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New classes containing Sălăgean operator and Ruscheweyh derivative

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Abstract

In this paper we introduce new classes containing the linear operator $RS_\alpha^n : \mathcal{A} \to \mathcal{A}$, $RS_\alpha^n f(z) = (1-\alpha)R^n f(z) + \alpha S^n f(z)$, $z \in U$***, where*** $R^n f(z)$ ***is the Ruscheweyh derivative,*** $S^n f(z)$ ***the Sălăgean operator and*** $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ ***is the class of normalized analytic functions with*** $\mathcal{A}_1 = \mathcal{A}$***. Characterization and other properties of these classes are studied.***

Keywords: *differential operator, distortion theorem.*
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1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1.1. *(Sălăgean [7]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} S^0 f(z) &= f(z) \\ S^1 f(z) &= z f'(z), \ \ldots \\ S^{n+1} f(z) &= z\left(S^n f(z)\right)', \quad z \in U. \end{aligned}$$

Remark 1.2. *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, *for* $z \in U$.

Definition 1.3. *(Ruscheweyh [6]) For* $f \in \mathcal{A}$, $n \in \mathbb{N}$, *the operator* R^n *is defined by* $R^n : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z), \ \dots \\ (n+1) R^{n+1} f(z) &= z \left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.4. *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.

Definition 1.5. *[1] Let* $\gamma \geq 0$, $n \in \mathbb{N}$. *Denote by* L_γ^n *the operator given by* $L_\gamma^n : \mathcal{A} \to \mathcal{A}$,

$$L_\gamma^n f(z) = (1-\gamma) R^n f(z) + \gamma S^n f(z), \qquad z \in U.$$

Remark 1.6. *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $L_\gamma^n f(z) = z + \sum_{j=2}^{\infty} \left(\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right) a_j z^j$, $z \in U$.
This operator was studied also in [2], [3], [4].

Definition 1.7. *Let* $f \in \mathcal{A}$. *Then* $f(z)$ *is in the class* $\mathcal{S}_{\lambda,\alpha}^n(\mu)$ *if and only if*

$$Re\left(\frac{z \left(L_\gamma^n f(z)\right)'}{L_\gamma^n f(z)} \right) > \mu, \ \ 0 \leq \mu < 1, \ \ z \in U.$$

Definition 1.8. *Let* $f \in \mathcal{A}$. *Then* $f(z)$ *is in the class* $\mathcal{C}_{\lambda,\alpha}^n(\mu)$ *if and only if*

$$Re\left(\frac{\left[z \left(L_\gamma^n f(z)\right)' \right]'}{\left(L_\gamma^n f(z)\right)'} \right) > \mu, \ \ 0 \leq \mu < 1, \ \ z \in U.$$

We study the charaterization and distortion theorems, and other properties of these classes, following the paper of M. Darus and R. Ibrahim [5].

2 General properties of L_γ^n

In this section we study the characterization properties and distortion theorems for the function $f(z) \in \mathcal{A}$ to belong to the classes $\mathcal{S}_\alpha^n(\mu)$ and $\mathcal{C}_\alpha^n(\mu)$ by obtaining the coefficient bounds.

Theorem 2.1. *Let $f \in \mathcal{A}$. If*

$$\sum_{j=2}^{\infty}(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq 1-\mu, \quad 0 \leq \mu < 1, \tag{1}$$

then $f(z) \in \mathcal{S}_{\alpha}^{n}(\mu)$. The result (1) is sharp.

Proof Suppose that (1) holds. Since
$1-\mu \geq \sum_{j=2}^{\infty}(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \geq$
$\mu\sum_{j=2}^{\infty}\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| - \sum_{j=2}^{\infty}j\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|$
then this implies that $\frac{1+\sum_{j=2}^{\infty}j\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|}{1+\sum_{j=2}^{\infty}\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|} > \mu$. So, we deduce that $Re\left(\frac{z\left(L_{\gamma}^{n}f(z)\right)'}{L_{\gamma}^{n}f(z)}\right) > \mu, \quad 0 \leq \mu < 1, \quad z \in U$. We have $f(z) \in \mathcal{S}_{\alpha}^{n}(\mu)$, which evidently completes the proof.

The assertion (1) is sharp and the extremal function is given by $f(z) = z+\sum_{j=2}^{\infty}\frac{(1-\mu)}{(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$.

Corollary 2.2. *Let the hypotheses of Theorem 2.1 satisfy. Then*

$$|a_j| \leq \frac{1-\mu}{(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad \forall\, j \geq 2.$$

Theorem 2.3. *Let $f \in \mathcal{A}$. If*

$$\sum_{j=2}^{\infty}j(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq 1-\mu, \quad 0 \leq \mu < 1, \tag{2}$$

then $f(z) \in \mathcal{C}_{\alpha}^{n}(\mu)$. The result (2) is sharp.

Proof Suppose that (2) holds. Since
$1-\mu \geq \sum_{j=2}^{\infty}j(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \geq$
$\mu\sum_{j=2}^{\infty}j\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| - \sum_{j=2}^{\infty}j^2\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|$
then this implies that $\frac{1+\sum_{j=2}^{\infty}j^2\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|}{1+\sum_{j=2}^{\infty}j\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|} > \mu$. So, we deduce that $Re\left(\frac{\left[z\left(L_{\gamma}^{n}f(z)\right)'\right]'}{\left(L_{\gamma}^{n}f(z)\right)'}\right) > \mu, \quad 0 \leq \mu < 1, \quad z \in U$. We have $f(z) \in \mathcal{C}_{\alpha}^{n}(\mu)$, which evidently completes the proof.

The assertion (2) is sharp and the extremal function is given by $f(z) = z+\sum_{j=2}^{\infty}\frac{(1-\mu)}{j(j-\mu)\left\{\alpha j^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$.

Corollary 2.4. *Let the hypotheses of Theorem 2.3 be satisfied. Then*

$$|a_j| \leq \frac{1-\mu}{j\left(j-\mu\right)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad \forall\ j \geq 2.$$

Also, we have the following inclusion results:

Theorem 2.5. *Let* $0 \leq \mu_1 \leq \mu_2 < 1$. *Then* $\mathcal{S}_\alpha^n\left(\mu_1\right) \supseteq \mathcal{S}_\alpha^n\left(\mu_2\right)$.

Proof By Theorem 2.1.

Theorem 2.6. *Let* $0 \leq \mu_1 \leq \mu_2 < 1$. *Then* $\mathcal{C}_\alpha^n\left(\mu_1\right) \supseteq \mathcal{C}_\alpha^n\left(\mu_2\right)$.

Proof By Theorem 2.3.

We introduce the following distortion theorems.

Theorem 2.7. *Let the function* $f \in \mathcal{A}$ *and*

$$\sum_{j=2}^{\infty}(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq 1-\mu, \quad 0 \leq \mu < 1.$$

Then for $z \in U$ *and* $0 \leq \mu < 1$,

$$\left|L_\gamma^n f(z)\right| \geq |z| - \frac{1-\mu}{2-\mu}|z|^2$$

and

$$\left|L_\gamma^n f(z)\right| \leq |z| + \frac{1-\mu}{2-\mu}|z|^2.$$

Proof By using Theorem 2.1, one can verify that
$(2-\mu)\sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq$
$\sum_{j=2}^{\infty}(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq 1-\mu$. Hence,
$\sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq \frac{1-\mu}{2-\mu}$. We obtain
$\left|L_\gamma^n f(z)\right| = \left|z + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \leq$
$|z| + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^j \leq$
$|z| + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^2 \leq |z| + \frac{1-\mu}{2-\mu}|z|^2$.

The other assertion can be proved as follows
$\left|L_\gamma^n f(z)\right| = \left|z + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \geq$
$|z| - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^j \geq$
$|z| - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^2 \geq |z| - \frac{1-\mu}{2-\mu}|z|^2$. This completes the proof.

Theorem 2.8. *Let the function $f \in \mathcal{A}$ and*

$$\sum_{j=2}^{\infty} j(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq 1-\mu, \quad 0 \leq \mu < 1.$$

Then for $z \in U$ and $0 \leq \mu < 1$,

$$\left|L_\gamma^n f(z)\right| \geq |z| - \frac{1-\mu}{2\,(2-\mu)}\,|z|^2$$

and

$$\left|L_\gamma^n f(z)\right| \leq |z| + \frac{1-\mu}{2\,(2-\mu)}\,|z|^2.$$

Proof By using Theorem 2.3, one can verify that
$2\,(2-\mu)\sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq$
$\sum_{j=2}^{\infty} j(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq 1-\mu$. Hence,
$\sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq \frac{1-\mu}{2(2-\mu)}$. We obtain
$\left|L_\gamma^n f(z)\right| = \left|z + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \leq$
$|z| + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^j \leq$
$|z| + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^2 \leq |z| + \frac{1-\mu}{2(2-\mu)}\,|z|^2.$

The other assertion can be proved as follows
$\left|L_\gamma^n f(z)\right| = \left|z + \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j\right| \geq$
$|z| - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^j \geq$
$|z| - \sum_{j=2}^{\infty}\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|\,|z|^2 \geq |z| - \frac{1-\mu}{2(2-\mu)}\,|z|^2$. This completes the proof.

Also, we have the following distortiin results.

Theorem 2.9. *Let the hypotheses of Theorem 2.1 be satisfied. Then*

$$|f(z)| \geq |z| - \frac{1-\mu}{(2-\mu)\left[\alpha 2^n + (1-\alpha)(n+1)\right]}\,|z|^2$$

and

$$|f(z)| \leq |z| + \frac{1-\mu}{(2-\mu)\left[\alpha 2^n + (1-\alpha)(n+1)\right]}\,|z|^2.$$

Proof In virtue of Theorem 2.1, we have
$(2-\mu)\left[\alpha 2^n + (1-\alpha)(n+1)\right]\sum_{j=2}^{\infty}|a_j| \leq$
$\sum_{j=2}^{\infty}(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \leq 1-\mu$, thus,

$\sum_{j=2}^{\infty} |a_j| \leq \frac{1-\mu}{(2-\mu)[\alpha 2^n+(1-\alpha)(n+1)]}$. We obtain $|f(z)| = \left|z + \sum_{j=2}^{\infty} a_j z^j\right| \leq |z| + \sum_{j=2}^{\infty} |a_j| |z|^2 \leq |z| + \frac{1-\mu}{(2-\mu)[\alpha 2^n+(1-\alpha)(n+1)]} |z|^2$. The other assertion can be proved as follows: $|f(z)| \geq |z| - \sum_{j=2}^{\infty} |a_j| |z|^2 \geq |z| - \frac{1-\mu}{(2-\mu)[\alpha 2^n+(1-\alpha)(n+1)]} |z|^2$. This completes the proof.

In the same way we can get the following result.

Theorem 2.10. *Let the hypotheses of Theorem 2.3 be satisfied. Then* $(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\} \geq 0$ *and* $0 \leq \mu < 1$ *poses*

$$|f(z)| \geq |z| - \frac{1-\mu}{2(2-\mu)[\alpha 2^n + (1-\alpha)(n+1)]} |z|^2$$

and

$$|f(z)| \leq |z| + \frac{1-\mu}{2(2-\mu)[\alpha 2^n + (1-\alpha)(n+1)]} |z|^2 .$$

Proof In virtue of Theorem 2.3, we have
$2(2-\mu)[\alpha 2^n + (1-\alpha)(n+1)] \sum_{j=2}^{\infty} |a_j| \leq$
$\sum_{j=2}^{\infty} j(j-\mu)\left\{\alpha j^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\} |a_j| \leq 1-\mu$, thus,
$\sum_{j=2}^{\infty} |a_j| \leq \frac{1-\mu}{2(2-\mu)[\alpha 2^n+(1-\alpha)(n+1)]}$. We obtain $|f(z)| = \left|z + \sum_{j=2}^{\infty} a_j z^j\right| \leq |z| + \sum_{j=2}^{\infty} |a_j| |z|^2 \leq |z| + \frac{1-\mu}{2(2-\mu)[\alpha 2^n+(1-\alpha)(n+1)]} |z|^2$. The other assertion can be proved as follows: $/ |f(z)| = \left|z + \sum_{j=2}^{\infty} a_j z^j\right| \geq |z| - \sum_{j=2}^{\infty} |a_j| |z|^2 \geq |z| - \frac{1-\mu}{2(2-\mu)[\alpha 2^n+(1-\alpha)(n+1)]} |z|^2$. This completes the proof.

3 Open Problem

For the defined classes find distortion theorems, extreme points, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to these classes. Use another differential operator and define and study the other classes of analytic functions using the definitions 1.7 and 1.8.

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