On a New Class of p-Valent Functions with Negative Coefficients Defined by Catas Operator


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Abstract

In this paper, we introduce a new class of analytic p-valent functions with negative coefficients defined in the open unit disc by using Cătaş operator and obtain some results including coefficient inequality, distortion theorems, Hadamard products, radii of close-to-convexity, starlikeness and convexity, closure theorems and extreme points of functions in this class.

Keywords: Analytic functions, p-valent functions, Hadamard product, radii of starlikeness and convexity, distortion theorem, coefficient inequality, closure theorem, Cătaş operator.

2000 Mathematical Subject Classification: 30C45.

1 Introduction

Let $S_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; \ p, k \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. 
A function $f$ belonging to the class $S_p$ is said to be $p$-valent starlike of order $\alpha$ in $U$ if and only if
\[ \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; \ z \in U). \] (1.2)

Also a function $f$ belonging to the class $S_p$ is said to be $p$-valent convex of order $\alpha$ in $U$ if and only if
\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; \ z \in U). \] (1.3)

We denote by $S^*_p(\alpha)$ the class of all functions in $S_p$ which are $p$-valent starlike of order $\alpha$ in $U$ and by $K_p(\alpha)$ the class of all functions in $S_p$ which are $p$-valent convex of order $\alpha$ in $U$. We note that $S^*_p(0) = S^*_1(0)$, $K_p(0) = K_1(0)$, and
\[ f(z) \in K_p(\alpha) \iff \frac{zf'(z)}{p} \in S^*_p(\alpha) \] (1.4)

Let $f \in S_p$ be given by (1.1) and $g \in S_p$ given by
\[ g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n. \] (1.5)

We define the Hadamard product (or convolution) of $f$ and $g$ by
\[ (f \ast g)(z) = z^p + \sum_{n=k}^{\infty} a_nb_n z^n = (g \ast f)(z). \] (1.6)

For positive real parameters ($\lambda \geq 0, \ \ell \geq 0, \ p \in \mathbb{N}, \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$), Cătăș[3] defined the linear operator $I_p^m(\lambda, \ell) : S_p \to S_p$ by:
\[ I_p^0(\lambda, \ell) f(z) = f(z) \]
\[ I_p^1(\lambda, \ell) f(z) = I_p(\lambda, \ell) f(z) = (1 - \lambda) f(z) + \frac{\lambda}{(p+\ell)z^{p+\ell}} \left( z^\ell f(z) \right)' \]
\[ = z^p + \sum_{n=k}^{\infty} \left( \frac{p+\ell+\lambda(n-p)}{p+\ell} \right) a_n z^n; \]
\[ I_p^2(\lambda, \ell) f(z) = I_p(\lambda, \ell) \left( I_p(\lambda, \ell) f(z) \right) \]
\[ = z^p + \sum_{n=k}^{\infty} \left( \frac{p+\ell+\lambda(n-p)}{p+\ell} \right)^2 a_n z^n; \]
and (in general)
\[ I_p^m(\lambda, \ell) f(z) = I_p(\lambda, \ell) \left( I_p^{m-1}(\lambda, \ell) f(z) \right) \]
\[ = z^p + \sum_{n=k}^{\infty} c_{n,p}^m(\lambda, \ell) a_n z^n, \] (1.7)
where

\[ c_{n,p}^m (\lambda, \ell) = \left( \frac{p + \ell + \lambda (n - p)}{p + \ell} \right)^m. \]  

(1.8)

It is known that the operator generalizes many other operators such as:

1. \( I_p^m (\lambda, 0) f (z) = D_p^m (\lambda) f (z) \) (see El-Ashwah and Aouf [6]);
2. \( I_p^m (1, 0) f (z) = D_p^m f (z) \) (see Kamali and Orhan [8] Aouf and Mostafa [2]);
3. \( I_1^m (\lambda, \ell) f (z) = I^m (\lambda, \ell) f (z) \) (see Cătăş et al. [4]);
4. \( I_1^m (1, \ell) f (z) = I^m f (z) \) (see Cho and Srivastava [5]);
5. \( I_1^m (1, 1) f (z) = I^m f (z) \) (see Uralgaddi and Somanatha [13]);
6. \( I_1^m (\lambda, 0) f (z) = D_0^m f (z) \) (see Al-Oboudi [1]);
7. \( I_1^m (1, 0) f (z) = D_0^m f (z) \) (see Şalăgean [11]).

For \( 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, 0 \leq \gamma < \frac{p}{2}, \lambda \geq 0, \ell \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 \) and, \( f \in S_p \) we define the class \( S_p^m (\lambda, \ell, \gamma, \beta, \xi) \) by

\[
\left| \frac{z f' (z)}{f (z)} - p \right| < \beta.
\]

(1.9)

For \( m = 0, k = p + 1, p \in \mathbb{N} \) in (1.9), the class \( S_p^m (\lambda, \ell, \gamma, \beta, \xi) \) reduces to the class \( S_p^0 (\lambda, \ell, \gamma, \beta, \xi) = S_p (\gamma, \beta, \xi) \) (see Kulkarni et al. [10]).

Let \( T_p \) denote the subclass of \( S_p \) consisting of functions of the form

\[
f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n, \quad a_n \geq 0; z \in U.
\]

(1.10)

Further, we define the class \( T_p^m (\lambda, \ell, \gamma, \beta, \xi) \) by

\[
T_p^m (\lambda, \ell, \gamma, \beta, \xi) = S_p^m (\lambda, \ell, \gamma, \beta, \xi) \cap T_p.
\]

We note that:

1. For \( m = 0 \), in (1.9), the class \( T_p^m (\lambda, \ell, \gamma, \beta, \xi) \) reduces to the class \( T_p^0 (\lambda, \ell, \gamma, \beta, \xi) = T_p (\gamma, \beta, \xi) \),
2. \( T_p (\gamma, \beta, \xi) = \\{ f \in T_p : \left| \frac{z f' (z)}{f (z)} - p \right| < \beta, \\
\left| \frac{z f' (z)}{f (z)} - \gamma \right| - \left| \frac{z f' (z)}{f (z)} - p \right| < \beta \}
\}

\[ \left( 0 \leq \gamma < \frac{p}{2}, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, z \in U \right) \}
\}

which for \( p = 1 \) reduces to \( T (\gamma, \beta, \xi) \) studied by Kulkarni [9];
(2) For \( \ell = 0 \), in (1.9), the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \) reduces to the class \( T^m_p ([\lambda, 0], \gamma, \beta, \xi) = T^m_p (\lambda, \gamma, \beta, \xi) \),

\[
T^m_p (\lambda, \gamma, \beta, \xi) = \left\{ f \in T_p : \begin{vmatrix} \frac{z (D^m_{p,\lambda} f (z))'}{D^m_{p,\lambda} f (z)} - p \\ 2^\xi \left[ \frac{z (D^m_{p,\lambda} f (z))'}{D^m_{p,\lambda} f (z)} - \gamma \right] - \left[ \frac{z (D^m_{p,\lambda} f (z))'}{D^m_{p,\lambda} f (z)} - p \right] \end{vmatrix} < \beta, \right. \\
0 \leq \gamma < \frac{p}{2}, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, \lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0, z \in U \}
\]

which for \( p = 1 \) reduces to \( D^m (\gamma, \beta, \xi) \) studied by Juma and Kulkarni [7];

(3) For \( \ell = 0, \lambda = 1, \) in (1.9), the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \) reduces to the class \( T^m_p ([1, 0], \gamma, \beta, \xi) = T^m_p (\gamma, \beta, \xi) \),

\[
T^m_p (\gamma, \beta, \xi) = \left\{ f \in T_p : \begin{vmatrix} \frac{z (D^m f (z))'}{D^m f (z)} - p \\ 2^\xi \left[ \frac{z (D^m f (z))'}{D^m f (z)} - \gamma \right] - \left[ \frac{z (D^m f (z))'}{D^m f (z)} - p \right] \end{vmatrix} < \beta, \right. \\
0 \leq \gamma < \frac{\xi}{2}, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, m \in \mathbb{N}_0, p \in \mathbb{N}, z \in U \}
\]

(4) For \( p = 1 \) in (1.9) the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \) reduces to the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \)

\[
= \left\{ f \in T : \begin{vmatrix} \frac{z (I^m_p f (z))'}{I^m_p f (z)} - 1 \\ 2^\xi \left[ \frac{z (I^m_p f (z))'}{I^m_p f (z)} - \gamma \right] - \left[ \frac{z (I^m_p f (z))'}{I^m_p f (z)} - 1 \right] \end{vmatrix} < \beta, \right. \\
0 \leq \gamma < \frac{1}{2}, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, \lambda \geq 0, \ell \geq 0, m \in \mathbb{N}_0, z \in U \}
\]

\section{Coefficient Inequality}

Unless otherwise mentioned, we shall assume in the reminder of this paper that \( 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, 0 \leq \gamma < \frac{\xi}{2}, \lambda \geq 0, \ell \geq 0, p < k, n \geq k, m \geq 0, z \in U, \) and \( c^m_{k,p} (\lambda, \ell) \) is defined by (1.8).

\textbf{Theorem 1.} Let the function \( f \) be defined by (1.10). Then \( f \) is in the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \) if and only if

\[
\sum_{n=k}^{\infty} [(n-p) (1-\beta) + 2 \beta \xi (n-\gamma)] c^m_{n,p} (\lambda, \ell) a_n \leq 2 \xi \beta (p-\gamma). \quad (2.1)
\]
Proof. Assume that the inequality (2.1) holds true, we find from (1.10) that

\[
\left| z \left[ I_p^m (\lambda, \ell) f (z) \right]' - p I_p^m (\lambda, \ell) f (z) \right| - \beta \left| 2 \xi \left\{ z \left[ I_p^m (\lambda, \ell) f (z) \right]' - \gamma I_p^m (\lambda, \ell) f (z) \right\} \right|
\]

\[
= \left| \sum_{n=k}^{\infty} (n - p) c_{n, p}^m (\lambda, \ell) a_n z^n \right| - \beta \left| 2 \xi \left( (p - \gamma) z^p - \sum_{n=k}^{\infty} (n - \gamma) c_{n, p}^m (\lambda, \ell) a_n z^n \right) \right|
\]

\[
\leq \sum_{n=k}^{\infty} [(n - p) + 2 \xi \beta (n - \gamma) - \beta (n - p)] c_{n, p}^m (\lambda, \ell) a_n - 2 \beta \xi (p - \gamma)
\]

\[
= \sum_{n=k}^{\infty} [(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)] c_{n, p}^m (\lambda, \ell) a_n - 2 \beta \xi (p - \gamma) \leq 0.
\]

Hence, by the maximum modulus theorem, we have \( f \in T_p^m (\lambda, \ell, \gamma, \beta, \xi) \).

Conversely, let \( f \in T_p^m (\lambda, \ell, \gamma, \beta, \xi) \). Then

\[
\left| \frac{z \left( I_p^m (\lambda, \ell) f (z) \right)'}{I_p^m (\lambda, \ell) f (z)} - p \right| < \beta,
\]

that is, that

\[
\left| \sum_{n=k}^{\infty} (n - p) c_{n, p}^m (\lambda, \ell) a_n z^n \right| - \beta \left| 2 \xi \left( (p - \gamma) z^p - \sum_{n=k}^{\infty} (n - \gamma) c_{n, p}^m (\lambda, \ell) a_n z^n \right) \right|
\]

\[
\leq \sum_{n=k}^{\infty} [(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)] c_{n, p}^m (\lambda, \ell) a_n - 2 \beta \xi (p - \gamma) \leq 0.
\]

Now since \( \text{Re} \ f (z) \leq | f (z) | \) for all \( z \), we have

\[
\text{(2.2)}
\]
Re \[ \left\{ \frac{\sum_{n=k}^{\infty} (n - p) c_{n,p}^m (\lambda, \ell) a_n z^n}{2 \xi (p - \gamma) z^p - \sum_{n=k}^{\infty} (n - \gamma) c_{n,p}^m (\lambda, \ell) a_n z^n} + \sum_{n=k}^{\infty} (n - p) c_{n,p}^m (\lambda, \ell) a_n z^n} \right\} < \beta. \] (2.3)

Choose values of \( z \) on the real axis so that \( \frac{z (I_p^m (\lambda, \ell) f (z))'}{I_p^m (\lambda, \ell) f (z)} \) is real. Then upon clearing the denominator in (2.3) and letting \( z \to 1^- \) through real values, we have

\[ \frac{\sum_{n=k}^{\infty} (n - p) c_{n,p}^m (\lambda, \ell) a_n}{2 \xi (p - \gamma) - \sum_{n=k}^{\infty} (n - \gamma) c_{n,p}^m (\lambda, \ell) a_n} \leq \beta. \]

That is

\[ \sum_{n=k}^{\infty} [(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell) a_n \leq 2 \xi \beta (p - \gamma). \] (2.4)

This gives the required condition.

**Corollary 1.** Let the function \( f \) defined by (1.10) be in the class \( T_p^m (\lambda, \ell, \gamma, \beta, \xi) \), then we have

\[ a_n \leq \frac{2 \xi \beta (p - \gamma)}{[(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)]} c_{n,p}^m (\lambda, \ell) \] \( (n \geq k) \). (2.5)

The result is sharp for the function \( f \) given by

\[ f(z) = z^p - \frac{2 \xi \beta (p - \gamma)}{[(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)]} c_{n,p}^m (\lambda, \ell) z^n \] \( (n \geq k) \). (2.6)

### 3 Growth and Distortion Theorems

**Theorem 2.** Let the function \( f \) defined by (1.10) be in the class \( T_p^m (\lambda, \ell, \gamma, \beta, \xi) \). Then for \( |z| = r < 1 \), we have

\[ |f(z)| \geq r^p - \frac{2 \xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2 \beta \xi (k - \gamma)]} c_{k,p}^m (\lambda, \ell) r^k \] (3.1)
|f(z)| \leq r^p + \frac{2\xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)]} c_{k,p}^{m} (\lambda, \ell) r^k. \quad (3.2)

The equalities in (3.1) and (3.2) are attained for the function $f$ given by

$$f(z) = z^p - \frac{2\xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)]} c_{k,p}^{m} (\lambda, \ell) z^k,$$  

at $z = r$ and $z = re^{i(2n+1)\pi}$ $(n \geq k)$.

**Proof.** Since for $n \geq k$,

$$\sum_{n=k}^{\infty} [(n - p) (1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}^{m} (\lambda, \ell) a_n \leq 2\xi \beta (p - \gamma),$$

then

$$\sum_{n=k}^{\infty} a_n \leq \frac{2\xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)]} c_{k,p}^{m} (\lambda, \ell). \quad (3.4)$$

From (1.10) and (3.4), we have

$$|f(z)| \geq r^p - r^k \sum_{n=k}^{\infty} a_n \geq r^p - \frac{2\xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)]} c_{k,p}^{m} (\lambda, \ell) r^k,$$

and

$$|f(z)| \leq r^p + r^k \sum_{n=k}^{\infty} a_n \leq r^p + \frac{2\xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)]} c_{k,p}^{m} (\lambda, \ell) r^k.$$

This completes the proof of Theorem 2.

**Theorem 3.** Let the function $f$ defined by (1.10) be in the class $T_{p}^{m} (\lambda, \ell, \alpha, \beta, \xi)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq pr^{p-1} - \frac{2k\xi \beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)]} c_{k,p}^{m} (\lambda, \ell) r^{k-1} \quad (3.5)$$

and
On a New Class of \( p \)-Valent Functions

\[
\left| f'(z) \right| \leq pr^{p-1} + \frac{2k\xi\beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)] c_{k,p}(\lambda, \ell)} \lambda^{k-1}. \tag{3.6}
\]

The equalities in (3.5) and (3.6) are attained for the function \( f \) given by (3.3).

**Proof.** For \( n \geq k \),

\[
\left| f'(z) \right| \leq pr^{p-1} - r^{k-1} \sum_{n=k}^{\infty} na_n, \tag{3.7}
\]

and by Theorem 1, we have

\[
\sum_{n=k}^{\infty} na_n \leq \frac{2k\xi\beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)] c_{k,p}(\lambda, \ell)}. \tag{3.8}
\]

From (3.7) and (3.8), we have

\[
\left| f'(z) \right| \geq pr^{p-1} - r^{k-1} \sum_{n=k}^{\infty} na_n \geq pr^{p-1} - \frac{2k\xi\beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)] c_{k,p}(\lambda, \ell)} \lambda^{k-1}
\]

and

\[
\left| f'(z) \right| \leq pr^{p-1} + r^{k-1} \sum_{n=k}^{\infty} na_n \leq pr^{p-1} + \frac{2k\xi\beta (p - \gamma)}{[(k - p) (1 - \beta) + 2\beta \xi (k - \gamma)] c_{k,p}(\lambda, \ell)} \lambda^{k-1}.
\]

This completes the proof of Theorem 3.

### 4 Radii of Starlikeness, Convexity and Close-to-Convexity.

In this section we obtain the radii of \( p \)-valent starlikeness, \( p \)-valent convexity and \( p \)-valent close-to-convexity for functions in the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \).

**Theorem 4.** Let the function \( f \) defined by (1.10) be in the class \( T^m_p (\lambda, \ell, \gamma, \beta, \xi) \). Then \( f \) is \( p \)-valent starlike of order \( \delta, 0 \leq \delta < p \) in disc \( |z| < R_1 \) where

\[
R_1 = \inf_{n \geq k} \left\{ \frac{(p - \delta) [(n - p) (1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}(\lambda, \ell)}{2\beta \xi (p - \gamma) (n - \delta)} \right\}^{\frac{1}{p-1}}. \tag{4.1}
\]

The result is sharp, with the extremal function \( f \) given by (2.6).
Proof. It is sufficient to show that
\[
\left| \frac{zf''(z)}{f(z)} - p \right| \leq p - \delta \text{ for } |z| < R_1,
\] (4.2)
where \( R_1 \) is given by (4.1). Indeed we find, again from (1.10) that
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq \sum_{n=k}^{\infty} \frac{(n-p) a_n |z|^{n-p}}{1 - \sum_{n=k}^{\infty} a_n |z|^{n-p}}.
\]
Thus
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta,
\]
if
\[
\sum_{n=k}^{\infty} \frac{(n-\delta)}{(p-\delta)} a_n |z|^{n-p} \leq 1.
\] (4.3)
But, by Theorem 1, (4.3) will be true if
\[
\frac{(n-\delta)}{(p-\delta)} |z|^{n-p} \leq \left[ \frac{(n-p) (1-\beta) + 2\beta \xi (n-\gamma)}{2\beta \xi (p-\gamma)} \right] c_{n,p}^m (\lambda, \ell),
\] (4.4)
that is, if
\[
R_1 = |z| \leq \left\{ \frac{(p-\delta) [(n-p) (1-\beta) + 2\beta \xi (n-\gamma)] c_{n,p}^m (\lambda, \ell)}{2\beta \xi (p-\gamma) (n-\delta)} \right\}^{-\frac{1}{n-p}} \quad (n \geq k).
\] (4.5)
Theorem 4 follows easily from (4.5).

Theorem 5. Let the function \( f \) defined by (1.10) be in the class \( T_{p}^{m} (\lambda, \ell, \gamma, \beta, \xi) \). Then \( f \) is \( p \)-valent convex of order \( \delta, (0 \leq \delta < p) \) in the disc \( |z| < R_2 \), where
\[
R_2 = \inf_{n \geq k} \left\{ \frac{p (p-\delta) [(n-p) (1-\beta) + 2\beta \xi (n-\gamma)] c_{n,p}^m (\lambda, \ell)}{2\beta \xi n (p-\gamma) (n-\delta)} \right\}^{-\frac{1}{n-p}}.
\] (4.6)
The result is sharp for the function \( f \) given by (2.6).

Proof. We must show that
\[
\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \delta, \text{ for } |z| < R_2,
\]
where $R_2$ is given by (4.6). Indeed we find from (1.10) that
\[
\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq \frac{\sum_{n=k}^{\infty} n (n-p) a_n |z|^{n-p}}{p - \sum_{n=2}^{\infty} n a_n |z|^{n-p}}.
\]
Thus
\[
\left| \frac{zf''(z)}{f'(z)} + (1-p) \right| \leq p - \delta,
\]
if
\[
\sum_{n=p}^{\infty} \frac{n (n-p) a_n}{p (p - \delta)} |z|^{n-p} \leq 1.
\]
But, by Theorem 1, (4.7) will be true if
\[
\frac{(n - \delta)}{p (p - \delta)} |z|^{n-p} \leq \left[ \frac{(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)}{2 \beta \xi n (p - \gamma)} \right] c_{n,p}^m (\lambda, \ell),
\]
that is, if
\[
|z| \leq \left\{ \frac{p (p - \delta) [(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)}{2 \beta \xi n (p - \gamma) (n - \delta)} \right\}^{1/p}, \ n \geq k.
\]
Theorem 5 follows easily from (4.8).

**Corollary 2.** Let the function $f$ defined by (1.10) be in the class $T_p^m (\lambda, \ell, \gamma, \beta, \xi)$. Then $f$ is $p$-valent close-to-convex of order $\delta$, $(0 \leq \delta < p)$ in the disc $|z| < R_3$, where
\[
R_3 = \inf_{n \geq 2} \left\{ \frac{(p - \delta) [(n - p) (1 - \beta) + 2 \beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)}{2 \beta \xi n (p - \gamma)} \right\}^{1/p}.
\]
The result is sharp, with the extremal function $f$ given by (2.6).

## 5 Closure Theorems

**Theorem 6.** Let $\mu_j \geq 0$ for $j = 1, 2, \ldots, m$, and $\sum_{j=1}^{m} \mu_j \leq 1$. If the functions $f_j$ defined by
\[
f_j (z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n \ (a_{n,j} \geq 0; \ j = 1, 2, \ldots, m),
\]

are in the class $T^m_p (\lambda, \ell, \gamma, \beta, \xi)$, for every $j = 1, 2, ..., m$. Then the function $h(z)$ defined by
\[
h(z) = z^p - \sum_{n=k}^{\infty} \left( \sum_{j=1}^{m} \mu_j a_{n,j} \right) z^n,
\]
is in the class $T^m_p (\lambda, \ell, \gamma, \beta, \xi)$.

**Proof.** Since $f_j \in T^m_p (\lambda, \ell, \gamma, \beta, \xi)$, it follows from Theorem 1, that
\[
\sum_{n=k}^{\infty} \left( (n - p) (1 - \beta) + 2\beta\xi (n - \gamma) \right) c_{n,p}^m (\lambda, \ell) a_{n,j} \leq 2\xi\beta (p - \gamma),
\]
for every $j = 1, 2, ..., m$. Hence
\[
\sum_{n=k}^{\infty} \left( (n - p) (1 - \beta) + 2\beta\xi (n - \gamma) \right) c_{n,p}^m (\lambda, \ell) a_{n,j} \leq 2\beta\xi (p - \gamma) \sum_{j=1}^{m} \mu_j \leq 2\beta\xi (p - \gamma).
\]
By Theorem 1, it follows that $h(z) \in T^m_p (\lambda, \ell, \gamma, \beta, \xi)$, and so the proof of Theorem 6 is completed.

**Theorem 7.** Let $f_{k-1} (z) = z^p$ and
\[
f_n (z) = z^p - \frac{2\xi\beta (p - \gamma)}{(n - p) (1 - \beta) + 2\beta\xi (n - \gamma)} c_{n,p}^m (\lambda, \ell) z^n \quad (n \geq k).
\]
Then $f$ is in the class $T^m_p (\lambda, \ell, \gamma, \beta, \xi)$, if and only if it can be expressed in the form:
\[
f(z) = \sum_{n=k-1}^{\infty} \mu_n f_n (z),
\]
where $\mu_n \geq 0 \quad (n \geq k - 1)$ and $\sum_{n=k-1}^{\infty} \mu_n = 1$.

**Proof.** Assume that
\[
f(z) = \sum_{n=k-1}^{\infty} \mu_n f_n (z)
\]
\[
= z^p - \sum_{n=k}^{\infty} \frac{2\xi\beta (p - \gamma)}{(n - p) (1 - \beta) + 2\beta\xi (n - \gamma)} c_{n,p}^m (\lambda, \ell) \mu_n z^n.
\]
Then it follows that
\[
\sum_{n=k}^{\infty} \frac{[\lambda - \beta (n - \gamma)]}{2\lambda (p - \gamma)} = \sum_{n=k}^{\infty} \frac{\mu_n}{2\lambda (p - \gamma)} = 1 - \mu_{k-1} \leq 1.
\]

So, by Theorem 1, \( f \in T_{\lambda} (\lambda, \ell, \gamma, \beta, \xi) \).

Conversely, assume that the functions \( f \) defined by (1.10) belongs to the class \( T_{\lambda} (\lambda, \ell, \gamma, \beta, \xi) \). Then
\[
a_n \leq \frac{2\xi (p - \gamma)}{[\lambda - \beta (n - \gamma)] \lambda (p - \gamma) a_n (n \geq k)}.
\]

Setting
\[
\mu_n = \frac{[\lambda - \beta (n - \gamma)]}{2\lambda (p - \gamma) a_n (n \geq k)},
\]
and
\[
\mu_{k-1} = 1 - \sum_{n=k}^{\infty} \mu_n.
\]

We can see that \( f \) can by expressed in the form (5.3). This completes the proof of Theorem 7.

**Corollary 3.** The extreme points of the class \( T_{\lambda} (\lambda, \ell, \gamma, \beta, \xi) \) are the functions \( f_{k-1} = z^p \) and \( f_n \) \((n \geq k) \) given by (5.2).

### 6 Modified Hadamard Product

For the functions
\[
f_j (z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n (a_{n,j} \geq 0; j = 1, 2; p < k; p, k \in \mathbb{N}),
\]
we denote by \((f_1 \ast f_2)\) the modified Hadamard product (or convolution) of the functions \( f_1 \) and \( f_2 \), that is,
\[
(f_1 \ast f_2) (z) = z^p - \sum_{n=k}^{\infty} a_{n,1} a_{n,2} z^n.
\]
Theorem 8. Let the functions $f_j (j = 1, 2)$, defined by (6.1) be in the class $T^m_p (\lambda, \ell, \gamma, \beta, \xi)$. Then $(f_1 * f_2) \in T^m_p (\lambda, \ell, \mu, \beta, \xi)$, where

$$\mu = p - \frac{2\beta \xi (p - \gamma)^2 (k - p) [(1 - \beta) + 2\beta \xi]}{[(k - p)(1 - \beta) + 2\beta \xi (k - \gamma)]^2 c_{n,p}^m (\lambda, \ell) - 4\beta^2 \xi^2 (p - \gamma)^2}. \quad (6.3)$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest $\mu$ such that

$$\sum_{n=k}^{\infty} \frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \mu)] c_{n,p}^m (\lambda, \ell)}{2\xi \beta (p - \mu)} a_{n,1} a_{n,2} \leq 1. \quad (6.4)$$

Since $f_j (z) \in T^m_p (\lambda, \ell, \gamma, \beta, \xi) (j = 1, 2)$, we readily see that

$$\sum_{n=k}^{\infty} \frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} a_{n,1} \leq 1, \quad (6.5)$$

and

$$\sum_{n=k}^{\infty} \frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} a_{n,2} \leq 1, \quad (6.6)$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=k}^{\infty} \frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} \sqrt{a_{n,1} a_{n,2}} \leq 1. \quad (6.7)$$

Thus it is sufficient to show that

$$\frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \mu)] c_{n,p}^m (\lambda, \ell)}{2\xi \beta (p - \mu)} a_{n,1} a_{n,2} \leq \frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} \sqrt{a_{n,1} a_{n,2}} \quad (n \geq k), \quad (6.8)$$

that is, that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(p - \mu) [(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)]}{(p - \gamma) [(n - p)(1 - \beta) + 2\beta \xi (n - \mu)]} \quad (n \geq k). \quad (6.9)$$

From (6.7) we have

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{2\xi \beta (p - \gamma)}{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,p}^m (\lambda, \ell)} \quad (n \geq k). \quad (6.10)$$
Consequently, we need only to prove that
\[
\frac{2\xi \beta (p-\gamma)}{[(n-p)(1-\beta)+2\beta \xi (n-\gamma)] c_{n,p}^m (\lambda, \ell)} \leq \frac{(p-\mu)[(n-p)(1-\beta)+2\beta \xi (n-\mu)]}{(p-\gamma)[(n-p)(1-\beta)+2\beta \xi (n-\mu)]},
\]
(6.11)
or, equivalently, that
\[
\mu \leq p - \frac{2\beta \xi (p-\gamma)^2 (n-p) [(1-\beta)+2\beta \xi]}{[(n-p)(1-\beta)+2\beta \xi (n-\gamma)]^2 c_{n,p}^m (\lambda, \ell) - 4\beta^2 \xi^2 (p-\gamma)^2 (n \geq k)}.
\]
(6.12)
Since
\[
\Phi(n) = p - \frac{2\beta \xi (p-\gamma)^2 (n-p) [(1-\beta)+2\beta \xi]}{[(k-p)(1-\beta)+2\beta \xi (k-\gamma)]^2 c_{n,p}^m (\lambda, \ell) - 4\beta^2 \xi^2 (p-\gamma)^2},
\]
is an increasing function of \(n\) \((n \geq k)\), letting \(n = k\) in (6.13), we obtain
\[
\mu \leq \Phi(k) = p - \frac{2\beta \xi (p-\gamma)^2 (k-p) [(1-\beta)+2\beta \xi]}{[(k-p)(1-\beta)+2\beta \xi (k-\gamma)]^2 c_{n,p}^m (\lambda, \ell) - 4\beta^2 \xi^2 (p-\gamma)^2}.
\]
(6.14)
Finally, by taking the functions \(f_j (j = 1, 2)\) given by
\[
f_j(z) = z^p - \frac{2\xi \beta (p-\gamma)}{[(k-p)(1-\beta)+2\beta \xi (k-\gamma)] c_{n,p}^m (\lambda, \ell)} z^k, \quad (j = 1, 2),
\]
(6.15)
we can see that the result is sharp.

**Theorem 9.** Let the function \(f_j (j = 1, 2)\) defined by (6.1), \(f_1 \in T_p^m (\lambda, \ell, \mu_1, \beta, \xi)\) and \(f_2 \in T_p^m (\lambda, \ell, \mu_2, \beta, \xi)\). Then \((f_1 * f_2) \in T_p^m (\lambda, \ell, \mu, \beta, \xi)\), where
\[
\mu = p - \frac{2\xi \beta (p-\gamma_1) (p-\gamma_2) (k-p) [(1-\beta)+2\beta \xi]}{A_1 (\mu_1, p, \beta, \xi, k) \cdot A_2 (\mu_2, p, \beta, \xi, k) c_{k,p}^m (\lambda, \ell) - 4\beta^2 \xi^2 (p-\mu_1) (p-\mu_2)}
\]
(6.16)
and
\[
\begin{align*}
A_1 (\mu_1, p, \beta, \xi, k) &= [(k-p)(1-\beta)+2\beta \xi (k-\mu_1)] \\
A_2 (\mu_2, p, \beta, \xi, k) &= [(k-p)(1-\beta)+2\beta \xi (k-\mu_2)].
\end{align*}
\]
(6.17)

**Proof.** We need to find the largest \(\mu\) such that
\[
\sum_{n=k}^{\infty} \frac{[(n-p)(1-\beta)+2\beta \xi (n-\mu)]}{2\xi \beta (p-\mu)} c_{n,p}^m (\lambda, \ell) a_{n,1} a_{n,2} \leq 1,
\]

Since
\[
(f_1 \in T_p^m (\lambda, \ell, \mu_1, \beta, \xi) \text{ and } f_2 \in T_p^m (\lambda, \ell, \mu_2, \beta, \xi))
\]
Then
\[
\sum_{n=k}^{\infty} \frac{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu_1) \right] c_n^m (\lambda, \ell) a_{n,1}}{2\xi \beta (p - \mu_1)} \leq 1
\]
and
\[
\sum_{n=k}^{\infty} \frac{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu_2) \right] c_n^m (\lambda, \ell) a_{n,2}}{2\xi \beta (p - \mu_2)} \leq 1
\]
Therefore, by the Cauchy-Schwarz inequality, we obtain
\[
\sum_{n=k}^{\infty} \frac{\left[ A_1 (\mu_1, p, \beta, \xi, n) \right]^\frac{1}{2} \left[ A_2 (\mu_2, p, \beta, \xi, n) \right]^\frac{1}{2} c_n^m (\lambda, \ell)}{2\xi \sqrt{(p - \mu_1) (p - \mu_2)}} a_{n,1} a_{n,2} \leq 1, \quad (6.18)
\]
where
\[
A_1 (\mu_1, p, \beta, \xi, n) = \left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu_1) \right] \\
A_2 (\mu_2, p, \beta, \xi, n) = \left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu_2) \right].
\]
Thus we only need to show that find largest \( \mu \) such that
\[
\sum_{n=k}^{\infty} \frac{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu) \right] c_n^m (\lambda, \ell) a_{n,1} a_{n,2}}{2\xi \beta (p - \mu)} \leq 1
\]
or, equivalently, that
\[
\sqrt{a_{n,1} a_{n,2}} \leq \frac{p - \mu}{\sqrt{(p - \mu_1) (p - \mu_2)}} \left[ A_1 (\mu_1, p, \beta, \xi, n) \right]^\frac{1}{2} \left[ A_2 (\mu_2, p, \beta, \xi, n) \right]^\frac{1}{2} \frac{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu) \right]}{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu) \right]} \quad (n \geq k).
\]
Hence, in light of inequality (6.18), it is sufficient to prove that
\[
\frac{2\xi \beta \sqrt{(p - \mu_1) (p - \mu_2)}}{\left[ A_1 (\mu_1, p, \beta, \xi, n) \right]^\frac{1}{2} \left[ A_2 (\mu_2, p, \beta, \xi, n) \right]^\frac{1}{2} c_n^m (\lambda, \ell)} \leq \frac{p - \mu}{\sqrt{(p - \mu_1) (p - \mu_2)}} \left[ A_1 (\mu_1, p, \beta, \xi, n) \right]^\frac{1}{2} \left[ A_2 (\mu_2, p, \beta, \xi, n) \right]^\frac{1}{2} \frac{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu) \right]}{\left[ (n - p) \left( 1 - \beta \right) + 2\beta \xi (n - \mu) \right]} \quad (6.19)
\]
It follows from (6.19) that
\[
\mu = p - \frac{2\xi \beta (p - \mu_1) (p - \mu_2) (n - p) \left[ (1 - \beta) + 2\beta \xi \right]}{A_1 (\mu_1, p, \beta, \xi, n) . A_2 (\mu_2, p, \beta, \xi, n) c_n^m (\lambda, \ell) - 4\xi^2 \beta^2 (p - \mu_1) (p - \mu_2)}.
\]
Now, defining the function $\Phi (n)$ by

$$
\Phi (n) = p - \frac{2\xi \beta (p - \mu_1) (p - \mu_2) (n - p) [(1 - \beta) + 2\beta \xi]}{A_1 (\mu_1, p, \beta, \xi, n). A_2 (\mu_2, p, \beta, \xi, n) c_{n,\ell}^m (\lambda, \ell) - 4\xi^2 \beta^2 (p - \mu_1) (p - \mu_2)}.
$$

We see that $\Phi (n)$ is an increasing function of $n \ (n \geq k)$. Therefore, we conclude that

$$
\mu = \Phi (k) = p - \frac{2\xi \beta (p - \mu_1) (p - \mu_2) (k - p) [(1 - \beta) + 2\beta \xi]}{A_1 (\mu_1, p, \beta, \xi, k). A_2 (\mu_2, p, \beta, \xi, k) c_{n,\ell}^m (\lambda, \ell) - 4\xi^2 \beta^2 (p - \mu_1) (p - \mu_2)},
$$

where $A_1 (\mu_1, p, \beta, \xi, k)$ and $A_2 (\mu_2, p, \beta, \xi, k)$ are given by (6.17), which evidently completes the proof of Theorem 9.

**Theorem 10.** Let the functions $f_j \ (j = 1, 2)$ defined by (6.1) be in the class $T_p^m (\lambda, \ell, \gamma, \beta, \xi)$. Then the function

$$
h (z) = z^p - \sum_{n=k}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n
$$

belongs to the class $T_p^m (\lambda, \ell, \tau, \beta, \xi)$, where

$$
\tau \leq p - \frac{4\beta \xi (p - \gamma)^2 (n - p) [(1 - \beta) + 2\beta \xi]}{[(n - p) (1 - \beta) + 2\beta \xi (n - \gamma)]^2 c_{n,\ell}^m (\lambda, \ell) - 8\beta^2 \xi^2 (p - \gamma)^2}.
$$

The result is sharp for the functions $f_j \ (j = 1, 2)$ defined by (6.15).

**Proof.** By virtue of Theorem 1, we obtain

$$
s \sum_{n=k}^{\infty} \left[ \frac{[(n - p) (1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,\ell}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} \right]^2 a_{n,1}^2 \leq 1
$$

and

$$
s \sum_{n=k}^{\infty} \left[ \frac{[(n - p) (1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,\ell}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} \right]^2 a_{n,2}^2 \leq 1.
$$

It follows from (6.22) and (6.23) that

$$
s \sum_{n=k}^{\infty} \frac{1}{2} \left[ \frac{[(n - p) (1 - \beta) + 2\beta \xi (n - \gamma)] c_{n,\ell}^m (\lambda, \ell)}{2\xi \beta (p - \gamma)} \right]^2 \left( a_{n,1}^2 + a_{n,2}^2 \right) \leq 1.
$$
Therefore, we need to find the largest $\tau$ such that
\[
\frac{[(n - p)(1 - \beta) + 2\beta \xi (n - \tau)] c_{n,p}^m(\lambda, \ell)}{2\beta (p - \tau)} \leq 1 \quad (n \geq k),
\]
that is, that
\[
\tau \leq p - \frac{4\beta \xi (p - \gamma)^2 (n - p) (1 - \beta + 2\beta \xi)}{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)]^2 c_{n,p}^m(\lambda, \ell) - 8\beta^2 \xi^2 (p - \gamma)^2}. \quad (6.26)
\]

Since
\[
D(n) = p - \frac{4\beta \xi (p - \gamma)^2 (n - p) (1 - \beta + 2\beta \xi)}{[(n - p)(1 - \beta) + 2\beta \xi (n - \gamma)]^2 c_{n,p}^m(\lambda, \ell) - 8\beta^2 \xi^2 (p - \gamma)^2},
\]
is an increasing function of $n$ ($n \geq k$), we readily have
\[
\tau \leq D(k) = p - \frac{4\beta \xi (p - \gamma)^2 (k - p) (1 - \beta + 2\beta \xi)}{[(k - p)(1 - \beta) + 2\beta \xi (k - \gamma)]^2 c_{k,p}^m(\lambda, \ell) - 8\beta^2 \xi^2 (p - \gamma)^2},
\]
and Theorem 9 follows at once.

**Remark.** Specializing the parameters $m, \lambda, \ell$ and $p$ in the above results, we obtained results corresponding to the subclasses maintain in the introduction.

### 7 Open problem

The authors suggest to study:

Neighbourhood problems partial sums for the class $T_p^m(\lambda, \ell, \gamma, \beta, \xi)$.

### References


