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Bounds, Asymptotic Behavior and Recurrence Relations for the Jacobi-Dunkl Polynomials

Frej CHOUCHENE

Department of Mathematics
Faculty of Sciences of Monastir
Street of the environment, 5019 Monastir, Tunisia
e-mail: frej.chouchene@fsm.rnu.tn

Abstract

In this paper, we establish explicit formulas and recurrence relations for the Jacobi-Dunkl polynomials. Bounds and asymptotic behavior of these polynomials are also given.

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1 Introduction

Jacobi polynomials are a class of classical orthogonal polynomials which have many important applications in such areas as mathematical physics, approximation theory, numerical analysis, combinatorics, and others, see [12, 8, 11, 2, 4, 1, 10].
S. Bochner characterized classical orthogonal polynomials in terms of their recurrence relations. These relations occur in a variety of mathematical contexts. They can be used to define the sequence of polynomials. They also lead to concise algorithms which are useful for either manual or automatic

calculations. In general, recurrences with more than three terms are harder to treat, see [13, 14, 3].

In the present paper our aim is to establish explicit formulas and recurrence relations, with more than three terms (six terms), for the Jacobi-Dunkl polynomials $\psi_m^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$, $m \in \mathbb{Z}$, see [5], defined on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by

$$\psi_m^{(\alpha,\beta)}(\theta) := \begin{cases} R_{|m|}^{(\alpha,\beta)}(\cos(2\theta)) + i\dfrac{\lambda_m^{(\alpha,\beta)}}{4(\alpha+1)}\sin(2\theta)R_{|m|-1}^{(\alpha+1,\beta+1)}(\cos(2\theta)) & \text{if } m \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if } m = 0, \end{cases} \tag{1}$$

where $R_n^{(\alpha,\beta)}$, $n \in \mathbb{N}$, is the normalized Jacobi polynomial of degree n such that $R_n^{(\alpha,\beta)}(1) = 1$, and

$$\lambda_m^{(\alpha,\beta)} := 2\,\mathrm{sgn}(m)\sqrt{|m|(|m|+\rho)}, \quad m \in \mathbb{Z}, \tag{2}$$

with

$$\rho := \alpha + \beta + 1. \tag{3}$$

Then, we study the asymptotic behavior of these polynomials and we give the upper and lower bounds of the module of each Jacobi-Dunkl polynomial $\psi_m^{(\alpha,\beta)}$, $m \in \mathbb{Z}$.

In the sequel, we take $\alpha > -1$, $\beta > -1$. We drop the exponents (α, β) when there is no confusion.

We consider the Jacobi polynomial

$$\varphi_n(\theta) = \varphi_n^{(\alpha,\beta)}(\theta) := R_n^{(\alpha,\beta)}(\cos(2\theta)), \quad n \in \mathbb{N}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Note that

$$\forall n \in \mathbb{N}, \quad \varphi_n(0) = 1, \tag{4}$$

and $\forall n \in \mathbb{N}$, $\forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\quad \varphi_n(-\theta) = \varphi_n(\theta)$.

For all $n \in \mathbb{N}$, the equation

$$\begin{cases} \Delta_{\alpha,\beta} u &= -4\gamma_n u, \\ u(0) &= 1, \\ u'(0) &= 0, \end{cases}$$

admits a unique solution on $\left[0, \frac{\pi}{2}\right[$ which is the fuction φ_n, where $\Delta_{\alpha,\beta}$, is the Jacobi operator defined on $C^2\left(\left]0, \frac{\pi}{2}\right[\right)$ by

$$\Delta_{\alpha,\beta} := \frac{d^2}{d\theta^2} + \left[(2\alpha+1)\cot - (2\beta+1)\tan\right]\frac{d}{d\theta},$$

with γ_n is given by
$$\gamma_n = \gamma_n^{(\alpha,\beta)} := n(n+\rho), \tag{5}$$
and ρ is given by (3).
We remark that
$$\forall u \in C^2\left(\left]0,\frac{\pi}{2}\right[\right),\ \forall \theta \in \left]0,\frac{\pi}{2}\right[,\quad \Delta_{\alpha,\beta}u(\theta) = \frac{1}{A(\theta)}\frac{d}{d\theta}\left[A(\theta)\frac{d}{d\theta}u(\theta)\right],$$
where A is the function given by
$$A(\theta) = A_{\alpha,\beta}(\theta) := \begin{cases} 2^{2\rho}(\sin|\theta|)^{2\alpha+1}(\cos\theta)^{2\beta+1} & \text{if } \theta \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\setminus \{0\}, \\ 0 & \text{if } \theta = 0. \end{cases} \tag{6}$$
Note that
$$\forall \theta \in \left]0,\frac{\pi}{2}\right[,\quad \frac{A'(\theta)}{A(\theta)} = (2\alpha+1)\cot\theta - (2\beta+1)\tan\theta.$$
In the following, we recall some basic properties of the Jacobi polynomials, see [12].
$$\forall n \in \mathbb{N},\quad \varphi_n\left(\frac{\pi}{2}\right) = (-1)^n\frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)}{\Gamma(\beta+1)\Gamma(\alpha+n+1)}. \tag{7}$$
$$\forall n \in \mathbb{N},\ \forall \theta \in \left[0,\frac{\pi}{2}\right],\quad \varphi_n^{(\beta,\alpha)}(\theta) = \frac{\varphi_n\left(\frac{\pi}{2}-\theta\right)}{\varphi_n\left(\frac{\pi}{2}\right)}. \tag{8}$$
In hypergeometric form, we have
$$\forall n \in \mathbb{N},\ \forall \theta \in \left[0,\frac{\pi}{2}\right[,\quad \varphi_n(\theta) = {}_2F_1\left(-n, n+\rho; \alpha+1; (\sin\theta)^2\right),$$
where ${}_2F_1$ is the Gaussian hypergeometric function defined by
$${}_2F_1(a,b;c;x) := \sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{k!\,(c)_k}x^k,\quad a,b,c \in \mathbb{R},\quad |x|<1,$$
with $(d)_0 = 1,\quad (d)_k = d(d+1)(d+2)...(d+k-1),\quad k \geq 1,\ d \in \mathbb{R}$.
We also have
$$\forall n \in \mathbb{N}\setminus\{0\},\ \forall \theta \in \left[0,\frac{\pi}{2}\right],\quad \frac{d}{d\theta}\varphi_n(\theta) = -\frac{\gamma_n}{\alpha+1}\sin(2\theta)\varphi_{n-1}^{(\alpha+1,\beta+1)}(\theta),$$
where γ_n is given by (5).
In particular, $\forall n \in \mathbb{N},\ \frac{d}{d\theta}\varphi_n(0) = 0$, and the Gegenbauer (ultraspherical) polynomials satisfy the following relations : $\forall n \in \mathbb{N},\ \forall \theta \in \left[0,\frac{\pi}{2}\right]$,
$$\varphi_{2n}^{(\alpha,\alpha)}\left(\frac{\theta}{2}\right) = \varphi_n^{\left(\alpha,-\frac{1}{2}\right)}(\theta), \tag{9}$$

and

$$\varphi_{2n+1}^{(\alpha,\alpha)}\left(\frac{\theta}{2}\right)=\cos\theta\,\varphi_n^{(\alpha,\frac{1}{2})}(\theta).$$

The orthogonality property is given by

$$\forall m\in\mathbb{N},\ \forall n\in\mathbb{N},\quad \int_0^{\frac{\pi}{2}}\varphi_m(\theta)\varphi_n(\theta)A(\theta)\,d\theta=k_n^{-1}\delta_{mn}, \tag{10}$$

where

$$k_n=k_n^{(\alpha,\beta)}:=\begin{cases}\dfrac{(2n+\rho)\Gamma(\alpha+n+1)\Gamma(\rho+n)}{2^{2\rho-1}(\Gamma(\alpha+1))^2 n!\,\Gamma(\beta+n+1)} & \text{if } n\in\mathbb{N}\setminus\{0\},\\ \dfrac{\Gamma(\rho+1)}{2^{2\rho-1}\Gamma(\alpha+1)\Gamma(\beta+1)} & \text{if } n=0.\end{cases} \tag{11}$$

A generating function

$$\sum_{n=0}^{+\infty}(-1)^n\varphi_n^{(0,\alpha)}\left(\frac{\pi}{2}\right)\varphi_n(\theta)x^n=S^{-1}\left(\frac{x+S-1}{2x(\sin\theta)^2}\right)^{\alpha}\left(\frac{x+S+1}{2}\right)^{-\beta},$$

where $S:=\sqrt{1-2x\cos(2\theta)+x^2}$, $\quad |x|<1,\quad 0<\theta<\dfrac{\pi}{2}$.

The operator $\Lambda_{\alpha,\beta}$ defined on $C^1\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ by

$$\Lambda_{\alpha,\beta}u(\theta):=\frac{d}{d\theta}u(\theta)+\Big[(2\alpha+1)\cot\theta-(2\beta+1)\tan\theta\Big]\frac{u(\theta)-u(-\theta)}{2}, \tag{12}$$

$u\in C^1\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$, is called the Jacobi-Dunkl operator on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$. It is the analogues on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ of the Jacobi-Dunkl operator on $\mathbb{R}$. It corresponds to the function A given by (6), see [5, 6].

The operator $\Lambda_{\alpha,-\frac{1}{2}}$ is the Opdam operator related to the roots system A_1, see [9].

From [5], for all $m\in\mathbb{Z}$, the equation

$$\begin{cases}\Lambda_{\alpha,\beta}u &= i\lambda_m u,\\ u(0) &= 1,\end{cases}$$

admits a unique solution sur $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ which is the Jacobi-Dunkl polynomial $\psi_m:=\psi_m^{(\alpha,\beta)}$ given by (1). It is related to the Jacobi polynomial $\varphi_{|m|}$ and to its derivative by

$$\forall\theta\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[,\quad \psi_m(\theta)=\begin{cases}\varphi_{|m|}(\theta)-\dfrac{i}{\lambda_m}\dfrac{d}{d\theta}\varphi_{|m|}(\theta) & \text{if } m\in\mathbb{Z}\setminus\{0\},\\ 1 & \text{if } m=0,\end{cases}$$

where $\Lambda_{\alpha,\beta}$ and λ_m are respectively given by (12) and (2).
The orthogonality property is given by

$$\forall m \in \mathbb{Z},\ \forall n \in \mathbb{Z}, \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_m(\theta)\overline{\psi_n(\theta)}A(\theta)\,d\theta = h_m^{-1}\delta_{mn}, \tag{13}$$

where

$$h_m := h_m^{(\alpha,\beta)} := \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\psi_m(\theta)|^2 A(\theta)\,d\theta\right)^{-1} = \begin{cases} \dfrac{k_{|m|}}{4} & \text{if } m \in \mathbb{Z}\setminus\{0\}, \\ \dfrac{k_0}{2} & \text{if } m = 0, \end{cases} \tag{14}$$

with $k_n,\ n \in \mathbb{N}$, is given by (11).
In [5], the author has studied the harmonic analysis associated with the Jacobi-Dunkl operator.
In [7], C.F. Dunkl has defined similar functions to the Jacobi-Dunkl polynomials.

2 Explicit representations

Notations : We denote by

- $$a(k,n,\alpha,\beta) := \begin{cases} \dbinom{n}{k}\dfrac{\Gamma(\alpha+1)\Gamma(\rho+n+k)}{\Gamma(\alpha+k+1)\Gamma(\rho+n)} & \text{if } n \geq 1,\ 0 \leq k \leq n, \\ 1 & \text{if } n = k = 0. \end{cases} \tag{15}$$

- $$b(k,n,\alpha,\beta) := \frac{a(k,n,\alpha,\beta)}{2^k}\,{}_2F_1\left(k-n, n+k+\rho; \alpha+k+1; \frac{1}{2}\right), \qquad 0 \leq k \leq n. \tag{16}$$

- $$c(k,n,\alpha,\beta) := \begin{cases} \dbinom{n}{k}\dfrac{\Gamma(\alpha+1)\Gamma(\beta+n+1)\Gamma(\rho+n+k)}{\Gamma(\alpha+n+1)\Gamma(\beta+k+1)\Gamma(\rho+n)} \\ \qquad\qquad \text{if } n \geq 1,\ 0 \leq k \leq n, \\ 1 \qquad\qquad \text{if } n = k = 0. \end{cases} \tag{17}$$

We can write

$$\begin{aligned} b(k,n,\alpha,\beta) &= \frac{(-1)^{n-k}\Gamma(\alpha+k+1)\Gamma(\beta+n+1)}{2^k\Gamma(\alpha+n+1)\Gamma(\beta+k+1)}a(k,n,\alpha,\beta) \\ &\times\ {}_2F_1\left(k-n, n+k+\rho; \beta+k+1; \frac{1}{2}\right), \quad n \geq 1,\ 0 \leq k \leq n. \end{aligned}$$

2.1 Explicit formulas of φ_n

From [12, 8, 11], for all $n \in \mathbb{N}$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned}
\varphi_n(\theta) &= \Gamma(\alpha+1)\Gamma(\beta+n+1) \\
&\times \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{\Gamma(\alpha+k+1)\Gamma(\beta+n-k+1)} (\sin\theta)^{2k}(\cos\theta)^{2(n-k)}. \quad (18)\\
&= (-1)^n \Gamma(\alpha+1)\Gamma(\beta+n+1) \\
&\times \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{\Gamma(\alpha+n-k+1)\Gamma(\beta+k+1)} (\sin\theta)^{2(n-k)}(\cos\theta)^{2k}. \quad (19)\\
&= \sum_{k=0}^{n} (-1)^k a(k,n,\alpha,\beta)(\sin\theta)^{2k}. \quad (20)\\
&= \sum_{k=0}^{n} b(k,n,\alpha,\beta)(\cos(2\theta))^k, \quad (21)
\end{aligned}$$

where $a(k,n,\alpha,\beta)$ and $b(k,n,\alpha,\beta)$ are respectively given by (15) and (16). In the following proposition, we give a new explicit forms of $\varphi_n,\ n \in \mathbb{N}$.

Proposition 2.1. $\forall n \in \mathbb{N},\ \forall \theta \in \left[0, \frac{\pi}{2}\right]$, *on a :*

$$\begin{aligned}
\varphi_n(\theta) &= \sum_{k=0}^{n} (-1)^{n-k} c(k,n,\alpha,\beta)(\cos\theta)^{2k}. \quad (22)\\
&= \sum_{k=-n}^{n} \left(\sum_{p=|k|}^{n} (-1)^{p-|k|} \binom{2p}{p+|k|} 2^{-2p} a(p,n,\alpha,\beta) \right) e^{2ik\theta}.\\
&= \sum_{k=-n}^{n} \left(\sum_{p=|k|}^{n} \frac{p!\, b(p,n,\alpha,\beta)}{2^p \Gamma\left(\frac{p+k}{2}+1\right) \Gamma\left(\frac{p-k}{2}+1\right)} \right) e^{2ik\theta}.\\
&= \sum_{k=-n}^{n} \left(\sum_{p=|k|}^{n} (-1)^{n-p} \binom{2p}{p+|k|} 2^{-2p} c(p,n,\alpha,\beta) \right) e^{2ik\theta},
\end{aligned}$$

where $a(k,n,\alpha,\beta), b(k,n,\alpha,\beta)$ *and* $c(k,n,\alpha,\beta)$ *are respectively given by* (15), (16) *and* (17).

Proof. We replace in (20), θ by $\frac{\pi}{2} - \theta$. Then we use (8), (7) and (15). Finally by interchanging α and β we obtain the first expression. For the others, it suffices to use $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$. $\square$

Examples 2.2. *For all $n \in \mathbb{N}$ et $\theta \in \left[0, \frac{\pi}{2}\right]$, we have*

1.

$$\begin{aligned}
\varphi_0(\theta) &= 1. \\
\varphi_1(\theta) &= \frac{(\rho+1)\cos(2\theta)+\alpha-\beta}{2(\alpha+1)} = 1 - \frac{\rho+1}{\alpha+1}(\sin\theta)^2. \\
\varphi_2(\theta) &= \frac{(\rho+2)(\rho+3)(\cos(2\theta))^2 + 2(\alpha-\beta)(\rho+2)\cos(2\theta)}{4(\alpha+1)(\alpha+2)} \\
&+ \frac{(\alpha-\beta)^2-\rho-3}{4(\alpha+1)(\alpha+2)}. \\
&= \frac{(\rho+2)(\rho+3)(\cos\theta)^4 - 2(\rho+2)(\beta+2)(\cos\theta)^2}{(\alpha+1)(\alpha+2)} \\
&+ \frac{(\beta+1)(\beta+2)}{(\alpha+1)(\alpha+2)}. \\
&= \frac{(\rho+2)(\rho+3)}{(\alpha+1)(\alpha+2)}(\sin\theta)^4 - \frac{2(\rho+2)}{\alpha+1}(\sin\theta)^2 + 1. \\
&= \frac{(\beta+1)(\beta+2)}{(\alpha+1)(\alpha+2)}(\sin\theta)^4 - \frac{2(\beta+2)}{\alpha+1}(\sin\theta)^2(\cos\theta)^2 + (\cos\theta)^4.
\end{aligned}$$

2. $\varphi_n^{(-\frac{1}{2},-\frac{1}{2})}(\theta) = \cos(2n\theta).$

3. $\varphi_n^{(\frac{1}{2},-\frac{1}{2})}(\theta) = \dfrac{\sin((2n+1)\theta)}{(2n+1)\sin\theta}, \quad \theta \neq 0.$

4. $\varphi_n^{(-\frac{1}{2},\frac{1}{2})}(\theta) = \dfrac{\cos((2n+1)\theta)}{\cos\theta}, \quad \theta \neq \dfrac{\pi}{2}.$

5. $\varphi_n^{(\frac{1}{2},\frac{1}{2})}(\theta) = \dfrac{\sin(2(n+1)\theta)}{(n+1)\sin(2\theta)}, \quad \theta \neq 0, \dfrac{\pi}{2}.$

Remarks 2.3.

1. $\forall n \in \mathbb{N}, \quad \varphi_n\left(\frac{\pi}{2}\right) = (-1)^n \dfrac{B(\alpha+1,\beta+n+1)}{B(\alpha+n+1,\beta+1)}.$

2. $\forall n \in \mathbb{N},\ \forall k \in \mathbb{N};\ 0 \leq k \leq n, \quad \varphi_{n-k}^{(\alpha+k,\beta+k)}\left(\frac{\pi}{2}\right) = \dfrac{\varphi_n\left(\frac{\pi}{2}\right)}{\varphi_k\left(\frac{\pi}{2}\right)}.$

3. $\forall n \in \mathbb{N} \setminus \{0\},\ \forall k \in \mathbb{N}, \quad \varphi_k^{(\beta,\rho+n-1)}\left(\frac{\pi}{2}\right) = (-1)^k \dfrac{\varphi_k^{(\alpha,\rho+n-1)}\left(\frac{\pi}{2}\right)}{\varphi_k\left(\frac{\pi}{2}\right)}.$

4. $\forall n \in \mathbb{N},\ \forall \delta > -1, \quad \varphi_n\left(\frac{\pi}{2}\right)\varphi_n^{(\beta,\delta)}\left(\frac{\pi}{2}\right) = (-1)^n \varphi_n^{(\alpha,\delta)}\left(\frac{\pi}{2}\right).$

5. $\forall n \in \mathbb{N},\ \forall k \in \mathbb{N};\ 0 \leq k \leq n, \quad \varphi_{n-k}^{(\alpha,\alpha+k)}\left(\frac{\pi}{2}\right) = (-1)^n \varphi_k^{(\alpha,\alpha+n-k)}\left(\frac{\pi}{2}\right).$

The relationships between the coefficients $a(k,n,\alpha,\beta)$, $b(k,n,\alpha,\beta)$ and $c(k,n,\alpha,\beta)$, $n,k\in\mathbb{N};\ 0\le k\le n$, are given in the following propositions :

Proposition 2.4. *For all $n,k\in\mathbb{N};\ 0\le k\le n$, we have*

$$\begin{aligned}
a(k,n,\alpha,\beta) &= \binom{n}{k}(-1)^k\varphi_k^{(\alpha,\rho+n-1)}\left(\frac{\pi}{2}\right), \quad n\ge 1.\\
&= \frac{2^k}{\varphi_{n-k}^{(\alpha+k,\beta+k)}\left(\frac{\pi}{4}\right)}b(k,n,\alpha,\beta).\\
&= (-1)^{n-k}\varphi_{n-k}^{(\beta+k,\alpha+k)}\left(\frac{\pi}{2}\right)c(k,n,\alpha,\beta).\\
&= 2^k\sum_{p=k}^{n}\binom{p}{k}b(p,n,\alpha,\beta).
\end{aligned}$$

Proof.

$$\forall n\in\mathbb{N},\ \forall p\in\mathbb{N};\ p\le n,\ \forall x\in\mathbb{R},\quad x^p=\sum_{k=0}^{p}\binom{p}{k}(x-1)^k.$$

Using $(15),(7),(16),(17),(20),(21)$ and (22), we get the above formulas. □

Proposition 2.5. *For all $n, k \in \mathbb{N};\ 0 \le k \le n$, we have*

$$\begin{aligned}
b(k,n,\alpha,\beta) &= 2^{-k}\varphi_{n-k}^{(\alpha+k,\beta+k)}\left(\frac{\pi}{4}\right)a(k,n,\alpha,\beta)\\
&= (-1)^{n-k}2^{-k}\varphi_{n-k}^{(\beta+k,\alpha+k)}\left(\frac{\pi}{4}\right)c(k,n,\alpha,\beta)\\
&= \binom{n}{k}2^{-k}(-1)^k\varphi_k^{(\alpha,\rho+n-1)}\left(\frac{\pi}{2}\right)\varphi_{n-k}^{(\alpha+k,\beta+k)}\left(\frac{\pi}{4}\right), \quad n \ge 1.\\
&= 2^{-n}\binom{n}{k}\\
&\times \sum_{p=0}^{n}\varphi_p^{(\alpha,\beta+n-p)}\left(\frac{\pi}{2}\right)\left(\sum_{q=\max\{k-p,0\}}^{\min\{k,n-p\}}(-1)^{k-q}\binom{k}{q}\binom{n-k}{p+q-k}\right).\\
&= 2^{-n}\binom{n}{k}\\
&\times \sum_{p=0}^{n}\varphi_{n-p}^{(\alpha,\beta+p)}\left(\frac{\pi}{2}\right)\left(\sum_{q=\max\{k-(n-p),0\}}^{\min\{k,p\}}(-1)^{k-q}\binom{k}{q}\binom{n-k}{p-q}\right).\\
&= 2^{-n}\binom{n}{k}\\
&\times \sum_{p=0}^{n}\varphi_p^{(\alpha,\beta+n-p)}\left(\frac{\pi}{2}\right)\left(\sum_{q=\max\{k-(n-p),0\}}^{\min\{k,p\}}(-1)^{q}\binom{k}{q}\binom{n-k}{p-q}\right).\\
&= 2^{-n}\binom{n}{k}\\
&\times \sum_{p=0}^{n}\varphi_{n-p}^{(\alpha,\beta+p)}\left(\frac{\pi}{2}\right)\left(\sum_{q=\max\{k-p,0\}}^{\min\{k,n-p\}}(-1)^{q}\binom{k}{q}\binom{n-k}{p+q-k}\right).\\
&= \sum_{p=k}^{n}(-1)^{p-k}2^{-p}\binom{p}{k}a(p,n,\alpha,\beta).\\
&= \sum_{p=k}^{n}(-1)^{n-p}2^{-p}\binom{p}{k}c(p,n,\alpha,\beta).
\end{aligned}$$

Proof. For all $n, p \in \mathbb{N};\ p \le n$ and $x \in \mathbb{R}$, we have

$$(1-x)^p = \sum_{k=0}^{p}(-1)^k\binom{p}{k}x^k, \quad (1+x)^p = \sum_{k=0}^{p}\binom{p}{k}x^k,$$

and

$$(1-x)^p(1+x)^{n-p} = \sum_{k=0}^{n}\left(\sum_{q=\max\{k-p,0\}}^{\min\{k,n-p\}}(-1)^{k-q}\binom{n-p}{q}\binom{p}{k-q}\right)x^k.$$

We complete the proof by using (18), (19), (20), (21), (22) and (7). $\square$

Proposition 2.6. *For all $n, k \in \mathbb{N};\ 0 \le k \le n$, we have*

$$\begin{aligned} c(k,n,\alpha,\beta) &= \binom{n}{k}(-1)^{n-k}\varphi_n\left(\frac{\pi}{2}\right)\varphi_k^{(\beta,\rho+n-1)}\left(\frac{\pi}{2}\right), \quad n \ge 1. \\ &= (-1)^{n-k}\varphi_{n-k}^{(\alpha+k,\beta+k)}\left(\frac{\pi}{2}\right)a(k,n,\alpha,\beta). \\ &= \frac{(-1)^{n-k}2^k}{\varphi_{n-k}^{(\beta+k,\alpha+k)}\left(\frac{\pi}{4}\right)}b(k,n,\alpha,\beta). \\ &= 2^k\sum_{p=k}^{n}(-1)^{n-p}\binom{p}{k}b(p,n,\alpha,\beta). \end{aligned}$$

Proof.

$$\forall n \in \mathbb{N},\ \forall p \in \mathbb{N};\ p \le n,\ \forall x \in \mathbb{R}, \quad x^p = \sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k}(x+1)^k.$$

Using (17), (7), (15), (16), (20), (21) and (22), we get the cited formulas. □

2.2 Explicit formulas of ψ_m

Proposition 2.7. *For all $m \in \mathbb{Z}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have*

1. $\psi_m(0) = 1.$
2. $\psi_{-m}(-\theta) = \psi_m(\theta).$
3. $\overline{\psi_m(\theta)} = \psi_{-m}(\theta) = \psi_m(-\theta).$
4. $\Re\psi_m(\theta) = \dfrac{\psi_m(\theta)+\psi_m(-\theta)}{2} = \varphi_{|m|}(\theta).$
5. $\Im\psi_m(\theta) = \dfrac{\psi_m(\theta)-\psi_m(-\theta)}{2} = \dfrac{\lambda_m}{4(\alpha+1)}\sin(2\theta)\varphi_{|m|-1}^{(\alpha+1,\beta+1)}(\theta), \quad m \ne 0.$
6. $\psi_{2m}^{(\alpha,\alpha)}\left(\dfrac{\theta}{2}\right) = \psi_m^{\left(\alpha,-\frac{1}{2}\right)}(\theta).$
7. $$\psi_m\left(\frac{\pi}{2}\right) = \psi_m\left(-\frac{\pi}{2}\right) = \varphi_{|m|}\left(\frac{\pi}{2}\right) = (-1)^m\frac{\Gamma(\alpha+1)\Gamma(\beta+|m|+1)}{\Gamma(\beta+1)\Gamma(\alpha+|m|+1)}. \tag{23}$$
8. $\psi_m^{(\beta,\alpha)}(\theta) = \dfrac{\psi_m\left(\theta-\frac{\pi}{2}\right)}{\psi_m\left(\frac{\pi}{2}\right)}, \quad 0 \le \theta \le \dfrac{\pi}{2}.$

9. $\psi_m^{(\beta,\alpha)}(\theta) = \dfrac{\psi_m\left(\theta + \frac{\pi}{2}\right)}{\psi_m\left(\frac{\pi}{2}\right)}, \quad -\dfrac{\pi}{2} \leq \theta < 0.$

Proof. We use $(4), (1), (9), (7)$ and (8), to get the above formulas. □

Remarks 2.8.

1. $\forall m \in \mathbb{Z}, \quad \overline{\psi_m\left(\dfrac{\pi}{2}\right)} = \psi_{-m}\left(\dfrac{\pi}{2}\right) = \psi_m\left(-\dfrac{\pi}{2}\right) = \psi_{-m}\left(-\dfrac{\pi}{2}\right) = \psi_m\left(\dfrac{\pi}{2}\right).$

2. $\forall m \in \mathbb{Z}, \quad \psi_m^{(\beta,\alpha)}\left(\dfrac{\pi}{2}\right) = \left(\psi_m\left(\dfrac{\pi}{2}\right)\right)^{-1}.$

Explicit forms of $\psi_m,\ m \in \mathbb{Z} \setminus \{0\}$, are given in the following theorem :

Theorem 2.9. *For all $m \in \mathbb{Z} \setminus \{0\}$ and $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we have*

1.

$$
\begin{aligned}
\psi_m(\theta) &= \Gamma(\alpha+1)\Gamma(\beta+|m|+1)\sum_{k=0}^{|m|}\frac{(-1)^k\binom{|m|}{k}}{\Gamma(\alpha+k+1)\Gamma(\beta+|m|-k+1)} \\
&\times \left[1 - i\frac{2}{\lambda_m}[k\cot\theta - (|m|-k)\tan\theta]\right](\sin\theta)^{2k}(\cos\theta)^{2(|m|-k)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta \neq 0, \pm\frac{\pi}{2}. \\
&= \psi_m\left(\frac{\pi}{2}\right)(\sin\theta)^{2|m|} + \Gamma(\alpha+1)\Gamma(\beta+|m|+1) \\
&\times \sum_{k=0}^{|m|-1}\frac{(-1)^k\binom{|m|}{k}}{\Gamma(\alpha+k+1)\Gamma(\beta+|m|-k+1)}(\sin\theta)^{2k}(\cos\theta)^{2(|m|-k)} \\
&\times \left[1 + i\frac{\lambda_m(|m|-k)}{2|m|(k+\alpha+1)}\tan\theta\right], \quad \theta \neq \pm\frac{\pi}{2}. \\
&= (\cos\theta)^{2|m|} + \Gamma(\alpha+1)\Gamma(\beta+|m|+1) \\
&\times \sum_{k=1}^{|m|}\frac{(-1)^k\binom{|m|}{k}}{\Gamma(\alpha+k+1)\Gamma(\beta+|m|-k+1)}(\sin\theta)^{2k}(\cos\theta)^{2(|m|-k)} \\
&\times \left[1 - i\frac{\lambda_m k}{2|m|(|m|-k+\beta+1)}\cot\theta\right], \quad \theta \neq 0.
\end{aligned}
$$

2.

$$\begin{aligned}
\psi_m(\theta) &= (-1)^{|m|}\Gamma(\alpha+1)\Gamma(\beta+|m|+1) \\
&\times \sum_{k=0}^{|m|} \frac{(-1)^k\binom{|m|}{k}}{\Gamma(\alpha+|m|-k+1)\Gamma(\beta+k+1)}(\sin\theta)^{2(|m|-k)}(\cos\theta)^{2k} \\
&\times \left[1-i\frac{2}{\lambda_m}[(|m|-k)\cot\theta-k\tan\theta]\right], \quad \theta\neq 0, \pm\frac{\pi}{2}. \\
&= \psi_m\left(\frac{\pi}{2}\right)(\sin\theta)^{2|m|}+(-1)^{|m|}\Gamma(\alpha+1)\Gamma(\beta+|m|+1) \\
&\times \sum_{k=1}^{|m|} \frac{(-1)^k\binom{|m|}{k}}{\Gamma(\alpha+|m|-k+1)\Gamma(\beta+k+1)}(\sin\theta)^{2(|m|-k)}(\cos\theta)^{2k} \\
&\times \left[1+i\frac{\lambda_m k}{2|m|(|m|-k+\alpha+1)}\tan\theta\right], \quad \theta\neq \pm\frac{\pi}{2}. \\
&= (\cos\theta)^{2|m|}+(-1)^{|m|}\Gamma(\alpha+1)\Gamma(\beta+|m|+1) \\
&\times \sum_{k=0}^{|m|-1} \frac{(-1)^k\binom{|m|}{k}}{\Gamma(\alpha+|m|-k+1)\Gamma(\beta+k+1)}(\sin\theta)^{2(|m|-k)}(\cos\theta)^{2k} \\
&\times \left[1-i\frac{\lambda_m(|m|-k)}{2|m|(k+\beta+1)}\cot\theta\right], \quad \theta\neq 0.
\end{aligned}$$

3.

$$\begin{aligned}
\psi_m(\theta) &= 1+\sum_{k=1}^{|m|}(-1)^k a(k,|m|,\alpha,\beta)\left(1-i\frac{2k}{\lambda_m}\cot\theta\right)(\sin\theta)^{2k}, \\
&\qquad\qquad \theta\neq 0. \\
&= (-1)^{|m|}\frac{\Gamma(\alpha+1)\Gamma(\rho+2|m|)}{\Gamma(\alpha+|m|+1)\Gamma(\rho+|m|)}(\sin\theta)^{2|m|} \\
&+ \sum_{k=0}^{|m|-1}(-1)^k a(k,|m|,\alpha,\beta)(\sin\theta)^{2k} \\
&\times \left[1+i\frac{(|m|-k)(|m|+k+\rho)}{\lambda_m(k+\alpha+1)}\sin(2\theta)\right].
\end{aligned}$$

4.

$$\begin{aligned}
\psi_m(\theta) &= \varphi_{|m|}\left(\frac{\pi}{4}\right)+\sum_{k=1}^{|m|} b(k,|m|,\alpha,\beta)\left(1+i\frac{2k}{\lambda_m}\tan(2\theta)\right)(\cos(2\theta))^k, \\
&\qquad\qquad \theta\neq\pm\frac{\pi}{4}. \\
&= \frac{\Gamma(\alpha+1)\Gamma(\rho+2|m|)}{2^{|m|}\Gamma(\alpha+|m|+1)\Gamma(\rho+|m|)}(\cos(2\theta))^{|m|} \\
&+ \sum_{k=0}^{|m|-1}\left[b(k,|m|,\alpha,\beta)+i\frac{2(k+1)b(k+1,|m|,\alpha,\beta)}{\lambda_m}\sin(2\theta)\right] \\
&\times (\cos(2\theta))^k.
\end{aligned}$$

5.

$$\begin{aligned}\psi_m(\theta) &= \frac{\Gamma(\alpha+1)\Gamma(\rho+2|m|)}{2^{|m|}\Gamma(\alpha+|m|+1)\Gamma(\rho+|m|)}(\cos(2\theta))^{|m|} \\ &+ \sum_{k=0}^{|m|-1}\frac{a(k,|m|,\alpha,\beta)}{2^k}(\cos(2\theta))^k \\ &\times \left[{}_2F_1\left(-(|m|-k),|m|+k+\rho;\alpha+k+1;\frac{1}{2}\right)\right. \\ &+ i\frac{(|m|-k)(|m|+k+\rho)}{\lambda_m(k+\alpha+1)}\sin(2\theta) \\ &\times \left.{}_2F_1\left(-(|m|-k)+1,|m|+k+\rho+1;\alpha+k+2;\frac{1}{2}\right)\right].\end{aligned}$$

6.

$$\begin{aligned}\psi_m(\theta) &= (-1)^m+\sum_{k=1}^{|m|}(-1)^{|m|-k}c(k,|m|,\alpha,\beta)\left(1+i\frac{2k}{\lambda_m}\tan\theta\right) \\ &\times (\cos\theta)^{2k}, \quad \theta\neq\pm\frac{\pi}{2}. \\ &= \frac{\Gamma(\alpha+1)\Gamma(\rho+2|m|)}{\Gamma(\alpha+|m|+1)\Gamma(\rho+|m|)}(\cos\theta)^{2|m|} \\ &+ \sum_{k=0}^{|m|-1}(-1)^{|m|-k}c(k,|m|,\alpha,\beta)(\cos\theta)^{2k} \\ &\times \left[1-i\frac{(|m|-k)(|m|+k+\rho)}{\lambda_m(k+\beta+1)}\sin(2\theta)\right].\end{aligned}$$

7.

$$\begin{aligned}\psi_m(\theta) &= \sum_{k=-|m|}^{|m|}\left(\sum_{p=|k|}^{|m|}(-1)^{p-|k|}\binom{2p}{p+|k|}2^{-2p}a(p,|m|,\alpha,\beta)\right) \\ &\times \left(1+\frac{2k}{\lambda_m}\right)e^{2ik\theta}. \\ &= \sum_{k=-|m|}^{|m|}\left(\sum_{p=|k|}^{|m|}\frac{p!\,b(p,|m|,\alpha,\beta)}{2^p\Gamma\left(\frac{p+k}{2}+1\right)\Gamma\left(\frac{p-k}{2}+1\right)}\right)\left(1+\frac{2k}{\lambda_m}\right)e^{2ik\theta}. \\ &= \sum_{k=-|m|}^{|m|}\left(\sum_{p=|k|}^{|m|}(-1)^{|m|-p}\binom{2p}{p+|k|}2^{-2p}c(p,|m|,\alpha,\beta)\right) \\ &\times \left(1+\frac{2k}{\lambda_m}\right)e^{2ik\theta},\end{aligned}$$

where $a(k,n,\alpha,\beta)$, $b(k,n,\alpha,\beta)$ *and* $c(k,n,\alpha,\beta)$, $n\in\mathbb{N}$, $0\leq k\leq n$, *are respectively given by* (15), (16) *and* (17).

Proof. We obtain the above results from (1), (18), (19), (20), (21) and (22). □

Remark 2.10. *The function ψ_m, $m \in \mathbb{Z}$, is a trigonometric polynomial in θ of degree $2|m|$, and of degree $|m|$ as a polynomial in 2θ.*

Proposition 2.11. *For all $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$, we have*

1. $\dfrac{d^{2n}}{d\theta^{2n}}\psi_m(0) = \dfrac{d^{2n}}{d\theta^{2n}}\varphi_{|m|}(0)$.
2. $\dfrac{d^{2n+1}}{d\theta^{2n+1}}\psi_m(0) = i\dfrac{(-1)^n 2^{2n-1}\lambda_m}{\alpha+1}\displaystyle\sum_{k=0}^{n}(-1)^k 2^{-2k}\binom{2n+1}{2k}\dfrac{d^{2k}}{d\theta^{2k}}\varphi_{|m|-1}^{(\alpha+1,\beta+1)}(0)$.

Proof. Let $m \in \mathbb{Z} \setminus \{0\}$, $\theta \in \left[0, \dfrac{\pi}{2}\right]$ and $n \in \mathbb{N}$. The equalities

$$\frac{d^n}{d\theta^n}\psi_m(\theta) = \frac{d^n}{d\theta^n}\varphi_{|m|}(\theta) + i\frac{\lambda_m}{4(\alpha+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{d\theta^{n-k}}[\sin(2\theta)]\frac{d^k}{d\theta^k}\varphi_{|m|-1}^{(\alpha+1,\beta+1)}(\theta),$$

$$\frac{d^{n-k}}{d\theta^{n-k}}[\sin(2\theta)] = 2^{n-k}\sin\left[2\theta + (n-k)\frac{\pi}{2}\right],$$

and $\sin\left[(n-k)\dfrac{\pi}{2}\right] = \dfrac{1-(-1)^{n-k}}{2}(-1)^{\left[\frac{n-k-1}{2}\right]}$, give the above formulas. □

Examples 2.12. *Let $m \in \mathbb{Z}$ and $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.*

1.

$$\begin{aligned}
\psi_0(\theta) &= 1.\\
\psi_1(\theta) &= 1 - \frac{\rho+1}{\alpha+1}(\sin\theta)^2 + i\frac{\sqrt{\rho+1}}{2(\alpha+1)}\sin(2\theta).\\
\psi_2(\theta) &= \frac{(\rho+2)(\rho+3)}{(\alpha+1)(\alpha+2)}(\sin\theta)^4 - \frac{2(\rho+2)}{\alpha+1}(\sin\theta)^2 + 1\\
&+ i\frac{\sqrt{2(\rho+2)}}{2(\alpha+1)(\alpha+2)}\sin(2\theta)\left[\alpha+2-(\rho+3)(\sin\theta)^2\right].
\end{aligned}$$

2. $\psi_m^{(-\frac{1}{2},-\frac{1}{2})}(\theta) = e^{2im\theta}$.

3. *If $m \neq 0$ and $\theta \neq 0$, then we have*

$$\begin{aligned}
\psi_m^{(\frac{1}{2},-\frac{1}{2})}(\theta) &= \frac{\sin((2|m|+1)\theta)}{(2|m|+1)\sin\theta} + \frac{i}{2(2|m|+1)\sqrt{|m|(|m|+1)}}\\
&\times \Big[\left(2|m|+1+(\cot\theta)^2\right)\sin(2m\theta) - 2m\cot\theta\cos(2m\theta)\Big].
\end{aligned}$$

4. *If* $m \neq 0$ *and* $\theta \neq \pm\dfrac{\pi}{2}$, *then we have*

$$\begin{aligned}\psi_m^{(-\frac{1}{2},\frac{1}{2})}(\theta) &= \frac{\cos((2|m|+1)\theta)}{\cos\theta} + \frac{i}{2\sqrt{|m|(|m|+1)}} \\ &\times \Big[\big(2|m|+1+(\tan\theta)^2\big)\sin(2m\theta) + 2m\tan\theta\cos(2m\theta)\Big].\end{aligned}$$

5. *If* $m \neq 0$ *and* $\theta \neq 0, \pm\dfrac{\pi}{2}$, *then we have*

$$\begin{aligned}\psi_m^{(\frac{1}{2},\frac{1}{2})}(\theta) &= \frac{\sin(2(|m|+1)\theta)}{(|m|+1)\sin(2\theta)} + \frac{i}{(|m|+1)\sqrt{|m|(|m|+2)}} \\ &\times \Big[\big(|m|+1+(\cot(2\theta))^2\big)\sin(2m\theta) - m\cot(2\theta)\cos(2m\theta)\Big].\end{aligned}$$

3 Recurrence relations

3.1 Recurrence relations of φ_n

From [12], the following formulas are given

$$\forall\theta \in \left[0, \frac{\pi}{2}\right], \quad \cos(2\theta)\varphi_0(\theta) = a_0\varphi_1(\theta) + b_0\varphi_0(\theta),$$

with

$$\begin{cases} a_0 = a_0^{(\alpha,\beta)} := \dfrac{2(\alpha+1)}{\rho+1}, \\ b_0 = b_0^{(\alpha,\beta)} := \dfrac{\beta-\alpha}{\rho+1}. \end{cases} \tag{24}$$

For all $n \in \mathbb{N} \setminus \{0\}$ and $\theta \in \left[0, \dfrac{\pi}{2}\right]$, we have

$$\begin{aligned}\varphi_{n+1}(\theta) &= \frac{(2n+\rho)[(2n+\rho+1)(2n+\rho-1)\cos(2\theta)+\alpha^2-\beta^2]}{2(n+\alpha+1)(n+\rho)(2n+\rho-1)}\varphi_n(\theta) \\ &- \frac{n(n+\beta)(2n+\rho+1)}{(n+\alpha+1)(n+\rho)(2n+\rho-1)}\varphi_{n-1}(\theta). \\ \cos(2\theta)\varphi_n(\theta) &= a_n\varphi_{n+1}(\theta) + b_n\varphi_n(\theta) + c_n\varphi_{n-1}(\theta), \end{aligned} \tag{25}$$

with

$$\begin{cases} a_n = a_n^{(\alpha,\beta)} := \dfrac{2(n+\alpha+1)(n+\rho)}{(2n+\rho)(2n+\rho+1)}, \\ b_n = b_n^{(\alpha,\beta)} := \dfrac{\beta^2-\alpha^2}{(2n+\rho)^2-1}, \\ c_n = c_n^{(\alpha,\beta)} := \dfrac{2n(n+\beta)}{(2n+\rho-1)(2n+\rho)}. \end{cases} \tag{26}$$

$$\frac{(\sin(2\theta))^2}{4(\alpha+1)}\varphi_{n-1}^{(\alpha+1,\beta+1)}(\theta) = a'_n\varphi_{n+1}(\theta) + b'_n\varphi_n(\theta) + c'_n\varphi_{n-1}(\theta), \tag{27}$$

with

$$\left\{\begin{array}{lll} a'_n = a_n^{(\alpha,\beta)\,\prime} := & -\dfrac{n+\alpha+1}{(2n+\rho)(2n+\rho+1)}, \\ b'_n = b_n^{(\alpha,\beta)\,\prime} := & \dfrac{\alpha-\beta}{(2n+\rho)^2-1}, \\ c'_n = c_n^{(\alpha,\beta)\,\prime} := & \dfrac{n+\beta}{(2n+\rho-1)(2n+\rho)}. \end{array}\right. \tag{28}$$

$$\begin{aligned} \sin(2\theta)\frac{d}{d\theta}\varphi_n(\theta) &= -\frac{2n[\alpha-\beta-(2n+\rho-1)\cos(2\theta)]}{2n+\rho-1}\varphi_n(\theta) \\ &- \frac{4n(n+\beta)}{2n+\rho-1}\varphi_{n-1}(\theta). \qquad (29) \\ (\sin\theta)^2\varphi_n^{(\alpha+1,\beta)}(\theta) &= \frac{(\alpha+1)}{2n+\rho+1}(\varphi_n(\theta)-\varphi_{n+1}(\theta)), \quad n\in\mathbb{N}. \qquad (30) \\ (\cos\theta)^2\varphi_n^{(\alpha,\beta+1)}(\theta) &= \frac{n+\alpha+1}{2n+\rho+1}\varphi_{n+1}(\theta)) + \frac{n+\beta+1}{2n+\rho+1}\varphi_n(\theta), \quad n\in\mathbb{N}. \\ \varphi_n(\theta) &= \frac{n+\alpha+1}{\alpha+1}(\sin\theta)^2\varphi_n^{(\alpha+1,\beta)}(\theta) + (\cos\theta)^2\varphi_n^{(\alpha,\beta+1)}(\theta), \\ &\qquad n\in\mathbb{N}. \\ \varphi_n^{(\alpha-1,\beta)}(\theta) &= \frac{(n+\alpha)(n+\rho-1)}{\alpha(2n+\rho-1)}\varphi_n(\theta) - \frac{n(n+\beta)}{\alpha(2n+\rho-1)}\varphi_{n-1}(\theta), \\ &\qquad \alpha>0. \\ \varphi_n^{(\alpha,\beta-1)}(\theta) &= \frac{n+\rho-1}{2n+\rho-1}\varphi_n(\theta) + \frac{n}{2n+\rho-1}\varphi_{n-1}(\theta), \quad \beta>0. \\ \varphi_n^{(\alpha,\beta-1)}(\theta) &= \frac{n}{n+\alpha}\varphi_{n-1}(\theta) + \frac{\alpha}{n+\alpha}\varphi_n^{(\alpha-1,\beta)}(\theta), \quad \alpha>0,\ \beta>0. \end{aligned}$$

Remarks 3.1.

1. $\forall\theta\in\left[0,\dfrac{\pi}{2}\right]$,

$$\varphi_1(\theta)\varphi_0(\theta) = \varphi_1(\theta) = \frac{\rho+1}{2(\alpha+1)}a_0\varphi_1(\theta) + \frac{(\rho+1)b_0+\alpha-\beta}{2(\alpha+1)}\varphi_0(\theta).$$

2. $\forall n\in\mathbb{N}\setminus\{0\},\ \forall\theta\in\left[0,\dfrac{\pi}{2}\right]$,

$$\begin{aligned} \varphi_1(\theta)\varphi_n(\theta) &= \frac{\rho+1}{2(\alpha+1)}a_n\varphi_{n+1}(\theta) + \frac{(\rho+1)b_n+\alpha-\beta}{2(\alpha+1)}\varphi_n(\theta) \\ &+ \frac{\rho+1}{2(\alpha+1)}c_n\varphi_{n-1}(\theta). \end{aligned}$$

3. $\forall x \in \mathbb{R}, \quad R_1^{(\alpha,\beta)}(x) R_0^{(\alpha,\beta)}(x) = R_1^{(\alpha,\beta)}(x)$

$$= \frac{\rho+1}{2(\alpha+1)} a_0 R_1^{(\alpha,\beta)}(x) + \frac{(\rho+1)b_0 + \alpha - \beta}{2(\alpha+1)} R_0^{(\alpha,\beta)}(x).$$

4. $\forall n \in \mathbb{N} \setminus \{0\}, \ \forall x \in \mathbb{R}, \quad R_1^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x)$

$$= \frac{\rho+1}{2(\alpha+1)} a_n R_{n+1}^{(\alpha,\beta)}(x) + \frac{(\rho+1)b_n + \alpha - \beta}{2(\alpha+1)} R_n^{(\alpha,\beta)}(x) + \frac{\rho+1}{2(\alpha+1)} c_n R_{n-1}^{(\alpha,\beta)}(x).$$

5. $a_0 + b_0 = 1.$

6. $\forall n \in \mathbb{N} \setminus \{0\}, \quad a_n + b_n + c_n = 1.$

7. $\dfrac{\rho+1}{2(\alpha+1)} a_0 + \dfrac{(\rho+1)b_0 + \alpha - \beta}{2(\alpha+1)} = 1.$

8. $\forall n \in \mathbb{N} \setminus \{0\}, \quad \dfrac{\rho+1}{2(\alpha+1)} a_n + \dfrac{(\rho+1)b_n + \alpha - \beta}{2(\alpha+1)} + \dfrac{\rho+1}{2(\alpha+1)} c_n = 1.$

9. $\forall n \in \mathbb{N} \setminus \{0\}, \quad a'_n + b'_n + c'_n = 0,$

10. $\forall n \in \mathbb{N} \setminus \{0\},$

$$\begin{aligned} a_n &= -2(n+\rho)a'_n. \\ b_n &= (1-\rho)b'_n = \frac{\alpha-\beta}{2n+\rho-1} - 2(n+\rho)b'_n. \\ c_n &= 2nc'_n = \frac{2(n+\beta)}{2n+\rho-1} - 2(n+\rho)c'_n. \end{aligned}$$

where $a_0, b_0, a_n, b_n, c_n, a'_n, b'_n$ *and* c'_n *are respectively given by* (24), (26) *and* (28).

3.2 Recurrence relations of ψ_m

Proposition 3.2. *For all* $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *we have*

1.
$$\cos(2\theta)\psi_0(\theta) = r_0\psi_1(\theta) + s''_0\psi_0(\theta) + w_0\psi_{-1}(\theta),$$
where $r_0 = w_0 := \dfrac{\alpha+1}{\rho+1}$ *and* $s''_0 := \dfrac{\beta-\alpha}{\rho+1}$.

2.
$$\sin(2\theta)\psi_0(\theta) = r''_0\psi_1(\theta) + s'_0\psi_0(\theta) + w''_0\psi_{-1}(\theta),$$
where $r''_0 = -w''_0 := -i\dfrac{(\alpha+1)\sqrt{\rho+1}}{\rho+1}$ *and* $s'_0 := 0.$

3.

$$e^{2i\theta}\psi_0(\theta) = (r_0 + ir''_0)\psi_1(\theta) + (s''_0 + is'_0)\psi_0(\theta) + (w_0 + iw''_0)\psi_{-1}(\theta).$$

4.

$$\begin{aligned}\cos(2\theta)\psi_1(\theta) &= r_1\psi_2(\theta) + s_1\psi_1(\theta) + (t_1+u_1)\psi_0(\theta) + v_1\psi_{-1}(\theta)\\ &+ w_1\psi_{-2}(\theta).\\ \cos(2\theta)\psi_{-1}(\theta) &= (r_{-1}+w_{-1})\psi_0(\theta) + s_{-1}\psi_{-1}(\theta) + t_{-1}\psi_{-2}(\theta)\\ &+ u_{-1}\psi_2(\theta) + v_{-1}\psi_1(\theta),\end{aligned}$$

where

$$\begin{aligned} r_1 = t_{-1} &:= \frac{(\alpha+2)\left[2(\rho+1)+\sqrt{2(\rho+1)(\rho+2)}\right]}{2(\rho+2)(\rho+3)},\\ s_1 = s_{-1} &:= \frac{\rho(\beta-\alpha)}{(\rho+1)(\rho+3)},\\ t_1 = u_1 = r_{-1} = w_{-1} &:= \frac{\beta+1}{(\rho+1)(\rho+2)},\\ v_1 = v_{-1} &:= \frac{\alpha-\beta}{(\rho+1)(\rho+3)},\\ w_1 = u_{-1} &:= \frac{(\alpha+2)\left[2(\rho+1)-\sqrt{2(\rho+1)(\rho+2)}\right]}{2(\rho+2)(\rho+3)}.\end{aligned}$$

5.

$$\begin{aligned}\sin(2\theta)\psi_1(\theta) &= r'_1\psi_2(\theta) + s'_1\psi_1(\theta) + (t'_1+u'_1)\psi_0(\theta) + v'_1\psi_{-1}(\theta)\\ &+ w'_1\psi_{-2}(\theta).\\ \sin(2\theta)\psi_{-1}(\theta) &= (r'_{-1}+w'_{-1})\psi_0(\theta) + s'_{-1}\psi_{-1}(\theta) + t'_{-1}\psi_{-2}(\theta)\\ &+ u'_{-1}\psi_2(\theta) + v'_{-1}\psi_1(\theta),\end{aligned}$$

where

$$\begin{aligned} r'_1 = -t'_{-1} &:= -i\frac{(\alpha+2)\sqrt{\rho+1}}{2(\rho+2)(\rho+3)}\left(2+\sqrt{2(\rho+1)(\rho+2)}\right),\\ s'_1 = -s'_{-1} &:= 0,\\ t'_1 = u'_1 = -r'_{-1} = -w'_{-1} &:= i\frac{(\beta+1)\sqrt{\rho+1}}{(\rho+1)(\rho+2)},\\ v'_1 = -v'_{-1} &:= i\frac{2(\alpha-\beta)\sqrt{\rho+1}}{(\rho+1)(\rho+3)},\\ w'_1 = -u'_{-1} &:= i\frac{(\alpha+2)\sqrt{\rho+1}}{2(\rho+2)(\rho+3)}\left(\sqrt{2(\rho+1)(\rho+2)}-2\right).\end{aligned}$$

6.

$$\begin{array}{rcl} e^{2i\theta}\psi_1(\theta) & = & (r_1 + ir'_1)\psi_2(\theta) + (s_1 + is'_1)\psi_1(\theta) \\ & + & [(t_1 + u_1) + i(t'_1 + u'_1)]\psi_0(\theta) + (v_1 + iv'_1)\psi_{-1}(\theta) \\ & + & (w_1 + iw'_1)\psi_{-2}(\theta). \\ e^{2i\theta}\psi_{-1}(\theta) & = & [(r_{-1} + w_{-1}) + i(r'_{-1} + w'_{-1})]\psi_0(\theta) + (s_{-1} + is'_{-1})\psi_{-1}(\theta) \\ & + & (t_{-1} + it'_{-1})\psi_{-2}(\theta) + (u_{-1} + iu'_{-1})\psi_2(\theta) \\ & + & (v_{-1} + iv'_{-1})\psi_1(\theta). \end{array}$$

Theorem 3.3. *For all* $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *we have*

1. $\forall n \in \mathbb{N} \setminus \{0, 1\}$,

$$\begin{array}{rcl} \cos(2\theta)\psi_n(\theta) & = & r_n\psi_{n+1}(\theta) + s_n\psi_n(\theta) + t_n\psi_{n-1}(\theta) \\ & + & u_n\psi_{-n+1}(\theta) + v_n\psi_{-n}(\theta) + w_n\psi_{-n-1}(\theta). \\ \cos(2\theta)\psi_{-n}(\theta) & = & r_{-n}\psi_{-n+1}(\theta) + s_{-n}\psi_{-n}(\theta) + t_{-n}\psi_{-n-1}(\theta) \\ & + & u_{-n}\psi_{n+1}(\theta) + v_{-n}\psi_n(\theta) + w_{-n}\psi_{n-1}(\theta), \end{array}$$

where

$$\begin{array}{rcl} r_n = t_{-n} & := & \dfrac{(n+\alpha+1)[4(n+1)(n+\rho) + \lambda_n\lambda_{n+1}]}{4(n+1)(2n+\rho)(2n+\rho+1)}, \\ s_n = s_{-n} & := & \dfrac{\rho(\beta-\alpha)}{(2n+\rho)^2 - 1}, \\ t_n = r_{-n} & := & \dfrac{(n+\beta)[4n(n+\rho-1) + \lambda_n\lambda_{n-1}]}{4(n+\rho-1)(2n+\rho)(2n+\rho-1)}, \\ u_n = w_{-n} & := & \dfrac{(n+\beta)[4n(n+\rho-1) - \lambda_n\lambda_{n-1}]}{4(n+\rho-1)(2n+\rho)(2n+\rho-1)}, \\ v_n = v_{-n} & := & \dfrac{\alpha-\beta}{(2n+\rho)^2 - 1}, \\ w_n = u_{-n} & := & \dfrac{(n+\alpha+1)[4(n+1)(n+\rho) - \lambda_n\lambda_{n+1}]}{4(n+1)(2n+\rho)(2n+\rho+1)}. \end{array}$$

2. $\forall n \in \mathbb{N} \setminus \{0, 1\}$,

$$\begin{array}{rcl} \sin(2\theta)\psi_n(\theta) & = & r'_n\psi_{n+1}(\theta) + s'_n\psi_n(\theta) + t'_n\psi_{n-1}(\theta) \\ & + & u'_n\psi_{-n+1}(\theta) + v'_n\psi_{-n}(\theta) + w'_n\psi_{-n-1}(\theta). \\ \sin(2\theta)\psi_{-n}(\theta) & = & r'_{-n}\psi_{-n+1}(\theta) + s'_{-n}\psi_{-n}(\theta) + t'_{-n}\psi_{-n-1}(\theta) \\ & + & u'_{-n}\psi_{n+1}(\theta) + v'_{-n}\psi_n(\theta) + w'_{-n}\psi_{n-1}(\theta), \end{array}$$

where

$$\begin{array}{rcl} r'_n = -t'_{-n} & := & -i\lambda_n \dfrac{(n+\alpha+1)[4n(n+1)+\lambda_n\lambda_{n+1}]}{8n(n+1)(2n+\rho)(2n+\rho+1)}, \\ s'_n = -s'_{-n} & := & 0, \\ t'_n = -r'_{-n} & := & i\lambda_n \dfrac{(n+\beta)[4(n+\rho)(n+\rho-1)+\lambda_n\lambda_{n-1}]}{8(n+\rho)(n+\rho-1)(2n+\rho)(2n+\rho-1)}, \\ u'_n = -w'_{-n} & := & i\lambda_n \dfrac{(n+\beta)[4(n+\rho)(n+\rho-1)-\lambda_n\lambda_{n-1}]}{8(n+\rho)(n+\rho-1)(2n+\rho)(2n+\rho-1)}, \\ v'_n = -v'_{-n} & := & i\lambda_n \dfrac{\alpha-\beta}{(2n+\rho)^2-1}, \\ w'_n = -u'_{-n} & := & i\lambda_n \dfrac{(n+\alpha+1)[\lambda_n\lambda_{n+1}-4n(n+1)]}{8n(n+1)(2n+\rho)(2n+\rho+1)}. \end{array}$$

3. $\forall m \in \mathbb{Z} \setminus \{-1,0,1\}$,

$$\begin{array}{rcl} e^{2i\theta}\psi_m(\theta) & = & (r_m + ir'_m)\psi_{m+1}(\theta) + (s_m + is'_m)\psi_m(\theta) \\ & + & (t_m + it'_m)\psi_{m-1}(\theta) + (u_m + iu'_m)\psi_{-m+1}(\theta) \\ & + & (v_m + iv'_m)\psi_{-m}(\theta) + (w_m + iw'_m)\psi_{-m-1}(\theta). \end{array}$$

Proof. Let $n \in \mathbb{N} \setminus \{0,1\}$ and $k \in \mathbb{Z} \setminus \{0\}$.

$$\begin{array}{rl} & \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta)\psi_n(\theta)\overline{\psi_k(\theta)}A(\theta)\,d\theta = 2\int_0^{\frac{\pi}{2}} \cos(2\theta)\varphi_n(\theta)\varphi_{|k|}(\theta)A(\theta)\,d\theta \\ + & \displaystyle\frac{\lambda_n\lambda_k}{32(\alpha+1)^2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\varphi_{n-1}^{(\alpha+1,\beta+1)}(\theta)\varphi_{|k|-1}^{(\alpha+1,\beta+1)}(\theta)4(\sin(2\theta))^2A(\theta)\,d\theta \\ = & \displaystyle 2\int_0^{\frac{\pi}{2}} \cos(2\theta)\varphi_n(\theta)\varphi_{|k|}(\theta)A(\theta)\,d\theta \\ + & \displaystyle\frac{\lambda_n\lambda_k}{32(\alpha+1)^2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\varphi_{n-1}^{(\alpha+1,\beta+1)}(\theta)\varphi_{|k|-1}^{(\alpha+1,\beta+1)}(\theta)A_{\alpha+1,\beta+1}(\theta)\,d\theta. \end{array}$$

Since $\cos(2\theta)\psi_n(\theta)$ is a trigonometric polynomial and by (25), (26), (10) and (13), we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta)\psi_n(\theta)\overline{\psi_k(\theta)}A(\theta)\,d\theta = 0, \quad |k| \notin \{n+1, n, n-1\},$$

and

$$\begin{array}{rcl} \cos(2\theta)\psi_n(\theta) & = & r_n\psi_{n+1}(\theta) + s_n\psi_n(\theta) + t_n\psi_{n-1}(\theta) \\ & + & u_n\psi_{-n+1}(\theta) + v_n\psi_{-n}(\theta) + w_n\psi_{-n-1}(\theta), \end{array}$$

with

$$\begin{array}{rcl}
2a_nk_{n+1}^{-1} + \dfrac{\lambda_n\lambda_{n+1}}{32(\alpha+1)^2}a_{n-1}^{(\alpha+1,\beta+1)}\left(k_n^{(\alpha+1,\beta+1)}\right)^{-1} & = & r_nh_{n+1}^{-1},\\
2a_nk_{n+1}^{-1} - \dfrac{\lambda_n\lambda_{n+1}}{32(\alpha+1)^2}a_{n-1}^{(\alpha+1,\beta+1)}\left(k_n^{(\alpha+1,\beta+1)}\right)^{-1} & = & w_nh_{n+1}^{-1},\\
2b_nk_n^{-1} + \dfrac{(\lambda_n)^2}{32(\alpha+1)^2}b_{n-1}^{(\alpha+1,\beta+1)}\left(k_{n-1}^{(\alpha+1,\beta+1)}\right)^{-1} & = & s_nh_n^{-1},\\
2b_nk_n^{-1} - \dfrac{(\lambda_n)^2}{32(\alpha+1)^2}b_{n-1}^{(\alpha+1,\beta+1)}\left(k_{n-1}^{(\alpha+1,\beta+1)}\right)^{-1} & = & v_nh_n^{-1},\\
2c_nk_{n-1}^{-1} + \dfrac{\lambda_n\lambda_{n-1}}{32(\alpha+1)^2}c_{n-1}^{(\alpha+1,\beta+1)}\left(k_{n-2}^{(\alpha+1,\beta+1)}\right)^{-1} & = & t_nh_{n-1}^{-1},\\
2c_nk_{n-1}^{-1} - \dfrac{\lambda_n\lambda_{n-1}}{32(\alpha+1)^2}c_{n-1}^{(\alpha+1,\beta+1)}\left(k_{n-2}^{(\alpha+1,\beta+1)}\right)^{-1} & = & u_nh_{n-1}^{-1}.
\end{array}$$

We use (14), (26) and (11) to get the expressions of r_n, s_n, t_n, u_n, v_n and w_n.

$$\begin{array}{rl}
& \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(2\theta)\psi_n(\theta)\overline{\psi_k(\theta)}A(\theta)\,d\theta\\
= & i\dfrac{\lambda_n}{2(\alpha+1)}\displaystyle\int_0^{\frac{\pi}{2}}(\sin(2\theta))^2\varphi_{n-1}^{(\alpha+1,\beta+1)}(\theta)\varphi_{|k|}(\theta)A(\theta)\,d\theta\\
- & i\dfrac{\lambda_k}{2(\alpha+1)}\displaystyle\int_0^{\frac{\pi}{2}}(\sin(2\theta))^2\varphi_{|k|-1}^{(\alpha+1,\beta+1)}(\theta)\varphi_n(\theta)A(\theta)\,d\theta.
\end{array}$$

Since $\sin(2\theta)\psi_n(\theta)$ is a trigonometric polynomial and by (27), (28), (10) and (13), we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(2\theta)\psi_n(\theta)\overline{\psi_k(\theta)}A(\theta)\,d\theta = 0, \quad |k| \notin \{n+1, n, n-1\},$$

and

$$\begin{array}{rcl}
\sin(2\theta)\psi_n(\theta) & = & r'_n\psi_{n+1}(\theta) + s'_n\psi_n(\theta) + t'_n\psi_{n-1}(\theta)\\
& + & u'_n\psi_{-n+1}(\theta) + v'_n\psi_{-n}(\theta) + w'_n\psi_{-n-1}(\theta),
\end{array}$$

with

$$\begin{array}{rcl}
2i\lambda_na'_nk_{n+1}^{-1} - 2i\lambda_{n+1}c'_{n+1}k_n^{-1} & = & r'_nh_{n+1}^{-1},\\
2i\lambda_na'_nk_{n+1}^{-1} - 2i\lambda_{n+1}c'_{n+1}k_n^{-1} & = & w'_nh_{n+1}^{-1},\\
2i\lambda_nb'_nk_n^{-1} - 2i\lambda_nb'_nk_n^{-1} & = & s'_nh_n^{-1},\\
2i\lambda_nb'_nk_n^{-1} + 2i\lambda_nb'_nk_n^{-1} & = & v'_nh_n^{-1},\\
2i\lambda_nc'_nk_{n-1}^{-1} - 2i\lambda_{n-1}a'_{n-1}k_n^{-1} & = & t'_nh_{n-1}^{-1},\\
2i\lambda_nc'_nk_{n-1}^{-1} + 2i\lambda_{n-1}a'_{n-1}k_n^{-1} & = & u'_nh_{n-1}^{-1}.
\end{array}$$

We use (14), (28) and (11) to get the expressions of $r'_n, s'_n, t'_n, u'_n, v'_n$ and w'_n. By the same way we get the other formulas. □

Remarks 3.4.

1. $r_0, s''_0, w_0 \in \mathbb{R}, \quad r_0 + s''_0 + w_0 = 1.$
2. $r''_0, s'_0, w''_0 \in (i\mathbb{R}), \quad r''_0 + s'_0 + w''_0 = 0.$
3. $(r_0+ir''_0), (s''_0+is'_0), (w_0+iw''_0) \in \mathbb{R}, \quad (r_0+ir''_0)+(s''_0+is'_0)+(w_0+iw''_0) = 1.$
4. $\forall m \in \mathbb{Z} \setminus \{0\}, \quad r_m, s_m, t_m, u_m, v_m, w_m \in \mathbb{R},$

   $r_m + s_m + t_m + u_m + v_m + w_m = 1.$
5. $\forall m \in \mathbb{Z} \setminus \{0\}, \quad r'_m, s'_m, t'_m, u'_m, v'_m, w'_m \in (i\mathbb{R}),$

   $r'_m + s'_m + t'_m + u'_m + v'_m + w'_m = 0.$
6. $\forall m \in \mathbb{Z} \setminus \{0\},$

   $(r_m+ir'_m), (s_m+is'_m), (t_m+it'_m), (u_m+iu'_m), (v_m+iv'_m), (w_m+iw'_m) \in \mathbb{R},$

   $(r_m+ir'_m)+(s_m+is'_m)+(t_m+it'_m)+(u_m+iu'_m)+(v_m+iv'_m)+(w_m+iw'_m) = 1.$

Proposition 3.5. *For all $m \in \mathbb{Z} \setminus \{0\}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have*

1. $$\frac{\sin(2\theta)}{\lambda_m} \Im\psi_m(\theta) = a'_{|m|}\varphi_{|m|+1}(\theta) + b'_{|m|}\varphi_{|m|}(\theta) + c'_{|m|}\varphi_{|m|-1}(\theta),$$

   where $a'_{|m|}, b'_{|m|}$ and $c'_{|m|}$ are given by (28).

2. $\Re\left[\left(\cos(2\theta) - i\frac{\lambda_m}{2|m|}\sin(2\theta)\right)\psi_m(\theta)\right]$
   $$= \frac{\alpha-\beta}{2|m|+\rho-1}\varphi_{|m|}(\theta) + \frac{2(|m|+\beta)}{2|m|+\rho-1}\varphi_{|m|-1}(\theta).$$

3. $\Re\left[\left(\cos(2\theta) + i\frac{\lambda_m}{2(|m|+\rho)}\sin(2\theta)\right)\psi_m(\theta)\right]$
   $$= \frac{2(|m|+\alpha+1)}{2|m|+\rho+1}\varphi_{|m|+1}(\theta) + \frac{\beta-\alpha}{2|m|+\rho+1}\varphi_{|m|}(\theta).$$

4. $\Re\left[\left(\cos(2\theta) - i\frac{\rho-1}{\lambda_m}\sin(2\theta)\right)\psi_m(\theta)\right]$
   $$= \frac{|m|+\alpha+1}{2|m|+\rho}\varphi_{|m|+1}(\theta) + \frac{|m|+\beta}{2|m|+\rho}\varphi_{|m|-1}(\theta).$$

Proof. We obtain the above formulas by using (1), (27), (29), (25), (26) and (28). □

4 Bounds and asymptotic behavior

First we study the sign of $\frac{A'}{A}$, where A is given by (6).

Lemma 4.1.

1. $\forall\theta \in \left]0, \frac{\pi}{2}\right[,\quad \frac{A'(\theta)}{A(\theta)} > 0$ *if and only if* $-1 < \beta \le -\frac{1}{2} \le \alpha$ *and* $(\alpha, \beta) \neq \left(-\frac{1}{2}, -\frac{1}{2}\right)$.

2. $\forall\theta \in \left]0, \frac{\pi}{2}\right[,\quad \frac{A'(\theta)}{A(\theta)} < 0$ *if and only if* $-1 < \alpha \le -\frac{1}{2} \le \beta$ *and* $(\alpha, \beta) \neq \left(-\frac{1}{2}, -\frac{1}{2}\right)$.

3. $\forall\theta \in \left]0, \frac{\pi}{2}\right[,\quad \frac{A'(\theta)}{A(\theta)} = 0$ *if and only if* $\alpha = \beta = -\frac{1}{2}$.

4. (a) *If* $\min\{\alpha, \beta\} > -\frac{1}{2}$, *then*

$$\forall\theta \in]0, \theta_0[,\quad \frac{A'(\theta)}{A(\theta)} > 0,\quad \frac{A'(\theta_0)}{A(\theta_0)} = 0,\quad \forall\theta \in \left]\theta_0, \frac{\pi}{2}\right[,\quad \frac{A'(\theta)}{A(\theta)} < 0.$$

   (b) *If* $-1 < \min\{\alpha, \beta\} \le \max\{\alpha, \beta\} < -\frac{1}{2}$, *then*

$$\forall\theta \in]0, \theta_0[,\quad \frac{A'(\theta)}{A(\theta)} < 0,\quad \frac{A'(\theta_0)}{A(\theta_0)} = 0,\quad \forall\theta \in \left]\theta_0, \frac{\pi}{2}\right[,\quad \frac{A'(\theta)}{A(\theta)} > 0,$$

   where

$$\theta_0 := \frac{1}{2}\arccos\left(\frac{\beta - \alpha}{\rho}\right). \tag{31}$$

Proof. It suffices to view the expression of

$$\frac{A'(\theta)}{A(\theta)} = (2\alpha + 1)\cot\theta - (2\beta + 1)\tan\theta = \frac{2[\alpha - \beta + \rho\cos(2\theta)]}{\sin(2\theta)},\quad \theta \in \left]0, \frac{\pi}{2}\right[.$$

□

4.1 Bounds and asymptotic behavior of φ_n

From [12, 8, 11, 1], we have

* If $\alpha \geq -\dfrac{1}{2}$, $-1 < \beta \leq \alpha$, then $\forall n \in \mathbb{N}$, $\displaystyle\max_{0\leq\theta\leq\frac{\pi}{2}} |\varphi_n(\theta)| = 1$,

  and

$$\forall k \in \mathbb{N}, \quad \frac{d^k}{d\theta^k}\varphi_n(\theta) = O(n^{2k}), \quad n \to +\infty, \quad 0 \leq \theta \leq \frac{\pi}{2}. \tag{32}$$

* If $\beta \geq -\dfrac{1}{2}$, $-1 < \alpha < \beta$, then $\forall n \in \mathbb{N}$, $\displaystyle\max_{0\leq\theta\leq\frac{\pi}{2}} |\varphi_n(\theta)| = \left|\varphi_n\left(\frac{\pi}{2}\right)\right|$,

  and

$$\forall k \in \mathbb{N}, \quad \frac{d^k}{d\theta^k}\varphi_n(\theta) = O(n^{\beta-\alpha+2k}), \quad n \to +\infty, \quad 0 \leq \theta \leq \frac{\pi}{2}. \tag{33}$$

* If $-1 < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} < -\dfrac{1}{2}$, then

$$\forall n \in \mathbb{N}, \quad \max_{0\leq\theta\leq\frac{\pi}{2}} |\varphi_n(\theta)| = |\varphi_n(\theta')|,$$

  where $x' := \cos(2\theta')$ is one of the two maximum points nearest $x_0 := \dfrac{\beta - \alpha}{\rho}$, and

$$\forall k \in \mathbb{N}, \quad \frac{d^k}{d\theta^k}\varphi_n(\theta) = O\left(n^{-\frac{1}{2}-\alpha+k}\right), \quad n \to +\infty, \quad 0 \leq \theta \leq \frac{\pi}{2}. \tag{34}$$

Remarks 4.2.

1. *If* $\alpha \geq -\dfrac{1}{2}$, $\quad -1 < \beta \leq \alpha$, *then*

$$\forall n \in \mathbb{N}, \quad \max_{0\leq\theta\leq\frac{\pi}{2}} |\varphi_n(\theta)| = \max_{0\leq\theta\leq\frac{\pi}{2}} \varphi_n(\theta) = \varphi_n(0) = 1.$$

2. *If* $-1 < \beta \leq \alpha$, *then* $\displaystyle\max_{0\leq\theta\leq\frac{\pi}{2}} |\varphi_1(\theta)| = \max_{0\leq\theta\leq\frac{\pi}{2}} \varphi_1(\theta) = \varphi_1(0) = 1.$

3. *If* $-1 < \alpha < \beta$, *then*

$$\max_{0\leq\theta\leq\frac{\pi}{2}} |\varphi_1(\theta)| = \left|\varphi_1\left(\frac{\pi}{2}\right)\right| = -\varphi_1\left(\frac{\pi}{2}\right) = \frac{\beta+1}{\alpha+1} > 1 = \varphi_1(0) = \max_{0\leq\theta\leq\frac{\pi}{2}} \varphi_1(\theta).$$

From [8], for all $\theta \in \left]0, \dfrac{\pi}{2}\right[$, we have

$$\varphi_n(\theta) \approx \frac{2^\rho\Gamma(\alpha+1)}{\sqrt{\pi}} \frac{n^{-\left(\alpha+\frac{1}{2}\right)}}{A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)} \cos\left[(2n+\rho)\theta - (2\alpha+1)\frac{\pi}{4}\right], \quad n \to +\infty. \tag{35}$$

4.2 Bounds and asymptotic behavior of ψ_m

From (23) it is clear that

Lemma 4.3. *For all $m \in \mathbb{Z} \setminus \{0\}$, we have*

1. *If $\alpha = \beta$, then* $\left|\psi_m\left(\frac{\pi}{2}\right)\right| = 1.$
2. *If $\beta < \alpha$, then* $0 < \left|\psi_m\left(\frac{\pi}{2}\right)\right| = \dfrac{B(\alpha-\beta, \beta+|m|+1)}{B(\alpha-\beta, \beta+1)} < 1.$
3. *If $\alpha < \beta$, then* $\left|\psi_m\left(\frac{\pi}{2}\right)\right| = \dfrac{B(\beta-\alpha, \alpha+1)}{B(\beta-\alpha, \alpha+|m|+1)} > 1.$

The following theorem gives the bounds of $|\psi_m|$, $m \in \mathbb{Z} \setminus \{0\}$.

Theorem 4.4. *For all $m \in \mathbb{Z} \setminus \{0\}$, we have*

1. $\displaystyle\max_{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}} |\psi_m(\theta)| = 1 \Longleftrightarrow \max\{\alpha, \beta\} = \alpha \ge -\frac{1}{2}.$
2. $\displaystyle\max_{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}} |\psi_m(\theta)| = \left|\psi_m\left(\frac{\pi}{2}\right)\right| \Longleftrightarrow \max\{\alpha, \beta\} = \beta \ge -\frac{1}{2}.$
3. $\displaystyle\max_{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}} |\psi_m(\theta)| = |\psi_m(\theta_0)| \Longleftrightarrow -1 < \min\{\alpha, \beta\} \le \max\{\alpha, \beta\} < -\frac{1}{2}.$
4. $\displaystyle\min_{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}} |\psi_m(\theta)| = 1 \Longleftrightarrow -1 < \min\{\alpha, \beta\} = \alpha \le -\frac{1}{2}.$
5. $\displaystyle\min_{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}} |\psi_m(\theta)| = \left|\psi_m\left(\frac{\pi}{2}\right)\right| \Longleftrightarrow -1 < \min\{\alpha, \beta\} = \beta \le -\frac{1}{2}.$
6. $\displaystyle\min_{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}} |\psi_m(\theta)| = |\psi_m(\theta_0)| \Longleftrightarrow \min\{\alpha, \beta\} > -\frac{1}{2},$

where θ_0 is given by (31).

Proof. For all $m \in \mathbb{Z} \setminus \{0\}$ and $\theta \in \left]0, \frac{\pi}{2}\right[$, we have

$$\begin{aligned}
\frac{d}{d\theta}\left(|\psi_m(\theta)|^2\right) &= 2\frac{d}{d\theta}\varphi_{|m|}(\theta)\left(\varphi_{|m|}(\theta) + \frac{1}{\lambda_m^2}\frac{d^2}{d\theta^2}\varphi_{|m|}(\theta)\right) \\
&= -\frac{2}{\lambda_m^2}\left(\frac{d}{d\theta}\varphi_{|m|}(\theta)\right)^2 \frac{A'(\theta)}{A(\theta)}.
\end{aligned}$$

Then Lemma 4.1 finishes the proof. □

Corollary 4.5. *For all $m \in \mathbb{Z} \setminus \{0\}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have*

1. *If $\alpha = \beta = -\frac{1}{2}$, then*
$$1 = \min_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = |\psi_m(\theta)| = \max_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = 1.$$

2. *If $-1 < \beta \le -\frac{1}{2} \le \alpha$ and $(\alpha, \beta) \ne \left(-\frac{1}{2}, -\frac{1}{2}\right)$, then*
$$0 < \left|\psi_m\left(\frac{\pi}{2}\right)\right| = \min_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| \le |\psi_m(\theta)| \le \max_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = 1,$$
and $\left|\psi_m\left(\frac{\pi}{2}\right)\right| < 1$.

3. *If $-\frac{1}{2} < \beta \le \alpha$, then*
$$0 < |\psi_m(\theta_0)| = \min_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| \le |\psi_m(\theta)| \le \max_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = 1,$$
and $|\psi_m(\theta_0)| < 1$.

4. *If $-1 < \alpha \le -\frac{1}{2} \le \beta$ and $(\alpha, \beta) \ne \left(-\frac{1}{2}, -\frac{1}{2}\right)$, then*
$$1 = \min_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| \le |\psi_m(\theta)| \le \max_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = \left|\psi_m\left(\frac{\pi}{2}\right)\right|,$$
and $1 < \left|\psi_m\left(\frac{\pi}{2}\right)\right|$.

5. *If $-\frac{1}{2} < \alpha < \beta$, then*
$$0 < |\psi_m(\theta_0)| = \min_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| \le |\psi_m(\theta)| \le \max_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = \left|\psi_m\left(\frac{\pi}{2}\right)\right|,$$
and $|\psi_m(\theta_0)| < 1 < \left|\psi_m\left(\frac{\pi}{2}\right)\right|$.

6. *If $-1 < \alpha \le \beta < -\frac{1}{2}$, then*
$$1 = \min_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| \le |\psi_m(\theta)| \le \max_{-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}} |\psi_m(\phi)| = |\psi_m(\theta_0)|,$$
and $1 < |\psi_m(\theta_0)|$.

7. If $-1<\beta<\alpha<-\dfrac{1}{2}$*, then*

$$0<\left|\psi_m\left(\frac{\pi}{2}\right)\right|=\min_{-\frac{\pi}{2}\le\phi\le\frac{\pi}{2}}|\psi_m(\phi)|\le|\psi_m(\theta)|\le\max_{-\frac{\pi}{2}\le\phi\le\frac{\pi}{2}}|\psi_m(\phi)|=|\psi_m(\theta_0)|,$$

$$\text{and } \left|\psi_m\left(\frac{\pi}{2}\right)\right|<1<|\psi_m(\theta_0)|,$$

where θ_0 *is given by* (31).

The following theorems give the asymptotic behavior of $\psi_m,\ m\in\mathbb{Z}$.

Theorem 4.6.

1. If $\alpha\ge-\dfrac{1}{2}$, $-1<\beta\le\alpha$*, then*

$$\forall k\in\mathbb{N},\quad \frac{d^k}{d\theta^k}\psi_m(\theta)=O(|m|^{2k+1}),\quad |m|\to\infty,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

2. If $\beta\ge-\dfrac{1}{2}$, $-1<\alpha<\beta$*, then*

$$\forall k\in\mathbb{N},\quad \frac{d^k}{d\theta^k}\psi_m(\theta)=O(|m|^{\beta-\alpha+2k+1}),\quad |m|\to\infty,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

3. If $-1<\min\{\alpha,\beta\}\le\max\{\alpha,\beta\}<-\dfrac{1}{2}$*, then*

$$\forall k\in\mathbb{N},\quad \frac{d^k}{d\theta^k}\psi_m(\theta)=O\left(|m|^{-\frac{1}{2}-\alpha+k}\right),\quad |m|\to\infty,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

Proof. For all $\theta\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ and $k\in\mathbb{N}$, we have

$$\frac{d^k}{d\theta^k}\psi_m(\theta)=\frac{d^k}{d\theta^k}\varphi_{|m|}(\theta)-\frac{i}{\lambda_m}\frac{d^{k+1}}{d\theta^{k+1}}\varphi_{|m|}(\theta)\text{ and }\lambda_m\approx 2|m|,\quad\text{as }|m|\to\infty.$$

Using (32), (33) and (34), we get the above relations. □

Theorem 4.7. *For all* $\theta\in\left]-\dfrac{\pi}{2},\dfrac{\pi}{2}\right[\setminus\{0\}$*, we have*

$$1.\ \Im\psi_m(|\theta|)\approx\frac{2^\rho\Gamma(\alpha+1)}{\sqrt{\pi}}\frac{|m|^{-\left(\alpha+\frac{1}{2}\right)}}{A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)}\sin\left[(2|m|+\rho)|\theta|-(2\alpha+1)\frac{\pi}{4}\right],$$

$|m|\longrightarrow\infty$.

$$2.\ \psi_m(|\theta|)\approx\frac{2^\rho\Gamma(\alpha+1)}{\sqrt{\pi}}\frac{|m|^{-\left(\alpha+\frac{1}{2}\right)}}{A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)}e^{i\left[(2|m|+\rho)|\theta|-(2\alpha+1)\frac{\pi}{4}\right]},\quad |m|\longrightarrow\infty.$$

$$3.\ |\psi_m(\theta)|\approx\frac{2^\rho\Gamma(\alpha+1)}{\sqrt{\pi}}\frac{|m|^{-\left(\alpha+\frac{1}{2}\right)}}{A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)},\quad |m|\longrightarrow\infty.$$

Proof. It suffices to use (1) and (35). □

5 Open problems

5.1 Problem 1

Find a positive sequence $(a_k)_{k\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \setminus \{0\},\ \forall \theta \in \left[0, \frac{\pi}{2}\right], \quad \varphi_n(\theta) = 1 - (\sin\theta)^2 \sum_{k=0}^{n-1} a_k \varphi_k^{(\alpha+1,\beta)}(\theta).$$

5.2 Problem 2

Find a positive sequence $(b_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N},\ \forall \theta \in \left[0, \frac{\pi}{2}\right], \quad \varphi_n(\theta) \geq 1 - b_n(\sin\theta)^2.$$

5.3 Problem 3

Write the Jacobi-Dunkl polynomial $\psi_m,\ m \in \mathbb{Z}$, in the following form :

$$\forall m \in \mathbb{Z},\ \forall \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \quad \psi_m(\theta) = \sum_{k=0}^{+\infty} \frac{i^k \lambda_m^k}{k!} M_k(\theta),$$

where $M_k,\ k \in \mathbb{N}$, is a C^∞ function on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

5.4 Problem 4

Find, for $n \in \mathbb{N}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the sign of the following expression :

$$\frac{2n+1}{n+1}\lambda_{n+1}\Re\psi_n(\theta)\Im\psi_{n+1}(\theta) - \frac{2(n+\rho)+1}{n+\rho}\lambda_n\Re\psi_{n+1}(\theta)\Im\psi_n(\theta).$$

5.5 Problem 5

What is about Rodrigues formula for the Jacobi-Dunkl polynomials $\psi_m,\ m \in \mathbb{Z}$?

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