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Some properties for higher-order derivatives of $p-$valent functions defined by a linear operator

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Abstract

By making use of the principle of subordination, we introduce a certain subclass of $p-$valent analytic functions. Such results as subordination properties, convolution properties and distortion theorems are proved.

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1 Introduction

Let $H[a,k]$ be the class of analytic functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \dots \quad (z \in U).$$

Also, let $\mathcal{A}(p,k)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n \quad (p,k \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For simplicity, we write $\mathcal{A}(p,1) = \mathcal{A}(p)$ and $\mathcal{A}(1,1) = \mathcal{A}$. If $f\ (z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ or $g(z)$ is superordinate to $f(z)$, written as $f \prec g$ in U or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that $f(z) = g(\omega(z))$ $(z \in U)$. Futher more, if the function $g(z)$ is univalent in U, then we have the following equivalence holds (see [8] and [9]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad and \quad f(U) \subset g(U).$$

For functions $f, g \in \mathcal{A}(p,k)$,where f given by (1) and g is defined by

$$g(z) = z^p + \sum_{n=p+k}^{\infty} b_n z^n \quad (p, k \in \mathbb{N}),$$

then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=p+k}^{\infty} a_n b_n z^n = (g * f)(z).$$

Upon differentiating both sides of (1) $j-$times with respect and to z, we have

$$f^{(j)}(z) = \delta(p;j)\, z^{p-j} + \sum_{n=p+k}^{\infty} \delta(n;j)\, a_n z^{n-j}, \tag{2}$$

where

$$\delta(p;j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{3}$$

For a function $f^{(j)}(z)$ given by (2), Aouf and Seoudy [4] defined the linear operator $D_p^n f^{(j)}$ by:

$$\begin{aligned}
D_p^0 f^{(j)}(z) &= f^{(j)}(z), \\
D_p^1 f^{(j)}(z) &= D\left(f^{(j)}(z)\right) = \delta(p;j)\, z^{p-j} + \sum_{n=p+k}^{\infty} \delta(n;j) \left(\frac{n-j}{p-j}\right) a_n z^{n-j}, \\
D_p^2 f^{(j)}(z) &= D\left(D_p^1 f^{(j)}(z)\right) = \delta(p;j)\, z^{p-j} + \sum_{n=p+k}^{\infty} \delta(n;j) \left(\frac{n-j}{p-j}\right)^2 a_{nk} z^{n-j},
\end{aligned}$$

and (in general)

$$\begin{aligned}
D_p^m f^{(j)}(z) &= D(D_p^{m-1} f^{(j)}(z)) = \delta(p;j)\, z^{p-j} + \sum_{n=p+k}^{\infty} \delta(n;j) \left(\frac{n-j}{p-j}\right)^m a_n z^{n-j} \\
&\quad (p > j; p, m \in \mathbb{N}; j \in \mathbb{N}_0; z \in U).
\end{aligned} \tag{4}$$

From (4), we can easily deduce that

$$z\left(D_p^m f^{(j)}(z)\right)^{'} = (p-j)\, D_p^{m+1} f^{(j)}(z) \quad (p>j; p\in\mathbb{N}; m,j\in\mathbb{N}_0; z\in U)\,. \quad (5)$$

The operator $D_p^m f^{(j)}(z)$ $(p>j, p\in\mathbb{N}, n, j\in\mathbb{N}_0)$ was introduced and studied by Aouf [1, 2] where

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \quad (a_n \geq 0)\,.$$

We note that:

(i) $D_p^n f^{(0)}(z) = D_p^m f(z)$ was introduced and studied by Kamali and Orhan [5] and Aouf and Mostafa [3];

(ii) $D_1^m f^{(0)}(z) = D^m f(z)$ was introduced by Sălăgean [10].

By making use of the linear operator $D_p^m f^{(j)}(z)$ and the above-mentioned principle of subordination between analytic functions, we now introduce the following subclass of $p-$valent non-Bazilevič functions.

Definition 1. A function $f\in\mathcal{A}(p,k)$ is said to be in the class $\mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$ if it satisfies the following subordination condition:

$$(1-\lambda)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\, z^{p-j}}\right)^{\mu} + \lambda \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\, z^{p-j}}\right)^{\mu} \prec \frac{1+Az}{1+Bz} \qquad (6)$$

$$(p,k\in\mathbb{N}; \lambda\in\mathbb{C};\ 0<\mu<1;\ -1\leq B\leq 1,\ A\neq B,\ A\in\mathbb{R})\,.$$

In the present paper, we aim at proving such results as subordination and superordination properties, convolution properties, distortion theorems and inequality properties of the class $\mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$.

2 Main results

In order to establish our main results, we need the following lemmas.

Lemma 1 [8]. *Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$. Suppose also that the function $g(z)$ given by*

$$g(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + ... \qquad (7)$$

is analytic in U. If

$$g(z) + \frac{zg^{'}(z)}{\gamma} \prec h(z) \quad (\Re(\gamma)>0; \gamma\neq 0)\,,$$

then

$$g(z) \prec q(z) = \frac{\gamma}{k} z^{-\frac{\gamma}{k}} \int_0^z h(t)\, t^{\frac{\gamma}{k}-1}\, dt \prec h(z),$$

and $q(z)$ *is the best dominant.*

Lemma 2 [6]. *Let* $\mathfrak{F}$ *be analytic and convex in* U. *If*

$$f, g \in \mathcal{A} \quad and \quad f, g \prec \mathfrak{F}$$

then

$$\gamma f + (1-\gamma)\, g \prec \mathfrak{F} \quad (0 \le \gamma \le 1).$$

Unless otherwise mentioned, we assume throughout this paper that $\mu > 0$, $-1 \le B \le 1$, $A \ne B$, $A \in \mathbb{R}$, $p, k \in \mathbb{N}$, $\delta(p; j)$ is given by (3) and all powers are understood as principle values. We begin by presenting our first subordination property given by Theorem 1 below.

Theorem 1. *Let* $f(z) \in \mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$ *with* $\Re(\lambda) > 0$. *Then*

$$\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j)\, z^{p-j}}\right)^{\mu} \prec q(z) = \frac{(p-j)\,\mu}{\lambda k} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p-j)\mu}{\lambda k}-1} du \prec \frac{1 + Az}{1 + Bz}, \quad (8)$$

and $q(z)$ *is the best dominant.*

Proof. Define the function $g(z)$ by

$$g(z) = \left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j)\, z^{p-j}}\right)^{\mu} \quad (z \in U). \quad (9)$$

Then the function $g(z)$ is of the form (7) and analytic in U . Differentiating (9) with respect to z and using the identity (5), we get

$$(1-\lambda)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j)\, z^{p-j}}\right)^{\mu} + \lambda \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)} \left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j)\, z^{p-j}}\right)^{\mu} = g(z) + \frac{\lambda z g'(z)}{\mu(p-j)}. \quad (10)$$

Since $f \in \mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$, we have

$$g(z) + \frac{\lambda z g'(z)}{\mu(p-j)} \prec \frac{1 + Az}{1 + Bz}.$$

Applying Lemma 1 with $\gamma = \frac{(p-j)\mu}{\lambda}$, we get

$$\begin{aligned}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j)\, z^{p-j}}\right)^{\mu} &\prec q(z) = \frac{(p-j)\,\mu}{\lambda k} z^{\frac{(p-j)\mu}{\lambda k}-1} \int_0^z \frac{1 + At}{1 + Bt}\, t^{\frac{(p-j)\mu}{\lambda k}-1} dt \\ &= \frac{(p-j)\,\mu}{\lambda k} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p-j)\mu}{\lambda k}-1} du \prec \frac{1 + Az}{1 + Bz}, \quad (11)\end{aligned}$$

and $q(z)$ is the best dominant. The proof of Theorem 1 is thus completed.

Theorem 2. *If* $\lambda > 0$ *and* $f \in \mathcal{B}_p^{j,\mu}(m; 1-2\rho, -1, 0)$ $(0 \leq \rho < 1)$. *Then* $f \in \mathcal{B}_p^{j,\mu}(m; 1-2\rho, -1, \lambda)$ *for* $|z| < R$, *where*

$$R = \left(\sqrt{\left(\frac{\lambda k}{(p-j)\mu} \right)^2 + 1} - \frac{\lambda k}{(p-j)\mu} \right)^{\frac{1}{k}}. \tag{12}$$

The bound R *is the best possible.*

Proof. We begin by writing

$$\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j) z^{p-j}} \right)^\mu = \rho + (1-\rho) g(z) \quad (z \in U; 0 \leq \rho < 1). \tag{13}$$

Then, clearly, the function $g(z)$ is of the form (7), is analytic and has a positive real part in U. Differentiating (13) with respect to z and using the identity (5), we get

$$\frac{1}{1-\rho} \left\{ (1-\lambda) \left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j) z^{p-j}} \right)^\mu + \lambda \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)} \left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j) z^{p-j}} \right)^\mu - \rho \right\}$$

$$= g(z) + \frac{\lambda z g'(z)}{(p-j)\mu}. \tag{14}$$

By making use of the following well-known estimate (see [7]):

$$\frac{|z g'(z)|}{\Re\{g(z)\}} \leq \frac{2kr^k}{1-r^{2k}} \quad (|z| = r < 1)$$

in (14), we obtain that

$$\Re \left(\frac{1}{1-\rho} \left\{ (1-\lambda) \left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j) z^{p-j}} \right)^\mu + \lambda \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)} \left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j) z^{p-j}} \right)^\mu - \rho \right\} \right)$$

$$\geq \Re\{g(z)\} \left(1 - \frac{2\lambda k r^k}{(p-j)\mu(1-r^{2k})} \right). \tag{15}$$

It is seen that the right-hand side of (15) is positive, provided that $r < R$, where R is given by (12).

In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}(p,k)$ defined by

$$\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j) z^{p-j}} \right)^\mu = \rho + (1-\rho) \frac{1+z^k}{1-z^k} \quad (z \in U; 0 \leq \rho < 1).$$

Noting that

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\, z^{p-j}}\right)^{\mu}+\lambda\frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\, z^{p-j}}\right)^{\mu}-\rho\right\}$$

$$=\frac{1+z^k}{1-z^k}+\frac{2\lambda k z^k}{(p-j)\,\mu\left(1-z^k\right)^2}=0, \tag{16}$$

for $|z|=R$, we conclude that the bound is the best possible. Theorem 2 is thus proved.

Theorem 3. *Let $f\in\mathcal{B}_p^{j,\mu}(m;A,B,\lambda)$ with $\Re(\lambda)>0$. Then*

$$f^{(j)}(z)=\delta(p;j)\left(z^{p-j}\left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\mu}}\right)*\left(z^{p-j}+\sum_{n=p+k}^{\infty}\frac{\left(\frac{p-j}{n-j}\right)^m}{\delta(n;j)}z^{n-j}\right), \tag{17}$$

where $\omega(z)$ is analytic function with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$.

Proof. Suppose that $f\in\mathcal{B}_p^{j,\mu}(m;A,B,\lambda)$ with $\Re(\lambda)>0$. It follows from (8) that

$$\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\, z^{p-j}}\right)^{\mu}=\frac{1+A\omega(z)}{1+B\omega(z)}, \tag{18}$$

where $\omega(z)$ is analytic function with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$. By virtue of (18), we easily find that

$$D_p^m f^{(j)}(z)=\delta(p;j)\, z^{p-j}\left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\mu}}. \tag{19}$$

Combining (4) and (19), we have

$$\left(\delta(p;j)\, z^{p-j}+\sum_{n=p+k}^{\infty}\delta(n;j)\left(\frac{n-j}{p-j}\right)^m z^{n-j}\right)*f^{(j)}(z)$$

$$=\delta(p;j)\, z^{p-j}\left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\mu}}. \tag{20}$$

The assertion (17) of Theorem 3 can now easily be derived from (20).

Theorem 4. *Let $f\in\mathcal{B}_p^{j,\mu}(m;A,B,\lambda)$ with $\Re(\lambda)>0$. Then*

$$\frac{1}{z^{p-j}}\Big[\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\left(\delta(p;j)\, z^{p-j}+\textstyle\sum_{n=p+k}^{\infty}\delta(n;j)\left(\frac{n-j}{p-j}\right)^m z^{n-j}\right)*f^{(j)}(z)$$
$$-\delta(p;j)\, z^{p-j}\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\Big]\neq 0 \qquad (z\in U;\ 0<\theta<2\pi). \tag{21}$$

Proof. Suppose that $f \in \mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$ with $\Re(\lambda) > 0$. We know that (8) holds true, which implies that

$$\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in U;\ 0 < \theta < 2\pi). \tag{22}$$

It is easy to see that the condition (22) can be written as follows:

$$\frac{1}{z^{p-j}}\left[D_p^m f^{(j)}(z)\left(1 + Be^{i\theta}\right)^{\frac{1}{\mu}} - \delta(p; j) z^{p-j}\left(1 + Ae^{i\theta}\right)^{\frac{1}{\mu}}\right] \neq 0 \ \ (0 < \theta < 2\pi). \tag{23}$$

Combining (4) and (23), we easily get the convolution property (21) asserted by Theorem 4.

Theorem 5. *Let* $\lambda_2 \geq \lambda_1 \geq 0$ *and* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$.*Then*

$$\mathcal{B}_p^{j,\mu}(m; A_2, B_2, \lambda_2) \subset \mathcal{B}_p^{j,\mu}(m; A_1, B_1, \lambda_1). \tag{24}$$

Proof. Suppose that $f \in \mathcal{B}_p^{j,\mu}(m; A_2, B_2, \lambda_2)$. We know that

$$(1 - \lambda_2)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} + \lambda_2 \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$\begin{aligned}(1 - \lambda_2)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} &+ \lambda_2 \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} \\ &\prec \frac{1 + A_2 z}{1 + B_2 z} \prec \frac{1 + A_1 z}{1 + B_1 z},\end{aligned} \tag{25}$$

that is $f \in \mathcal{B}_p^{j,\mu}(m; A_1, B_1, \lambda_2)$. Thus the assertion (24) holds for $\lambda_2 = \lambda_1 \geq 0$. If $\lambda_2 > \lambda_1 \geq 0$, by Theorem 1 and (25), we know that $f \in \mathcal{B}_p^{j,\mu}(m; A_1, B_1, 0)$, that is,

$$\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} \prec \frac{1 + A_1 z}{1 + B_1 z}, \tag{26}$$

At the same time, we have

$$\begin{aligned}&(1 - \lambda_1)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} + \lambda_1 \frac{D_p^{m+1} f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu} \\ = &\left(1 - \frac{\lambda_1}{\lambda_2}\right)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}}\right)^{\mu}\end{aligned}$$

$$+\frac{\lambda_1}{\lambda_2}\left((1-\lambda_2)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu}+\lambda_2\frac{D_p^{m+1}f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu}\right). \quad (27)$$

Moreover, since $0\le\frac{\lambda_1}{\lambda_2}<1$, and the function $\frac{1+A_1z}{1+B_1z}\,(-1\le B_1<A_1\le 1)$ is analytic and convex in U. Combining (25)-(27) and Lemma 2, we find that

$$(1-\lambda_1)\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu}+\lambda_1\frac{D_p^{m+1}f^{(j)}(z)}{D_p^m f^{(j)}(z)}\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu}\prec\frac{1+A_1z}{1+B_1z},$$

that is $f\in\mathcal{B}_p^{j,\mu}(m;A_1,B_1,\lambda_1)$, which implies that the assertion (24) of Theorem 5 holds.

Theorem 6. *Let $f\in\mathcal{B}_p^{j,\mu}(m;A,B,\lambda)$ with $\lambda>0$ and $-1\le B<A\le 1$. Then*

$$\frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{(p-j)\mu}{\lambda k}-1}du<\Re\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu}$$
$$<\frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{(p-j)\mu}{\lambda k}-1}du. \quad (28)$$

The extremal function of (28) is defined by

$$D_p^m F^{(j)}(z)=\delta(p;j)\,z^{p-j}\left(\frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(p-j)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}. \quad (29)$$

Proof. Let $f\in\mathcal{B}_p^{j,\mu}(m;A,B,\lambda)$ with $\lambda>0$. From Theorem 1, we know that (8) holds, which implies that

$$\begin{aligned}\Re\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu} &< \sup_{z\in U}\Re\left\{\frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(p-j)\mu}{\lambda k}-1}du\right\}\\ &\le \frac{(p-j)\mu}{\lambda k}\int_0^1\sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(p-j)\mu}{\lambda k}-1}du\\ &< \frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1+Au}{1+Bu}\,u^{\frac{(p-j)\mu}{\lambda k}-1}du, \quad (30)\end{aligned}$$

and

$$\begin{aligned}\Re\left(\frac{D_p^m f^{(j)}(z)}{\delta(p;j)\,z^{p-j}}\right)^{\mu} &> \inf_{z\in U}\Re\left\{\frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}\,u^{\frac{(p-j)\mu}{\lambda k}-1}du\right\}\\ &\ge \frac{(p-j)\mu}{\lambda k}\int_0^1\inf_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(p-j)\mu}{\lambda k}-1}du\\ &> \frac{(p-j)\mu}{\lambda k}\int_0^1\frac{1+Au}{1+Bu}\,u^{\frac{(p-j)\mu}{\lambda k}-1}du. \quad (31)\end{aligned}$$

Combining (30) and (31), we get (28). By noting that the function $D_p^m F^{(j)}(z)$ defined by (29) belongs to the class $\mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$, we obtain that equality (28) is sharp. The proof of Theorem 6 is evidently completed.

In view of Theorem 6, we easily derive the following distortion theorems for the class $\mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$.

Corollary 1. *Let $f \in \mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$ with $\lambda > 0$ and $-1 \leq B < A \leq 1$. Then for $|z| = r < 1$, we have*

$$\delta(p; j) r^{p-j} \left(\frac{(p-j)\mu}{\lambda k} \int_0^1 \frac{1 - Aur}{1 - Bur} u^{\frac{(p-j)\mu}{\lambda k} - 1} du \right)^{\frac{1}{\mu}} < \left| D_p^m f^{(j)}(z) \right|$$

$$< \delta(p; j) r^{p-j} \left(\frac{(p-j)\mu}{\lambda k} \int_0^1 \frac{1 + Aur}{1 + Bur} u^{\frac{(p-j)\mu}{\lambda k} - 1} du \right)^{\frac{1}{\mu}}. \qquad (32)$$

The extremal function of (32) is defined by (29).

By noting that

$$(\Re(\upsilon))^{\frac{1}{2}} \leq \Re\left(\upsilon^{\frac{1}{2}}\right) \leq |\upsilon|^{\frac{1}{2}} \quad (\upsilon \in \mathbb{C}; \Re(\upsilon) \geq 0).$$

Form Theorem 6 and Corollary 1, we easily get the following results.

Corollary 2. *Let $f \in \mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$ with $\lambda > 0$ and $-1 \leq B < A \leq 1$. Then*

$$\left(\frac{(p-j)\mu}{\lambda k} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{(p-j)\mu}{\lambda k} - 1} du \right)^{\frac{1}{2}} < \Re\left(\frac{D_p^m f^{(j)}(z)}{\delta(p; j) z^{p-j}} \right)^{\frac{\mu}{2}}$$

$$< \left(\frac{(p-j)\mu}{\lambda k} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{(p-j)\mu}{\lambda k} - 1} du \right)^{\frac{1}{2}}. \qquad (33)$$

The extremal function of (33) is defined by (29).

Remark. Taking $m = 0$ in the above results, we obtain the corresponding results for the function $f^{(j)}(z)$.

3 Open Problem

Study and calculate Fekete-Sezgö problems and other properties for functions belonging to the class $\mathcal{B}_p^{j,\mu}(m; A, B, \lambda)$.

References

[1] M. K. Aouf, Generalization of certain subclasses of multivalent functions with negative coefficients defined by using a differential operator, Math. Comput. Modelling, 50(2009), no. 9-10, 1367-1378.

[2] M. K. Aouf, On certain multivalent functions with negative coefficients defined by using a differential operator, Indian J. Math., 51 (2009), no. 2, 433-451.

[3] M. K. Aouf and A. O. Mostafa, On a subclass of $n-p-$valent prestarlike functions, Comput. Math. Appl., 55 (2008), no. 4, 851-861.

[4] M. K. Aouf and T. M. Seoudy, Differential sandwich theorems for higher-order derivatives of $p-$valent functions defined by linear operator, Bull. Korean Math. Soc.,48(2011), no. 3, 627-636.

[5] M. Kamali and H. Orhan, On a subclass of certain starlike functions with negative coefficients, Bull. Korean Math. Soc., 41 (2004), no. 1, 53-71.

[6] M.-S. Liu, On certain subclass of analytic functions, J. South China Normal Univ. 4(2002), 15-20 (in Chinese).

[7] T. H. Macgregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 14(1963), 514-520.

[8] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[9] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl., 48(2003), no. 10, 815–826.

[10] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag) 1013 , (1983), 362 - 372.