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On certain generalized class of Bazilevič function of type $\alpha + i\mu$ and of order β

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Abstract

In the present paper, the authors introduce a generalized class $\mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$ of Bazilevič function of type $\alpha + i\mu$ and of order β. The subordination relations and inequality properties are discussed by making use of differential subordination method. The results presented here generalize and improve some known results, and some other new results are obtained.

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1 Introduction

Let H_n denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{+\infty} a_k z^k,\ n = 1, 2, \cdots \tag{1}$$

that are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in H_n$ is said to be in the class $\mathcal{S}_n^*(\beta)$ of starlike functions of order β $(0 \le \beta < 1)$ in $\mathbb{U}$, if it satisfies the inequality

$$\mathrm{Re}(\frac{zf'(z)}{f(z)}) > 0,\ \beta(0 \le \beta < 1)\ z \in \mathbb{U}. \tag{2}$$

Let $\mathcal{B}_n(\alpha, \mu, \beta, g(z))$ denote the class of functions in H_n satisfying the inequality

$$\mathrm{Re}\Big\{\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu}\Big\} > \beta,\ \alpha \ge 0, \mu \in \mathbb{R}, 0 \le \beta < 1, z \in \mathbb{U} \tag{3}$$

where $g(z) \in \mathcal{S}_n^*(\beta)$. The function $f(z)$ in this class is said to be g-Bazilevič function of type $\alpha + i\mu$ and of order β. (See [1])

In the present paper, we define the following class of analytic functions.

Definition 1.1. *Let $\mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$ denote the class of functions in H_n satisfying the inequality*

$$\mathfrak{Re}\Big\{\Big(1 - \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu}\Big\} > \beta, z \in \mathbb{U}, \tag{4}$$

where $0 \le \lambda \le 1, \alpha \ge 0, \mu \in \mathbb{R}, 0 \le \beta < 1, g(z) \in \mathcal{S}_n^*(\beta)$.

All the powers in (1.4) are principal values, below we apply this agreement. The function $f(z)$ in this class is said to be (λ, g)-Bazilevič function of type $\alpha + i\mu$ and of order β. Clearly, the class $\mathcal{B}_n(1, \alpha, \mu, \beta, z)$ is the class of Bazilevič functions of type $\alpha + i\mu$ and of order β, the class $\mathcal{B}_n(1, \alpha, 0, \beta, z)$ is the class of Bazilevič functions of order β, the class $\mathcal{B}_n(1, 0, 0, \beta, z)$ is the class $\mathcal{S}_n^*(\beta)$ of starlike functions of order β.

Let $f(z)$ and $F(z)$ be analytic in U, then we say that the function $f(z)$ is subordinate to $F(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)| \le |z|$, and $f(z) \equiv F(w(z))$, denoted $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in U, then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$. (See [2])

In this paper, we will discuss the subordination relations and inequality properties of $\mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$. The results presented here generalize and improve some known results, and some other new results are obtained.

2 Some lemmas

To prove our main result, we need the following lemmas:

Lemma 2.1. [3] *Let $f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{U}$, $h(z)$ be analytic and convex in $\mathbb{U}$, $h(0) = 1$. If*

$$f(z) + \frac{1}{c}zf'(z) \prec h(z) \tag{5}$$

where $c \neq 0$ *and* $\mathfrak{Re}\ c \geq 0$, *then*

$$f(z) \prec \frac{c}{n} z^{-\frac{c}{n}} \int_0^z t^{-\frac{c}{n}-1} h(t)\mathrm{d}t \prec h(z)$$

and $\frac{c}{n} z^{-\frac{c}{n}} \int_0^z t^{-\frac{c}{n}-1} h(t)\mathrm{d}t$ *is the best dominant for differential subordination (2.1).*

Lemma 2.2. [4] *Let* $u = u_1 + u_2 i, v = v_1 + v_2 i$ *and* $\psi(u, v) : \mathbb{D}(\subset \mathbb{C}^2) \to \mathbb{C}$ *be a complex-valued function satisfying:*

(1) $\theta(u, v)$ *is continuous in a domain* $\mathbb{D} \subset \mathbb{C}^2$.

(2) $(1, 0) \in \mathbb{D}$ *and* $\mathfrak{Re}\ \theta(1, 0) > 0$,

(3) $\mathfrak{Re}\ \theta(iu_2, v_1) \leq 0$ *when* $(iu_2, v_1) \in \mathbb{D}$, $v_1 \leq -\frac{n(1+u_2^2)}{2}$.

Let $f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *be a function regular in* $\mathbb{U}$ *with* $(f(z), zf'(z)) \in \mathbb{D}$ *and* $\mathfrak{Re}\ \theta(f(z), zf'(z)) > 0, z \in \mathbb{U}$, *Then* $\mathfrak{Re} f(z) > 0, z \in \mathbb{U}$.

Lemma 2.3. *Let* $0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}, \alpha + i\mu \neq 0, 0 \leq \beta < 1, g(z) \in \mathcal{S}_n^*(\beta)$. *Then* $f(z) \in \mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$ *if and only if*

$$\mathfrak{Re}\{F(z) + \frac{\lambda}{\alpha + i\mu} zF'(z)\} > \beta \tag{6}$$

where

$$F(z) = \Big(\frac{f(z)}{g(z)}\Big)^{\alpha + i\mu}. \tag{7}$$

Proof. If $\lambda = 0$, we obtain the result from the definition of $\mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$.

If $\lambda > 0$. Let $F(z) = (\frac{f(z)}{g(z)})^{\alpha+i\mu}$, then $F(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$. By taking the derivatives in the both sides, we have

$$\Big(1 - \lambda \frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} + \lambda \frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} = F(z) + \frac{\lambda}{\alpha + i\mu} zF'(z). \tag{8}$$

Since $f(z) \in \mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$, we have

$$\mathfrak{Re}\{F(z) + \frac{\lambda}{\alpha + i\mu} zF'(z)\} > \beta. \tag{9}$$

3 Main Results

Theorem 3.1. *Let* $0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}, \alpha + i\mu \neq 0, 0 \leq \beta < 1$, *and* $g(z) \in \mathcal{S}_n^*(\beta)$. *If* $f(z) \in \mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$, *then*

$$\mathfrak{Re}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} > \frac{2\beta(\alpha^2 + \mu^2) + n\lambda\alpha}{2(\alpha^2 + \mu^2) + n\lambda\alpha} \tag{10}$$

Proof. Let $F(z) = (\frac{f(z)}{g(z)})^{\alpha+i\mu}$, then $F(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$. Suppose that $f(z) \in \mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z))$, by Lemma 2.3, we know

$$\mathfrak{Re}\Big\{F(z) + \frac{\lambda}{\alpha+i\mu} zF'(z)\Big\} > \beta. \tag{11}$$

Let

$$F(z) = \rho + (1-\rho)p(z) \tag{12}$$

where $\rho = \frac{2\beta(\alpha^2+\mu^2)+n\lambda\alpha}{2(\alpha^2+\mu^2)+n\lambda\alpha} \le 1$, then $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1}$ is analytic in $\mathbb{U}$. Thus that

$$F(z) + \frac{\lambda}{\alpha+i\mu} zF'(z) - \beta = \rho - \beta + (1-\rho)p(z) + \frac{\lambda(1-\rho)}{\alpha+i\mu} zp'(z). \tag{13}$$

Now, we taking

$$\theta(u,v) = \rho - \beta + (1-\rho)p(z) + \frac{\lambda(1-\rho)}{\alpha+i\mu} zp'(z), \tag{14}$$

then $\theta(u, v)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C}^2$, $(1, 0) \in \mathbb{D}$, $\mathrm{Re}\psi(1, 0) = 1-\beta > 0$, and

$$\mathfrak{Re}\ \theta(iu_2, v_1) = \rho - \beta + \frac{\lambda\alpha(1-\rho)}{\alpha^2+\mu^2} v_1 \quad \le \rho - \beta + \tfrac{n\lambda\alpha(1-\rho)(1+u_2^2)}{\alpha^2+\mu^2} \tag{15}$$

$$= -\tfrac{n\lambda\alpha(1-\rho)(1+u_2^2)}{2(\alpha^2+\mu^2)} \le 0. \tag{16}$$

where $(iu_2, v_1) \in \mathbb{D}$, $v_1 \le -\frac{n(1+u_2^2)}{2}$. From (3.4) we have $(p(z), zp'(z)) \in \mathbb{D}$ and $\mathfrak{Re}\psi(p(z), zp'(z)) > 0$, then by Lemma 2.2, we obtain $\mathfrak{Re}p(z) > 0, z \in \mathbb{U}$.

Thus, by applying (3.3), our proof of Theorem 3.1 is completed.

Corollary 3.1. *Let* $0 \le \lambda \le 1, \alpha \ge 0, \mu \in \mathbb{R}, \alpha + i\mu \ne 0$ *and* $0 \le \beta < 1$, *If* $f(z) \in \mathcal{B}_n(\lambda, \alpha, \mu, \beta, z)$, *then*

$$\mathfrak{Re}\Big(\frac{f(z)}{z}\Big)^{\alpha+i\mu} > \frac{2\beta(\alpha^2+\mu^2)+n\lambda\alpha}{2(\alpha^2+\mu^2)+n\lambda\alpha} \tag{17}$$

Corollary 3.2. *Let* $\alpha \ge 0, \mu \in \mathbb{R}, \alpha + i\mu \ne 0$ *and* $0 \le \beta < 1$, *If* $f(z) \in \mathcal{B}_n(1, \alpha, \mu, \beta, z)$, *then*

$$\mathfrak{Re}\Big(\frac{f(z)}{z}\Big)^{\alpha+i\mu} > \frac{2\beta(\alpha^2+\mu^2)+n\alpha}{2(\alpha^2+\mu^2)+n\alpha} \tag{18}$$

Corollary 3.3. *Let $\alpha \geq 0, \mu \in \mathbb{R}, \alpha + i\mu \neq 0$ and $0 \leq \beta < 1$, If $f(z) \in \mathcal{B}_n(1, \alpha, 0, \beta, z)$, then*

$$\mathfrak{Re}\Big(\frac{f(z)}{z}\Big)^{\alpha} > \frac{2\beta\alpha + n}{2\alpha + n} \tag{19}$$

Since the proof of Theorem 3.2 is similar to the proof of Theorem 3.1, we state the theorem without proof.

Theorem 3.2. *Let $0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}, \alpha + i\mu \neq 0, \beta > 1, g(z) \in \mathcal{S}_n^*(\beta)$ and $f(z) \in H_n$ satisfies*

$$\mathfrak{Re}\Big\{\Big(1 - \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu}\Big\} < \beta, \tag{20}$$

then

$$\mathfrak{Re}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} < \frac{2\beta(\alpha^2 + \mu^2) + n\lambda\alpha}{2(\alpha^2 + \mu^2) + n\lambda\alpha}. \tag{21}$$

Theorem 3.3. *Let $0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}, \beta \neq 1, g(z) \in \mathcal{S}_n^*(\beta)$ and $f(z) \in H_n$ satisfies*

$$\Big(1 - \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} \prec \frac{1 + (1 - 2\beta)z}{1 - z}, z \in \mathbb{U}, \tag{22}$$

then

$$\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} \prec \beta + \frac{(1 - \beta)(\alpha + i\mu)}{n\lambda}\int_0^1 \frac{1 + zu}{1 - zu}u^{\frac{\alpha+i\mu}{n\lambda}-1}du, z \in \mathbb{U}, \tag{23}$$

Proof. Let $F(z) = (\frac{g(z)}{f(z)})^{\alpha+i\mu}$, then $F(z) = 1 + c_1z + c_2z^2 + \cdots$ is analytic in $\mathbb{U}$. By taking the derivatives in the both sides, we have

$$\Big(1 + \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{g(z)}{f(z)}\Big)^{\alpha+i\mu} - \lambda\frac{zf'(z)}{f(z)}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha+i\mu} = F(z) + \frac{\lambda}{\alpha + i\mu}zF'(z). \tag{24}$$

From (3.10) we have

$$F(z) + \frac{\lambda}{\alpha + i\mu}zF'(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z}, z \in \mathbb{U}. \tag{25}$$

Clearly, $h(z) = \frac{1+(1-2\beta)z}{1-z}$ is analytic in $\mathbb{U}$ and $h(0) = 1$.

Thus, by applying Lemma 2.2, our proof of Theorem 3.3 is completed.

Corollary 3.4. *Let $0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}$ and $0 \leq \beta < 1$, If $f(z) \in \mathcal{B}_n(\lambda, \alpha, \mu, \beta, g(z)), g(z) \in \mathcal{S}_n^*(\beta)$, then*

$$\mathfrak{Re}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} > \beta + (1-\beta)\inf_{z\in\mathbb{U}} \mathfrak{Re}\Big(\frac{\alpha+i\mu}{n\lambda}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n\lambda}-1}du\Big), z \in \mathbb{U}. \quad (26)$$

We can see that inequality (3.17) is sharp.

Corollary 3.5. *Let $0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}, \beta > 1, g(z) \in \mathcal{S}_n^*(\beta)$ and $f(z) \in H_n$ satisfies*

$$\mathfrak{Re}\Big\{\Big(1-\lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu}\Big\} < \beta, \quad (27)$$

then

$$\mathfrak{Re}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} < \beta + (1-\beta)\inf_{z\in\mathbb{U}} \mathfrak{Re}\Big(\frac{\alpha+i\mu}{n\lambda}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n\lambda}-1}du\Big), z \in \mathbb{U}. \quad (28)$$

We can see that inequality (3.19) is sharp.

Corollary 3.6. *Let $0 \leq \lambda \leq 1, \alpha \geq 0, \beta \neq 1, g(z) \in \mathcal{S}_n^*(\beta)$ and $f(z) \in H_n$ satisfies*

$$\Big(1-\lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha} \prec \frac{1+(1-2\beta)z}{1-z}, z \in \mathbb{U}, \quad (29)$$

then

$$\beta + \frac{(1-\beta)\alpha}{n\lambda}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{n\lambda}-1}du < \mathfrak{Re}\Big(\tfrac{f(z)}{g(z)}\Big)^{\alpha} \quad (30)$$

$$< \beta + \tfrac{(1-\beta)\alpha}{n\lambda}\int_0^1 \tfrac{1+u}{1-u}u^{\frac{\alpha}{n\lambda}-1}du, z \in \mathbb{U}. \quad (31)$$

4 Open Problem

We define the following class of analytic functions:

Definition *Let $\mathcal{B}_n(\lambda, \alpha, \mu, \mathrm{A}, \mathrm{B}, g(z))$ denote the class of functions in H_n satisfying the condition:*

$$\Big(1-\lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{g(z)}\Big)^{\alpha+i\mu} \prec \frac{1+\mathrm{A}z}{1+\mathrm{B}z}, z \in \mathrm{U}. \quad (32)$$

where

$$0 \leq \lambda \leq 1, \alpha \geq 0, \mu \in \mathbb{R}, -1 \leq \mathrm{B} \leq 1, \mathrm{A} \neq \mathrm{B}, \mathrm{A} \in \mathbb{R}, g(z) \in \mathrm{S}^*.$$

We can using the differential subordination method introduced by Miller and Mocanu consider the subordination relations, inclusion relations, distortion theorems and inequality properties, etc.

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