. The Multivariable Bessel transform $\mathcal{F}_B$ is a topological isomorphism from $S_*(\mathbb{R}^n)$ onto itself.
Theorem 4.1. 1) (Plancherel formula). For all f, g in $S_*(\mathbb{R}^n)$. We have

$$\text{i)}\;\; \int_{\Omega_n} f(x)\overline{g(x)}d\mu_\alpha(x) = \int_{\Omega_n} \mathcal{F}_B(f)(\lambda)\overline{\mathcal{F}_B(g)(\lambda)}d\mu_\alpha(\lambda) \tag{4.6}$$

$$\text{ii)}\;\; \int_{\Omega_n} |f(x)|^2 d\mu_\alpha(x) = \int_{\Omega_n} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda) \tag{4.7}$$

2) The transform $\mathcal{F}_B$ can be extended to an isometric isomorphism of $L^2_\alpha(\Omega_n)$ onto itself.
Definition 4.2. i) The Multivariable Bessel transform of a distribution τ in $S'_*(\mathbb{R}^n)$ is defined by

$$\langle \mathcal{F}_B(\tau), \phi\rangle = \langle \tau, \mathcal{F}_B(\phi)\rangle, \phi \in S_*(\mathbb{R}^n).$$

ii) The Multivariable Bessel transform of a distribution τ in $\mathcal{E}'_*(\mathbb{R}^n)$ is the function given by

$$\forall\, y \in \mathbb{R}^n,\; \mathcal{F}_B(\tau)(y) = \langle \tau, \Lambda_\alpha(y, .)\rangle$$

Theorem 4.2. The Multivariable Bessel transform $\mathcal{F}_B$ is a topological isomorphism

i) From $S'_*(\mathbb{R}^n)$ onto itself

ii) From $\mathcal{E}'_*(\mathbb{R}^n)$ onto $\mathcal{H}_*(\mathbb{C}^n)$.

Remark. For all τ in $S_*(\mathbb{R}^n)$ we have

$$\mathcal{F}_B(\ell_{\alpha_i}\tau) = -\lambda_i^2 \mathcal{F}_B(\tau) \tag{4.8}$$

$$\mathcal{F}_B(\Delta_\alpha \tau) = -\|\lambda\|^2 \mathcal{F}_B(\tau). \tag{4.9}$$

For all f in $L^2_\alpha(\Omega_n)$, we define the distribution T_f in $S'_*(\mathbb{R}^n)$ by

$$\langle T_f, \varphi\rangle = \int_{\Omega_n} f(x)\varphi(x) d\mu_\alpha(x), \quad \varphi \in S_*(\mathbb{R}^n). \tag{4.10}$$

In the following T_f will be denoted by f.

Proposition 4.4. Let f be in $L^2_\alpha(\Omega_n)$. Then we have

$$\mathcal{F}_B(\Delta_\alpha f) = -\|\lambda\|^2 \mathcal{F}_B(f). \tag{4.11}$$

Proof. Let f be in $L^2_\alpha(\Omega_n)$, for all $\varphi \in S_*(\mathbb{R}^n)$ we have

$$\begin{aligned}
\langle \mathcal{F}_B(\Delta_\alpha f), \varphi\rangle &= \int_{\Omega_n} \mathcal{F}_B(\Delta_\alpha f)(\lambda)\varphi(\lambda) d\mu_\alpha(\lambda) \\
&= \int_{\Omega_n} \mathcal{F}_B(\sum_{i=1}^n \ell_{\alpha_i} f)(\lambda)\varphi(\lambda) d\mu_\alpha(\lambda) \\
&= \sum_{i=1}^n \int_{\Omega_n} \mathcal{F}_B(\ell_{\alpha_i} f)(\lambda)\varphi(\lambda) d\mu_\alpha(\lambda) \\
&= -\sum_{i=1}^n \int_{\Omega_n} \lambda_i^2 \mathcal{F}_B(f)(\lambda)\varphi(\lambda) d\mu_\alpha(\lambda) \\
&= \int_{\Omega_n} -\|\lambda\|^2 \mathcal{F}_B(f)(\lambda)\varphi(\lambda) d\mu_\alpha(\lambda) \\
&= -\langle \|x\|^2 \mathcal{F}_B(f), \varphi\rangle \\
\mathcal{F}_B(\Delta_\alpha f) &= -\|\lambda\|^2 \mathcal{F}_B(f).
\end{aligned}$$

Notations 4.1. We denote by

- $L^2_{\alpha,c}(\Omega_n)$ the space of functions in $L^2_\alpha(\Omega_n)$ with compact support.
- $\mathbb{H}_{L^2_\alpha}(\mathbb{C}^n)$ the space of entire functions on $\mathbb{C}^n$ of exponential type, and such that f/Ω_n belongs to $L^2_\alpha(\Omega_n)$.

Theorem 4.3. The Multivariable Bessel transform $\mathcal{F}_B$ is bijective from $L^2_{\alpha,c}(\Omega_n)$ onto $\mathbb{H}_{L^2_\alpha}(\mathbb{C}^n)$.

Proof. i) We consider the function f on $\mathbb{C}^n$ given by

$$\forall\, z \in \mathbb{C}^n \,, f(z) = \int_{\Omega_n} g(x)\Lambda_\alpha(x, z) d\mu_\alpha(x). \tag{4.12}$$

with $g \in L^2_{\alpha,c}(\Omega_n)$.
By derivation under the integral sign and by using the inequality (2.2) we deduce that the function f is entire on $\mathbb{C}^n$ and of exponential type. On the other hand the relation (4.12) can also be written in the form

$$\forall\, y \in \mathbb{R}^n, \quad f(y) = \mathcal{F}_B(g)(y).$$

Then from Theorem 3.1 the function $f|\Omega_n$ belongs to $L^2_\alpha(\Omega_n)$, thus f belongs to $\mathbb{H}_{L^2_\alpha}(\mathbb{C}^n)$.

ii) Reciprocally let ψ be in $\mathbb{H}_{L^2_\alpha}(\mathbb{C}^n)$.
From Theorem 4.2 ii) there exists $S \in \mathcal{E}'_*(\mathbb{R}^n)$ with support in the Ball $B(0,a)$ of center 0 and radius a, such that

$$\forall\, y \in \mathbb{R}^n,\ \psi(y) = \langle S_x, \Lambda_\alpha(x,y)\rangle. \tag{4.13}$$

On the other hand as $\psi_{/\Omega_n}$ belongs to $L^2_\alpha(\Omega_n)$ then from Theorem 4.1 there exists $h \in L^2_\alpha(\Omega_n)$ such that

$$\psi_{/\Omega_n} = \mathcal{F}_B(h). \tag{4.14}$$

From (4.13), for all $\varphi \in D_*(\mathbb{R}^n)$ we have

$$\int_{\Omega_n} \psi(y)\mathcal{F}_B(\varphi)(y)d\mu_\alpha(y) = \langle S_x, \int_{\Omega_n} \Lambda(x,y)\mathcal{F}_B\varphi(y)d\mu_\alpha(y)\rangle.$$

Thus using (4.5) we deduce that

$$\int_{\Omega_n} \psi(y)\mathcal{F}_B\varphi(y)d\mu_\alpha(y) = \langle S_x, \varphi\rangle. \tag{4.15}$$

On the other hand (4.14) implies

$$\int_{\Omega_n} \psi(y)\mathcal{F}_B\varphi(y)d\mu_\alpha(y) = \int_{\Omega_n} \mathcal{F}_B h(y)\mathcal{F}_B\varphi(y)d\mu_\alpha(y).$$

But from Theorem 4.1 we deduce that

$$\begin{aligned}\int_{\Omega_n} \mathcal{F}_B(h)(y)\mathcal{F}_B\varphi(y)d\mu_\alpha(y) &= \int_{\Omega_n} h(y)\varphi(y)d\mu_\alpha(y) \\ &= \langle T_h, \varphi\rangle\end{aligned} \tag{4.16}$$

Thus the relations (4.15), (4.16) imply the distribution S is given by the function h.
This relation shows that the support of h is compact, then $h \in L^2_{\alpha,c}(\Omega_n)$.

5 Generalized convolution product associated with the Bessel operator on Ω_n

.

Definition 5.1. The generalized translation operators $T_x, x \in \mathbb{R}^n$, associated with the Bessel operator on $\mathbb{R}^n$, are defined for f in $\mathcal{C}_*(\mathbb{R}^n)$ by

$$T_x f(y) = c_\alpha \int_{[0,\pi]^n} f(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, ..., \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n})$$

$$\times (\sin\theta_1)^{2\alpha_1} ... (\sin\theta_n)^{2\alpha_n} d\theta_1 ... d\theta_n. \tag{5.1}$$

Where $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n)$ and $c_\alpha = \displaystyle\prod_{i=1}^{n} \frac{\Gamma(\alpha_i + 1)}{\sqrt{\pi}\,\Gamma(\alpha_i + 1/2)}$.

Definition 5.2. The generalized convolution product associated with the Bessel operator in Ω_n of f and g in $\mathcal{C}_{*,c}(\mathbb{R}^n)$ is defined by

$$\forall\, x \in \mathbb{R}^n,\ f \ *_B\, g(x) = \int_{\Omega_n} T_x f(y) g(y) d\mu_\alpha(y). \tag{5.2}$$

Proposition 5.1. i) Let f be in $L^1_\alpha(\Omega_n)$. Then for all $x \in \Omega_n$, we have

$$\forall\, \lambda \in \mathbb{R}^n,\ \mathcal{F}_B(T_x f)(\lambda) = \Lambda_\alpha(\lambda, x) \mathcal{F}_B(f)(\lambda) \tag{5.3}$$

ii) Let $f \in L^1_\alpha(\Omega_n)$ and $g \in L^2_\alpha(\Omega_n)$ then $f \ *_B\, g$ is defined almost every where, belongs to $L^2_\alpha(\Omega_n)$ and we have

$$\mathcal{F}_B(f \ *_B\, g) = \mathcal{F}_B(f)\mathcal{F}_B(g). \tag{5.4}$$

Proposition 5.2. i) Let f be in $L^1_\alpha(\Omega_n)$ and g in $L^\infty_\alpha(\Omega_n)$. Then we have

$$\|f \ *_B\, g\|_{\alpha,\infty} \leq \|f\|_{\alpha,1} \|g\|_{\alpha,\infty} \tag{5.5}$$

ii) Let g be in $L^1_\alpha(\Omega_n)$ and f in $L^p_\alpha(\Omega_n) 1 \leq p \leq +\infty$, then

$$\|f \ *_B \ g\|_{\alpha,p} \leq \|f\|_{\alpha,p} \|g\|_{\alpha,1} \tag{5.6}$$

iii) Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If f is in $L^p_\alpha(\Omega_n)$, g in $L^q_\alpha(\Omega_n)$. Then $f \ *_B\, g \in L^r_\alpha(\Omega_n)$ and we have

$$\|f \ *_B\, g\|_{r,\alpha} \leq \|f\|_{\alpha,p} \|g\|_{\alpha,q}. \tag{5.7}$$

6 Paley-Wiener theorems for the Multivariable Bessel transform $\mathcal{F}_B$ on Ω_n

6.1 Paley-Wiener theorems for $\mathcal{F}_B$ using complex method

Notation. We denote by $\mathbb{H}_*(\mathbb{C}^n)$ the space of functions on $\mathbb{C}^n$ which are even with respect to each variable, entire, rapidly decreasing and of exponential type.

Theorem 6.1. The Multivariable Bessel transform $\mathcal{F}_B$ is a topological isomorphism from $D_*(\mathbb{R}^n)$ onto $\mathbb{H}_*(\mathbb{C}^n)$

6.2 Real Paley-Wiener theorem for Multivariable Bessel transform $\mathcal{F}_B$

To prove the main of this subsection result we need the following lemma.

Lemma 6.1. Let a probability measure m on a subsect E of $\mathbb{R}^n$, and φ a measurable function on E, such that f belongs to the Lebesgue space $L^{p_0}(E,m)$ for some $p_0 < +\infty$. Then

$$\lim_{p\to+\infty} \|\varphi\|_{L^p(E,m)} = \|\varphi\|_{L^\infty(E,m)}.$$

Let $\beta = (k, ..., k) \in \mathbb{N}^n$.

In the particular case will be denoted by $\mathcal{L}_\alpha^k$ inside of $\mathcal{L}_\alpha^\beta$.

Theorem 6.1. Let f be in $\mathcal{E}_*(\mathbb{R}^n)$ such that for all $k \in \mathbb{N}, \mathcal{L}_\alpha^k f$ belongs to $L_\alpha^2(\Omega_n)$. Then

$$\lim_{k\to+\infty} \|\mathcal{L}_\alpha^k f\|_{\alpha,2}^{1/2k} = \sigma_f \tag{6.1}$$

where

$$\sigma_f = \sup\{|\lambda_1...\lambda_n| : \lambda \in \text{ supp } \mathcal{F}_B(f)\}. \tag{6.2}$$

If the spectrum of f is bounded then $\sigma_f < \infty$, otherwise, $\sigma_f = +\infty$.

Proof of Theorem 6.1. We can assume that $\|f\|_{\alpha,2} > 0$, otherwise $\sigma_f = 0$ and then

$$\lim_{k\to+\infty} \|\mathcal{L}_\alpha^k f\|_{\alpha,2}^{1/2k} = 0.$$

Since $\mathcal{L}_\alpha^k f \in L_\alpha^2(\Omega_n)$ for any $k \in \mathbb{N}$, then their Multivariable Bessel transform exists and form (4.3) we have

$$\begin{aligned}\mathcal{F}_B(\mathcal{L}_\alpha^\beta f)(\lambda) &= (-1)^{nk}\lambda^{2(k,k,...k)}\mathcal{F}_B(f)(\lambda)\\ &= (-1)^{nk}(\lambda_1, ..., \lambda_n)^{2k}\mathcal{F}_B(f)(\lambda).\end{aligned}$$

Using theorem 4.1 we have

$$\begin{aligned}\|\mathcal{L}_\alpha^k f\|_{\alpha,2}^2 &= \int_{\Omega_n} |\lambda_1...\lambda_n|^{4k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda)\\ &= \int_{\text{supp }\mathcal{F}_B(f)} |\lambda_1...\lambda_n|^{4k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda).\end{aligned}$$

Consequently

$$\left\|\mathcal{L}_\alpha^k f\right\|_{\alpha,2}^{1/2k} = \|\mathcal{F}(f)\|_{\alpha,2}^{1/2k} \left[\int_{\text{supp}\mathcal{F}_B(f)} |\lambda_1...\lambda_n|^{4k} |\mathcal{F}_B(f)(\lambda)|^2 \frac{d\mu_\alpha(\lambda)}{\|\mathcal{F}(f)\|_{\alpha,2}^2}\right]^{\frac{1}{4k}}$$

we now apply lemma 6.1 for $E = \text{supp } \mathcal{F}_B(f)$ $\varphi(\lambda) = |\lambda_1...\lambda_n|$ and

$$dm(\lambda) = \|\mathcal{F}_B(f)\|_{\alpha,2}^{-2} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Then we obtain

$$\lim_{k\to+\infty} \|\mathcal{L}_\alpha^k f\|_{\alpha,2}^{1/2k} = \sup_{\lambda\in\text{supp}\mathcal{F}_B(f)} |\lambda_1...\lambda_n| = \sigma_f.$$

7 The Multivariable Bessel transform of functions vanishing on a disc

We consider the Gauss kernel associated with the Bessel operator $\mathcal{L}_k$ defined by

$$h_k(x) = e^{-\frac{\|x\|^2}{4k}} \tag{7.1}$$

The following proposition gives the radius of the maximum disc on which the Multivariable Bessel transform of function vanishes almost everywhere.
Theorem 7.1. Let f be in $L_\alpha^2(\Omega_n)$ and we consider the sequence

$$g_k(x) = f *_B h_k(x), \quad k \in \mathbb{N}^* \tag{7.2}$$

Then

$$\lim_{k\to+\infty} \sqrt{-\frac{1}{k} Log(\|g_k\|_{\alpha,2})} = \delta_f. \tag{7.3}$$

Where

$$\delta_f = \inf\{|\lambda_1...\lambda_n| : \quad \lambda \in \text{ supp } \mathcal{F}_B(f)\}. \tag{7.4}$$

Proof. First remark that from (5.7), the function g_k is well defined. We can assume that $\|f\|_{\alpha,2} > 0$, otherwise the relation (7.3) is clear.
To prove (7.3) it is sufficient to prove the equivalent identity

$$\lim_{k\to+\infty} \|g_k\|_{\alpha,2}^{1/k} = e^{-\delta_f^2}. \tag{7.5}$$

Since $f \in L^2_\alpha(\Omega_n)$ and $h_k \in L^1_\alpha(\Omega_n)$ we have from 4.4

$$\mathcal{F}_B(g_k)(x) = \mathcal{F}_B(f)(x)\mathcal{F}_B(h_k)(x).$$

We have also from Proposition 5.2 and (5.6) that $g_k \in L^2_\alpha(\Omega_n)$.
Then by applying the Parseval equality, we deduce that

$$\|g_k\|_{\alpha,2} = \|\mathcal{F}_B(g_k)\|_{\alpha,2} = \left(\int_{\mathrm{supp}\mathcal{F}_B(f)} |\mathcal{F}_B(f)(x)|^2 |\mathcal{F}_B(h_k)(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{2}}.$$

From the fact that $\mathcal{F}_B(h_k)(x) = k^{n/2}(4k)^\alpha(\prod_{i=1}^n \Gamma(\alpha_i+1))e^{-k\|x\|^2}$ and Lemma 6.1 applied for the set E given by

$$E = \text{ supp } \mathcal{F}_B(f), dm(\lambda) = \|\mathcal{F}_B(f)\|^{-2}_{\alpha,2}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda)$$

and the function $\varphi(\lambda) = e^{-\|\lambda\|^2}$, we obtain

$$\lim_{k\to+\infty} \|g_k\|^{1/k}_{\alpha,2} = \sup\{e^{-\|x\|^2} : x \in \text{ supp } \mathcal{F}_B(f)\}.$$

A function $f \in L^2_\alpha(\Omega_n)$ is the Multivariable Bessel transform of a function vanishing in a neighborhood of the origin if and only if $\delta_f > 0$, or equivalently if the limit (7.5) is loss then 1 hence it, holds the theorem.

Corollary 7.1. A necessary and sufficient condition for a function f in $L^2_\alpha(\Omega_n)$ to have its Multivariable Bessel transform vanishing in a neighborhood of the origin is

$$\lim_{k\to+\infty} \|g_k\|^{1/k}_{\alpha,2} < 1. \tag{7.6}$$

Remark 1. Since $\delta_f \leq \sigma_f$ it is clear that the following inequality is always true

$$\lim_{k\to+\infty} \|\mathcal{L}^k_\alpha f\|^{1/k}_{\alpha,2} \geq -\lim_{k\to+\infty} \frac{1}{k} Log\|g_k\|_{\alpha,2}. \tag{7.7}$$

Remark 2. From Proposition 6.1 and Theorem 7.1, it follows that the support of Multivariable Bessel transform of a function in $L^2_\alpha(\Omega_n)$ is in the tore $\delta_f \leq \|\lambda\| \leq \sigma_f$ if and only if

$$\delta_f \leq \lim_{k\to+\infty} \sqrt{-\frac{1}{k} Log\|g_k\|_{\alpha,2}} \leq \lim_{k\to+\infty} \|\Delta^k_\alpha f\|^{\frac{1}{2k}}_{\alpha,2} \leq \sigma_f.$$

8 Characterization of functions with compact spectrum

Definition 8.1. i) We define the support of $f \in L^2_\alpha(\Omega_n)$ and we denote it by supp$f$, the smallest closed set, outside the function f vanishes almost everywhere

ii) We denote by $R_f = \sup_{\lambda \in \text{ supp } f} \|\lambda\|$ the radius of the support of f.

Remark It is clear that R_f is finite if and only if f has a compact support.

Proposition 8.1. Let $f \in L^2_\alpha(\Omega_n)$ such that for all $k \in \mathbb{N}$ the function $\|\lambda\|^{2k} f(\lambda)$ belongs to $L^2_\alpha(\Omega_n)$. Then

$$R_f = \lim_{k \to +\infty} \left(\int_{\Omega_n} \|\lambda\|^{4k} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right)^{\frac{1}{4k}}. \tag{8.1}$$

Proof. We suppose that $\|f\|_{\alpha,2} \neq 0$ otherwise $R_f = 0$ and formula (8.1) is trivial. Assume now that f has compact support with $R_f > 0$.
Then

$$\left[\int_{\Omega_n} \|\lambda\|^{4k} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right]^{\frac{1}{4k}} \leq \left[\int_{\|\lambda\| \leq R_f} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right]^{\frac{1}{4k}} R_f.$$

Thus we deduce that

$$\begin{aligned} &\lim_{k \to +\infty} \sup \left[\int_{\Omega_n} \|\lambda\|^{4k} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right]^{\frac{1}{4k}} \\ &\qquad \leq \lim_{k \to +\infty} \sup \left[\int_{\|\lambda\| \leq R_f} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right]^{\frac{1}{4k}} R_f = R_f. \end{aligned}$$

On the other hand, for any positive ε we have

$$\int_{R_f - \varepsilon \leq \|\lambda\| \leq R_f} |f(\lambda)|^2 d\mu_\alpha(\lambda) > 0.$$

$$\begin{aligned} &\lim_{k \to +\infty} \inf \left\{ \int_{\Omega_n} \|\lambda\|^{4k} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{4k}} \\ &\qquad \geq \lim_{k \to +\infty} \inf \left\{ \int_{R_f - \varepsilon \leq \|\lambda\| \leq R_f} \|\lambda\|^{4k} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{4k}} \\ &\qquad \geq R_f - \varepsilon. \end{aligned}$$

Thus

$$R_f = \lim_{k \to +\infty} \left[\int_{\Omega_n} \|\lambda\|^{4k} |f(\lambda)|^2 d\mu_\alpha(\lambda) \right]^{\frac{1}{4k}}.$$

We prove now the assertion in the case where f has unbounded support. Indeed for any positive N, we have

$$\int_{\|\lambda\| \geq N} |f(\lambda)|^2 d\mu_\alpha(\lambda) > 0.$$

Thus

$$\lim_{k\to+\infty}\inf\left\{\int_{\Omega_n}\|\lambda\|^{4k}|f(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{4k}}$$

$$\geq \lim_{k\to+\infty}\inf\left\{\int_{\|\lambda\|\geq N}\|\lambda\|^{4k}\|f(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{4k}} \geq N.$$

Thus implies that

$$\lim_{k\to+\infty}\inf\left\{\int_{\Omega_n}\|\lambda\|^{4k}|f(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{4k}} = +\infty.$$

Notations We denote by
- $L^2_{\alpha,R}(\Omega_n) = \big\{f \in L^2_{\alpha,c}(\Omega_n) : R_f = R\big\}$, for $R \geq 0$.
- $D_R(\mathbb{R}^n) = \{f \in D_*(\mathbb{R}^n) : R_f = R\}$, for $R \geq 0$.

Definition 8.2. We define the Paley-Wiener spaces $PW^2_\alpha(\mathbb{R}^n)$ and $PW^2_{\alpha,R}(\mathbb{R}^n)$ as follows

i) $PW^2_\alpha(\mathbb{R}^n) = \{f \in \mathcal{E}_*(\mathbb{R}^n) : \Delta^m_\alpha f \in L^2_\alpha(\Omega_n)$ for all $m \in \mathbb{N}$ and $\lim\limits_{m\to+\infty}\|\Delta^m_\alpha f\|^{\frac{1}{2m}}_{\alpha,2} = R^{\Delta_\alpha}_f < +\infty\}$.

ii) $PW^2_{\alpha,R}(\mathbb{R}^n) = \{f \in PW^2_\alpha(\mathbb{R}^n) : R^{\Delta_\alpha}_f = R\}$.

We formulate now the real L^2-Paley-Wiener Theorem for the Multivariable Bessel transform .

Theorem 8.1. The Multivariable Bessel transform $\mathcal{F}_B$ is a bijection

i) From $PW^2_{\alpha,R}(\mathbb{R}^n)$ onto $L^2_{\alpha,R}(\Omega_n)$.

ii) From $PW^2_\alpha(\mathbb{R}^n)$ onto $L^2_{\alpha,c}(\Omega_n)$.

Proof. i) Let $f \in PW^2_{\alpha,R}(\mathbb{R}^n)$. Then from Proposition 4.3, the function $\mathcal{F}_B(\Delta^k_\alpha f)(\lambda) = (-1)^k\|\lambda\|^{2k}\mathcal{F}_B(f)(\lambda)$ belongs to $L^2_\alpha(\Omega_n)$ for all $k \in \mathbb{N}$. On the other hand from Theorem 4.1. we deduce that

$$\lim_{k\to+\infty}\left\{\int_{\Omega_n}\|\lambda\|^{4k}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{4k}} = \lim_{k\to+\infty}\left\{\int_{\Omega_n}|\Delta^k_\alpha f(x)|^2 d\mu_\alpha(x)\right\}^{\frac{1}{4k}} = R.$$

Thus using Proposition 8.1 we conclude that $\mathcal{F}_B(f)$ has compact support with $R_{\mathcal{F}_B(f)} = R$.

Conversely let g be in $L^2_{\alpha,R}(\Omega_n)$. Then $\|\lambda\|^k g \in L^1_\alpha(\Omega_n)$ for any $k \in \mathbb{N}$ and $\mathcal{F}_B^{-1}(g) \in \mathcal{E}_*(\mathbb{R}^n)$. On the other hand from Theorem 3.1 we have

$$\lim_{k\to+\infty}\left[\int_{\Omega_n}\Delta^k_\alpha|\mathcal{F}_B^{-1}(g)(x)|^2 d\mu_\alpha(x)\right]^{\frac{1}{4k}} = \lim_{k\to+\infty}\left[\int_{\Omega_n}\|\lambda\|^{4k}|g(\lambda)|^2 d\mu_\alpha(\lambda)\right]^{\frac{1}{4k}} = R.$$

Thus
$$\mathcal{F}_B^{-1}(g) \in PW^2_{\alpha,R}(\mathbb{R}^n).$$

ii) We deduce ii) from i).

Corollary 8.1. The Multivariable Bessel transform $\mathcal{F}_B$ is a bijection from $PW^2_\alpha(\mathbb{R}^n)$ onto $\mathbb{H}_{L^2_\alpha}(\mathbb{C}^n)$.

Proof. We deduce the result from Theorem 8.1. ii) and Theorem 4.4 ii) and Theorem 4.3.

Definition 8.4 i) The Paley-Wiener space $PW_\alpha(\mathbb{R}^n)$ is the space of function $f \in \mathcal{E}_*(\mathbb{R}^n)$ satisfying.

a) $(1+\|x\|)^m \Delta^k_\alpha f \in L^2_\alpha(\Omega_n)$ for all $k, m \in \mathbb{N}$.

b) $R_f^{\Delta_\alpha} = \lim\limits_{k\to+\infty} \|\Delta^k_\alpha f\|_{\alpha,2}^{\frac{1}{2k}} < \infty$.

ii) Let $R \geq 0$. We define the space $PW_{\alpha,R}(\mathbb{R}^n)$ by
$$PW_{\alpha,R}(\mathbb{R}^n) = \left\{ f \in PW_\alpha(\mathbb{R}^n) : R_f^{\Delta_\alpha} = R \right\}.$$

Theorem 8.2. The Multivariable Bessel transform $\mathcal{F}_B$ is a bijection

i) from $PW_{\alpha,R}(\mathbb{R}^n)$ onto $D_R(\mathbb{R}^n)$.

ii) from $PW_\alpha(\mathbb{R}^n)$ onto $D_*(\mathbb{R}^n)$.

Proof. i) Let $g \in PW_{\alpha,R}(\mathbb{R}^n) \subset PW^2_{\alpha,R}(\mathbb{R}^n)$. Then $\mathcal{F}_B(g) \in \mathcal{E}_*(\mathbb{R}^n)$ since g has polynomial decay and by Theorem 8.1, then function $\mathcal{F}_B(g)$ has compact support with $R_{\mathcal{F}_B(g)} = R$.
Conversely let f be in $D_R(\mathbb{R}^n)$, then $\mathcal{F}_B^{-1}(f) \in S_*(\mathbb{R}^n)$, and $\mathcal{F}_B^{-1}(f) \in PW^2_{\alpha,R}(\mathbb{R}^n)$ by Theorem 8.1. So it only remain to show that $\mathcal{F}_B^{-1}(f)$ satisfy, the polynomial decay condition for any $f \in D_R(\mathbb{R}^n)$. We have from (4.11) and binomial formula
$$(1+\|x\|^2)^n \mathcal{F}_B^{-1}(f)(x) = \int_{\Omega_n} (I-\Delta_\alpha)^n f(\lambda)\Lambda(\lambda,x)d\mu_\alpha(\lambda).$$
Thus we obtain the result.

ii) We deduce the result from i).

9 Characterization of functions with symmetric body spectrum

According to [1], a convex compact and symmetric set on $\mathbb{R}^n$ with non empty interior is called a symmetric body (symmetric means $-x \in K$ if $x \in K$). Let

K be a symmetric body in $\mathbb{R}^n$. The set $K^* = \{y \in \mathbb{R}^n : |x.y| \leq 1$ for all $x \in K\}$ is called the polar set of K. Then K^* is also a symmetric body and $(K^*)^* = K$.
We state in the following a new real Paley-Wiener type theorem for functions with symmetric body-spectrum.

Theorem 9.1. A function f in $\mathcal{E}_*(\mathbb{R}^n)$ is the Multivariable Bessel transform of a square integrable function vanishing outside a symmetric body K if and only if, $\mathcal{L}_\alpha^\beta f$ belongs to $L^2_\alpha(\Omega_n)$ for all multi-indices $\beta = (k, ..., k)$ and

$$\sup_{a \in K^*} \|(a^2 \mathcal{L}_\alpha)^k f\|_{\alpha,2} \leq M; k = 1, 2, ... \tag{9.1}$$

M is a positive constant independent of k, and $\mathcal{L}_\alpha^k = \ell_{\alpha_n}^k \otimes ... \otimes \ell_{\alpha_n}^k$.
Proof. Let the Multivariable Bessel transform of $f \in L^2_\alpha(\Omega_n)$ vanish outside the symmetric body K. Then $\mathcal{L}_\alpha^\beta f$ exists for all $\beta \in \mathbb{N}^n$ and $\mathcal{L}_\alpha^\beta f \in L^2_\alpha(\Omega_n)$. We can assume that $f \neq 0$ otherwise it is trivial. From the relation (3.3) and the Parseval equality we obtain

$$\|(a^2 \mathcal{L}_\alpha)^k f\|_{\alpha,2} = \|a^{2k} \lambda^{2k} \mathcal{F}_B(f)(\lambda)\|_{\alpha,2} \ , \tag{9.2}$$

where $a^{2k} = a_1^{2k} ... a_n^{2k}$ and $\lambda^{2k} = \lambda_1^{2k} ... \lambda_n^{2k}$. Since K is a symmetric body

$$|a.\lambda| \leq 1, \quad \text{for all } \lambda \in K, \quad \text{and } a \in K^*.$$

Hence

$$\begin{aligned} \|(a^2\lambda^2)^k \mathcal{F}_B(f)(\lambda)\|^2_{\alpha,2} &= \int_{\Omega_n} |(a^2\lambda^2)^k|^2 |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda) \\ &= \int_K |(a^2\lambda^2)^k \mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda) \\ &\leq \int_K |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda) = \|\mathcal{F}_B(f)(\lambda)\|^2_{\alpha,2} \\ &= \|f\|^2_{\alpha,2}. \end{aligned} \tag{9.3}$$

That means

$$\sup_{a \in K^*} \|(a^2 \mathcal{L}_\alpha)^k f\|_{\alpha,2} \leq \|f\|_{\alpha,2} = M. \tag{9.4}$$

Conversely suppose now that the inequality (9.1) is valid for all $k \in \mathbb{N}$.
Since $\mathcal{L}_\alpha^\beta f \in L^2_\alpha(\Omega_n)$ for all multi-indices β its Multivariable Bessel transform exists and we have from the relation (4.3)

$$\forall\, \lambda \in \mathbb{R}^n \ , \mathcal{F}_B((a^2\mathcal{L}_\alpha)^k f)(\lambda) = (-1)^{nk} a^{2k} \lambda^{2k} \mathcal{F}_B(f)(\lambda).$$

Then from Parseval equality and inequality (9.1)

$$\sup_{a \in K^*} \|(a^2\lambda^2)^k \mathcal{F}_B(f)\|_{\alpha,2} = \sup_{\alpha \in K^*} \|(a^2 \mathcal{L}_\alpha)^k f\|_{\alpha,2} \leq M. \tag{9.5}$$

Let $\lambda_0 \notin K$, then $\lambda_0 \notin (K^*)^*$, which means that there exist $a_0 \in K^*$ such that $|a_0\lambda_0| > 1$.
Then there is a neighborhood U_{λ_0} of λ_0 with the property $|\lambda a_0| > \frac{1+|\lambda_0 a_0|}{2} > 1$ for all $\lambda \in U_{\lambda_0}$ we have

$$\begin{aligned} M^2 &\geq \sup_{a\in K^*} \|(a^2\lambda^2)^k \mathcal{F}_B(f)(\lambda)\|^2_{\alpha,2} \\ &\geq \|(a_0^2\lambda^2)^k \mathcal{F}_B(f)(\lambda)\|^2_{\alpha,2} \geq \int_{U_{\lambda_0}} |(a_0^2\lambda^2)^k \mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda) \\ &\geq (\frac{1+|\lambda_0 a_0|}{2})^{2k} \int_{U_{\lambda_0}} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda)) \end{aligned} \tag{9.6}$$

Since $(\frac{1+|\lambda_0 a_0|}{2})^{2k} \underset{\text{as } k\to+\infty}{\longrightarrow} +\infty$, the relation (9.6) holds only if

$$\int_{U_{\lambda_0}} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_\alpha(\lambda) = 0.$$

That means λ_0 does not belongs to the support of $\mathcal{F}_B(f)$.
Hence supp$\mathcal{F}_B(f) \subseteq K$ and Theorem 9.1 is proved.

10 Multivariable Bessel transform of function with polynomial domain support

Let $P(x)$ be a non constant polynomial and

$$\Gamma_P = \{x \in \mathbb{R}^n : |P(x_1^2, x_2^2, ...x_n^2)| \leq 1\}.$$

The set Γ_P is called polynomial domain in $\mathbb{R}^n$.
- A disc is a polynomial domain.
- A polynomial domain (for example $U = \{x : |x_1^2....x_n^2| \leq 1\}$ may be unbounded and non convex).

Theorem 9.1. The Multivariable Bessel transform $\mathcal{F}_B(f)$ of $f \in S_*(\mathbb{R}^n)$ vanishes outside a polynomial domain Γ_P, if and only if

$$\overline{\lim_{k\to+\infty}} \|P^k(-\mathcal{L}_\alpha)f(x)\|^{\frac{1}{k}}_{\alpha,p} \leq 1, \quad 1 \leq p \leq +\infty \tag{10.1}$$

Proof. The theorem has to be proved only for $f \neq 0$. Let q be the conjugate exponent of p.
We see that

$$\mathcal{F}_B(P(-\mathcal{L}_\alpha)f(x)) = P(\lambda_1^2, \lambda_2^2..., \lambda_n^2)\mathcal{F}_B(f)(\lambda).$$

Indeed, let $P(x)=\sum_{|\beta|\leq N} C_\beta x^\beta$, $\beta\in\mathbb{N}^n$, $\quad \beta=(\beta_1,...,\beta_n)$.
We note by $-\mathcal{L}_\alpha=(-\ell_{\alpha_1})\otimes....\otimes(-\ell_{\alpha_n})$

$$P(-\mathcal{L}_\alpha)f(x)=\sum_{|\beta|\leq N} C_\beta(-\mathcal{L}_\alpha)^\beta f(x)$$

$$\begin{aligned}\mathcal{F}_B(P(-\mathcal{L}_\alpha)f)(\lambda) &= \sum_{|\beta|\leq N} C_\beta\mathcal{F}_B((-\mathcal{L}_\alpha)^\beta f)(\lambda)\\ &= \sum_{|\beta|\leq N} C_\beta\lambda^{2\beta}\mathcal{F}_B(f)(\lambda)\\ &= P(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda).\end{aligned}$$

Then

$$\mathcal{F}_B(P^k(-\mathcal{L}_\alpha)f)(\lambda)=P^k(\lambda_1^2...,\lambda_n^2)\mathcal{F}_B(f)(\lambda). \tag{10.2}$$

i) Let $1\leq p\leq 2$.

Suppose that (10.1) is valid.

Applying the Hausdorff-Young inequality

$$\|\mathcal{F}_B P^k(-\mathcal{L}_\alpha)f(\lambda)\|_{\alpha,q}\leq C\|P^k(-\mathcal{L}_\alpha)f(\lambda)\|_{\alpha,p}.$$

Then

$$\|P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)\|_{\alpha,q}\leq C\|P^k(-\mathcal{L}_\alpha)f(\lambda)\|_{\alpha,p}$$

$$\overline{\lim_{k\to+\infty}}\|P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)\|_{\alpha,q}^{\frac{1}{k}}\leq 1. \tag{10.3}$$

Let $\lambda_0\notin\Gamma_P$, that means $|P(\lambda_{0,1}^2,...,\lambda_{0,n}^2)|>1$ then there exists a neighborhood U_{λ_0} of λ_0 with the property $|P(\lambda_1^2,...,\lambda_n^2)|>\dfrac{1+|P(\lambda_{0,1}^2,...,\lambda_{0,n}^2)|}{2}$ for $\lambda\in U_{\lambda_0}$.

a) Suppose $p>1$, then we have

$$\begin{aligned}1 &\geq \overline{\lim_{k\to+\infty}}\|P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)\|_{\alpha,q}^{\frac{1}{k}}\\ &\geq \overline{\lim_{k\to+\infty}}\Big(\int_{U_{\lambda_0}}|P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)|^q d\mu_\alpha(\lambda)\Big)^{\frac{1}{qk}}\\ &\geq \frac{1+|(P(\lambda_{0,1}^2,...,\lambda_{0,n}^2)|}{2}\overline{\lim_{k\to+\infty}}\Big(\int_{U_{\lambda_0}}|\mathcal{F}_B(f)(\lambda)|^q d\mu_\alpha(\lambda)\Big)^{\frac{1}{qk}}.\end{aligned} \tag{10.4}$$

Because of $\dfrac{1+|P(\lambda_{0,1}^2,...,\lambda_{0,n}^2)|}{2} > 1$ and the last limit in (10.4) can be either 1 or 0 then

$$\overline{\lim_{k\to+\infty}}\left(\int_{U_{\lambda_0}}|\mathcal{F}_B(f)(\lambda)|^q d\mu_\alpha(\lambda)\right)^{\frac{1}{qk}} = 0.$$

that means $\displaystyle\int_{U_{\lambda_0}}|\mathcal{F}_B(f)(\lambda)|^q d\mu_\alpha(\lambda) = 0.$
Consequently $\lambda_0 \notin$ supp $\mathcal{F}_B(f)$ and, hence, supp $\mathcal{F}_B(f) \subseteq \Gamma_P$.

b) Assume now that $p = 1$, then

$$\begin{array}{rl} 1 & \geq \overline{\lim\limits_{k\to+\infty}}\|P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)\|_{\alpha,\infty}^{\frac{1}{k}} \\ & \geq \overline{\lim\limits_{k\to+\infty}}\ \sup\limits_{\lambda\in U_{\lambda_0}} ess|P(\lambda_1^2,...,\lambda_n^2||\mathcal{F}_B(f)(\lambda)|^{\frac{1}{k}} \\ & \geq \dfrac{1+|p(\lambda_{0,1}^2,...,\lambda_{0,n}^2)|}{2}\ \overline{\lim\limits_{k\to+\infty}}\ \sup\limits_{\lambda\in U_{\lambda_0}} ess|\mathcal{F}_B(f)(\lambda)|^{\frac{1}{k}} \end{array} \tag{9.5}$$

therefore $\text{supess}_{\lambda\in U_{\lambda_0}}|\mathcal{F}_B(f)(\lambda)| = 0.$

that means $\lambda_0 \notin$ supp $\mathcal{F}_B(f)$ and hence supp$\mathcal{F}_B(f) \subseteq \Gamma_P$.
Conversely, suppose that now supp$\mathcal{F}_B(f) \subseteq \Gamma_P$ we have

$$\begin{aligned} \|f\|_{\alpha,p}^p &= \int_{\Omega_n}(1+|x|^2)^{-mp}(1+|x|^2)^{mp}|f(x)|^p d\mu_\alpha(x) \\ &\leq \|(1+|x|^2)^m f(x)\|_{2,\alpha}^p\|(1+|x|^2)^{-mp}\|_{\frac{2}{2-p},\alpha} \\ &\leq C\|(1+|x|^2)^m f\|_{\alpha,2}^p \end{aligned}$$

where the Hölder inequality is been applied then using the Parseval equality we have

$$\begin{aligned} \|f\|_{\alpha,p}^p &\leq c\|\mathcal{F}_B((1+|x|^2)^m f)\|_{\alpha,2}^p \\ &\leq C\|\mathcal{F}_B\Big(\sum_{j=0}^m C_m^j|x|^{2j}f(x))\Big)\|_{2,\alpha}^p \\ &= C\|\sum_{j=0}^m C_m^j\mathcal{F}_B(|x|^{2j}f(x))\|_{2,\alpha}^p \\ &= C\|\mathcal{F}_B(f(x)) + \sum_{j=1}^m C_m^j(-1)^{nj}\mathcal{L}_\alpha^j\mathcal{F}_B f(x)\|_{2,\alpha}^p \\ &= C\|(I+(-1)^n\mathcal{L}_\alpha)^m\mathcal{F}_B f(x)\|_{2,\alpha}^p. \end{aligned}$$

Consequently

$$\begin{aligned}\|P^k(-\mathcal{L}_\alpha)f\|_{\alpha,p} &\leq C^{\frac{1}{p}}\|I+(-1)^n\mathcal{L}_\alpha)^m\mathcal{F}_B(P^k(-\mathcal{L}_\alpha)f)\|_{\alpha,2}\\ &\leq C^{\frac{1}{p}}\|(I+(-1)^n\mathcal{L}_\alpha)^m P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)\|_{\alpha,2}\end{aligned}$$

its clear that there exists positives integers $N(m)$ and $N'(m)$ such that

$$(I+(-1)^n\mathcal{L}_\alpha)^m(P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)) = k^{N(m)}P^{N'(m)}(\lambda_1^2,...,\lambda_n^2)\phi_k(\lambda).$$

with supp $\phi_k \subset$supp $\mathcal{F}_B(f)$, $\|\phi_k\|_{\alpha,2} \leq C_1$, where C_1 independent of k, hence

$$\|P^k(-\mathcal{L}_\alpha)f\|_{\alpha,p} \leq C^{\frac{1}{p}}C_1k^{N(m)}.$$

Thus the inequality (10.1) follows.

ii) Let now $2 < p < \infty$.

Suppose that supp$\mathcal{F}_B(f) \subset \Gamma_P$. Then $|P(\lambda_1^2,...,\lambda_n^2)| \leq 1$ on the support of $\mathcal{F}_B(f)$, and therefore, by the Hausdorff-Young inequality we have

$$\begin{aligned}\|P^k(-\mathcal{L}_\alpha f)\|_{\alpha,p} &\leq C_2\|P^k(\lambda_1^2,...,\lambda_n^2)\mathcal{F}_B(f)(\lambda)\|_{\alpha,q}\\ &\leq C_2\|\mathcal{F}_B(f)\|_{\alpha,q};,\end{aligned} \tag{10.6}$$

where C_2 is independent from k.

Then

$$\overline{\lim_{k\to+\infty}}\|P^k(-\mathcal{L}_\alpha)f\|_{\alpha,p}^{\frac{1}{k}} \leq 1$$

conversely, suppose now that (10.6) hold . Since $f \in S_*(\mathbb{R}^n)$ the function f and its derivatives vanish at infinity, therefore, integration by parts gives

$$\int_{\Omega_n} P^k(-\mathcal{L}_\alpha)\overline{f(x)}P^k(-\mathcal{L}_\alpha)f(x)d\mu_\alpha(x) = \int_{\Omega_n}\overline{f(x)}P^{2k}(-\mathcal{L}_\alpha)f(x)d\mu_\alpha(x). \tag{10.7}$$

Then by Hölder inequality

$$\|P^k(-\mathcal{L}_\alpha)f(x)\|_{\alpha,2}^2 \leq \|f\|_{\alpha,q}\|P^{2k}(-\mathcal{L}_\alpha)f(x)\|_{\alpha,p}. \tag{10.8}$$

Then

$$\overline{\lim_{k\to+\infty}}\|P^k(-L_\alpha)f(x)\|_{\alpha,2}^{\frac{1}{k}} \leq 1. \tag{10.9}$$

Applying now i) with $p = 2$ we conclude that supp$\mathcal{F}_B(f) \subseteq \Gamma_P$.

iii) Let $p = \infty$.

The same proof as ii).

Remark. Theorem 6.1 has been obtained for $p = 2$ by V.K.Tuan in [5].

11 Open Problem

In [4] Roe proved that if a doubly-infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies
$\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \leq M$ for all $j = 0, \pm1, \pm2, ...$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x+b)$ where a and b are real constants. This result was extended to $\mathbb{R}^d$ by Strichartz [5] where $\frac{d}{dx}$ is substituted by the Laplacian on $\mathbb{R}^d$ as follow.
Theorem. (Strichartz). Let $(f_j)_{j\in\mathbb{Z}}$ be a doubly infinite sequence of measurable functions on $\mathbb{R}^d$ such that for all $j \in \mathbb{Z}$, (i) $||f_j||_{L^\infty(\mathbb{R}^d)} \leq C$ for some constant $C > 0$ and (ii) for some $a > 0, \triangle f_j = af_{j+1}$. Then $\triangle f_0 = -af_0$.

The purpose of the future work is to generalize this theorem. In place of Laplace operator $\triangle$ of $\mathbb{R}^d$, we shall extended this to multivariable Laplace-Bessel operator l_α.

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