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On a New Class of p-Valent Functions with Negative Coefficients Defined by Catas Operator

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Abstract

In this paper, we introduce a new class of analytic p-valent functions with negative coefficients defined in the open unit disc by using Cãtaş operator and obtain some results including coefficient inequality, distortion theorems, Hadamard products, radii of close-to-convexity, starlikeness and convexity, closure theorems and extreme points of functions in this class.

Keywords: *Analytic functions, p-valent functions, Hadamard product, radii of starlikeness and convexity, distortion theorem, coefficient inequality, closure theorem, Cãtaş operator.*
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1 Introduction

Let S_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad \left(p < k;\ p,k \in \mathbb{N} = \{1,2,...\} \right), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z :\mid z \mid < 1\}$.

A function f belonging to the class S_p is said to be p-valent starlike of order α in U if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < p;\ z \in U)\,. \tag{1.2}$$

Also a function f belonging to the class S_p is said to be p-valent convex of order α in U if and only if

$$Re\left\{1 + \frac{zf\ ''(z)}{f\ '(z)}\right\} > \alpha \quad (0 \leq \alpha < p;\ z \in U)\,. \tag{1.3}$$

We denote by $S_p^*(\alpha)$ the class of all functions in S_p which are p-valent starlike of order α in U and by $K_p(\alpha)$ the class of all functions in S_p which are p-valent convex of order α in U. We note that $S_p^* = S_p^*(0)\,,\ S^*(\alpha) = S_1^*(\alpha)\,,\ K_p = K_p(0)\ ,\ K(\alpha) = K_1(\alpha)\,,$ and

$$f(z) \in K_p(\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha) \tag{1.4}$$

Let $f \in S_p$ be given by (1.1) and $g \in S_p$ given by

$$g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n. \tag{1.5}$$

We define the Hadmard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{n=k}^{\infty} a_n b_n z^n = (g * f)(z)\ . \tag{1.6}$$

For positive real parameters $(\lambda \geq 0,\ \ell \geq 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})\,,$ Cãtaş[3] defined the linear operator $I_p^m(\lambda, \ell) : S_p \to S_p$ by:

$I_p^0(\lambda, \ell)\, f(z) = f(z)$

$$I_p^1(\lambda, \ell)\, f(z) = I_p(\lambda, \ell)\, f(z) = (1-\lambda)\, f(z) + \tfrac{\lambda}{(p+\ell)z^{\ell-1}}\left(z^\ell f(z)\right)'$$
$$= z^p + \sum_{n=k}^{\infty}\left(\tfrac{p+\ell+\lambda(n-p)}{p+\ell}\right) a_n z^n;$$

$$I_p^2(\lambda, \ell)\, f(z) = I_p(\lambda, \ell)\left(I_p(\lambda, \ell)\, f(z)\right)$$
$$= z^p + \sum_{n=k}^{\infty}\left(\tfrac{p+\ell+\lambda(n-p)}{p+\ell}\right)^2 a_n z^n;$$

and (in general)

$$I_p^m(\lambda, \ell)\, f(z) = I_p(\lambda, \ell)\left(I_p^{m-1}(\lambda, \ell)\, f(z)\right)$$

$$= z^p + \sum_{n=k}^{\infty} c_{n,p}^m(\lambda, \ell)\, a_n z^n, \tag{1.7}$$

where
$$c_{n,p}^{m}(\lambda,\ell)=\left(\frac{p+\ell+\lambda(n-p)}{p+\ell}\right)^{m}. \tag{1.8}$$
It is known that the operator generalizes many other operators sach as:
(1) $I_p^m(\lambda,0)f(z)=D_p^m(\lambda)f(z)$ (see El-Ashwah and Aouf [6]);
(2) $I_p^m(1,0)f(z)=D_p^m f(z)$ (see Kamali and Orhan [8] Aouf and Mostafa [2]);
(3) $I_1^m(\lambda,\ell)f(z)=I^m(\lambda,\ell)f(z)$ (*see* Cãtaş et al.[4]);
(4) $I_1^m(1,\ell)f(z)=I_\ell^m f(z)$ (see Cho and Srivastava [5]);
(5) $I_1^m(1,1)f(z)=I^m f(z)$ (see Uralgaddi and Somanatha [13]);
(6) $I_1^m(\lambda,0)f(z)=D_\lambda^m f(z)$ (see AI-Oboudi [1]);
(7) $I^m(1,0)f(z)=D^m f(z)$ (see Sălăgean [11]).

For $0<\beta\leq 1,\frac{1}{2}\leq\xi\leq 1,0\leq\gamma<\frac{p}{2},\ \lambda\geq 0,\ell\geq 0,p\in\mathbb{N},m\in\mathbb{N}_0$ and, $f\in S_p$ we define the class $S_p^m(\lambda,\ell,\gamma,\beta,\xi)$ by

$$\left|\frac{\dfrac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{I_p^m(\lambda,\ell)f(z)}-p}{2\xi\left[\dfrac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{I_p^m(\lambda,\ell)f(z)}-\gamma\right]-\left[\dfrac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{I_p^m(\lambda,\ell)f(z)}-p\right]}\right|<\beta. \tag{1.9}$$

For $m=0,k=p+1,p\in\mathbb{N}$ in (1.9), the class $S_p^m(\lambda,\ell,\gamma,\beta,\xi)$ reduces to the class $S_p^0(\lambda,\ell,\gamma,\beta,\xi)=S_p^0(\gamma,\beta,\xi)$(see Kulkarni et al. [10]).
Let T_p denote the subclass of S_p consisting of functions of the form

$$f(z)=z^p-\sum_{n=k}^{\infty}a_nz^n\quad,\ a_n\geq 0;z\in U. \tag{1.10}$$

Further, we define the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ by

$$T_p^m(\lambda,\ell,\gamma,\beta,\xi)=S_p^m(\lambda,\ell,\gamma,\beta,\xi)\cap T_p.$$

We note that:
(1) For $m=0$, in (1.9), the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ reduces to the class $T_p^0(\lambda,\ell,\gamma,\beta,\xi)=T_p(\gamma,\beta,\xi)$,

$$T_p(\gamma,\beta,\xi)=\left\{f\in T_p:\left|\frac{\dfrac{zf'(z)}{f(z)}-p}{2\xi\left[\dfrac{zf'(z)}{f(z)}-\gamma\right]-\left[\dfrac{zf'(z)}{f(z)}-p\right]}\right|<\beta,\right.$$

$$\left.\left(0\leq\gamma<\tfrac{p}{2},\ 0<\beta\leq 1,\tfrac{1}{2}\leq\xi\leq 1,z\in U\right)\right\};$$
which for $p=1$ reduces to $T(\gamma,\beta,\xi)$ studied by Kulkarni [9];

(2) For $\ell = 0$, in (1.9) , the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ reduces to the class $T_p^m([\lambda,0],\gamma,\beta,\xi) = T_p^m(\lambda,\gamma,\beta,\xi)$,

$$T_p^m(\lambda,\gamma,\beta,\xi) = \left\{ f \in T_p : \left| \frac{\dfrac{z\left(D_{p,\lambda}^m f(z)\right)'}{D_{p,\lambda}^m f(z)} - p}{2\xi\left[\dfrac{z\left(D_{p,\lambda}^m f(z)\right)'}{D_{p,\lambda}^m f(z)} - \gamma\right] - \left[\dfrac{z\left(D_{p,\lambda}^m f(z)\right)'}{D_{p,\lambda}^m f(z)} - p\right]} \right| < \beta, \right.$$

$$\left. \left(0 \le \gamma < \frac{p}{2}, 0 < \beta \le 1, \frac{1}{2} \le \xi \le 1, \lambda \ge 0, p \in \mathbb{N}, m \in \mathbb{N}_0, z \in U\right) \right\};$$

which for $p = 1$ reduces to $D_\lambda^m(\gamma,\beta,\xi)$ studied by Juma and Kulkarni [7] ;
(3) For $\ell = 0, \lambda = 1$, in (1.9) , the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ reduces to the class $T_p^m([1,0],\gamma,\beta,\xi) = T_p^m(\gamma,\beta,\xi)$,

$$T_p^m(\gamma,\beta,\xi) = \left\{ f \in T_p : \left| \frac{\dfrac{z\left(D_p^m f(z)\right)'}{D_p^m f(z)} - p}{2\xi\left[\dfrac{z\left(D_p^m f(z)\right)'}{D_p^m f(z)} - \gamma\right] - \left[\dfrac{z\left(D_p^m f(z)\right)'}{D_p^m f(z)} - p\right]} \right| < \beta, \right.$$

$$\left. \left(0 \le \gamma < \tfrac{p}{2},\ 0 < \beta \le 1, \tfrac{1}{2} \le \xi \le 1, m \in \mathbb{N}_0, p \in \mathbb{N}, z \in U\right) \right\};$$

(4) For $p = 1$ in (1.9) the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ reduces to the class $T^m(\lambda,\ell,\gamma,\beta,\xi)$

$$= \left\{ f \in T : \left| \frac{\dfrac{z\left(I^m(\lambda,\ell) f(z)\right)'}{I^m(\lambda,\ell) f(z)} - 1}{2\xi\left[\dfrac{z\left(I^m(\lambda,\ell) f(z)\right)'}{I^m(\lambda,\ell) f(z)} - \gamma\right] - \left[\dfrac{z\left(I^m(\lambda,\ell) f(z)\right)'}{I^m(\lambda,\ell) f(z)} - 1\right]} \right| < \beta, \right.$$

$$\left. \left(0 \le \gamma < \frac{1}{2}, 0 < \beta \le 1, \frac{1}{2} \le \xi \le 1, \lambda \ge 0, \ell \ge 0, m \in \mathbb{N}_0, z \in U\right) \right\}.$$

2 Coefficient Inequality

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0 < \beta \le 1,\ \frac{1}{2} \le \xi \le 1,\ 0 \le \gamma < \frac{p}{2},\ \lambda \ge 0,\ \ell \ge 0, p < k, n \ge k,\ m \ge 0, z \in U$, and $c_{k,p}^m(\lambda,\ell)$ is defined by (1.8) .

Theorem 1. Let the function f be defined by (1.10). Then f is in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ if and only if

$$\sum_{n=k}^{\infty} \left[(n-p)(1-\beta) + 2\beta\xi(n-\gamma)\right] c_{n,p}^m(\lambda,\ell)\, a_n \le 2\xi\beta(p-\gamma). \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds true, we find from (1.10) that

$$\left|z\left[I_p^m(\lambda,\ell)f(z)\right]' - pI_p^m(\lambda,\ell)f(z)\right| - \beta\left|2\xi\left\{z\left[I_p^m(\lambda,\ell)f(z)\right]' - \gamma I_p^m(\lambda,\ell)f(z)\right\}\right.$$

$$\left. - \left\{z\left[I_p^m(\lambda,\ell)f(z)\right]' - pI_p^m(\lambda,\ell)f(z)\right\}\right|$$

$$= \left|\sum_{n=k}^{\infty}(n-p)\,c_{n,p}^m(\lambda,\ell)\,a_n z^n\right| - \beta\left|2\xi\left[(p-\gamma)z^p - \sum_{n=k}^{\infty}(n-\gamma)\,c_{n,p}^m(\lambda,\ell)\,a_n z^n\right]\right.$$

$$\left. + \sum_{n=k}^{\infty}(n-p)\,c_{n,p}^m(\lambda,\ell)\,a_n z^n\right|$$

$$\leq \sum_{n=k}^{\infty}\left[(n-p) + 2\xi\beta(n-\gamma) - \beta(n-p)\right]c_{n,p}^m(\lambda,\ell)\,a_n - 2\beta\xi(p-\gamma)$$

$$= \sum_{n=k}^{\infty}\left[(n-p)(1-\beta) + 2\beta\xi(n-\gamma)\right]c_{n,p}^m(\lambda,\ell)\,a_n - 2\beta\xi(p-\gamma) \leq 0.$$

Hence, by the maximum modulus theorem, we have $f \in T_p^m(\lambda,\ell,\gamma,\beta,\xi)$.
Conversely, let $f \in T_p^m(\lambda,\ell,\gamma,\beta,\xi)$. Then

$$\left|\frac{\dfrac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{I_p^m(\lambda,\ell)f(z)} - p}{2\xi\left[\dfrac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{I_p^m(\lambda,\ell)f(z)} - \gamma\right] - \left[\dfrac{z\left(I_p^m(\lambda,\ell)f(z)\right)'}{I_p^m(\lambda,\ell)f(z)} - p\right]}\right| < \beta,$$

that is, that

$$\frac{\left|\sum\limits_{n=k}^{\infty}(n-p)\,c_{n,p}^m(\lambda,\ell)\,a_n z^n\right|}{\left|2\xi\left[(p-\gamma)z^p - \sum\limits_{n=k}^{\infty}(n-\gamma)\,c_{n,p}^m(\lambda,\ell)\,a_n z^n\right] + \sum\limits_{n=k}^{\infty}(n-p)\,c_{n,p}^m(\lambda,\ell)\,a_n z^n\right|} < \beta. \tag{2.2}$$

Now since $Re\, f(z) \leq |\, f(z)|$ for all z, we have

$$Re\left\{\frac{\sum\limits_{n=k}^{\infty}(n-p)\ c_{n,p}^{m}(\lambda,\ell)\,a_n z^n}{2\xi\left[(p-\gamma)\,z^p-\sum\limits_{n=k}^{\infty}(n-\gamma)\,c_{n,p}^{m}(\lambda,\ell)\,a_n z^n\right]+\sum\limits_{n=k}^{\infty}(n-p)\,c_{n,p}^{m}(\lambda,\ell)\,a_n z^n}\right\}<\beta. \tag{2.3}$$

Choose values of z on the real axis so that $\frac{z\left(I_p^m(\lambda,\ell)\,f(z)\right)'}{I_p^m(\lambda,\ell)\,f(z)}$ is real. Then upon clearing the denominator in (2.3) and letting $z\to 1^-$ through real values, we have

$$\frac{\sum\limits_{n=k}^{\infty}(n-p)\,c_{n,p}^{m}(\lambda,\ell)\,a_n}{2\xi\left[(p-\gamma)-\sum\limits_{n=k}^{\infty}(n-\gamma)\,c_{n,p}^{m}(\lambda,\ell)\,a_n\right]+\sum\limits_{n=k}^{\infty}(n-p)\,c_{n,p}^{m}(\lambda,\ell)\,a_n}\leq\beta.$$

That is

$$\sum_{n=k}^{\infty}\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]\ c_{n,p}^{m}(\lambda,\ell)\,a_n\leq 2\xi\beta(p-\gamma). \tag{2.4}$$

This gives the required condition.
Corollary 1. Let the function f defined by (1.10) be in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$, then we have

$$a_n\leq\frac{2\xi\beta(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]\ c_{n,p}^{m}(\lambda,\ell)}\quad(n\geq k). \tag{2.5}$$

The result is sharp for the function f given by

$$f(z)=z^p-\frac{2\xi\beta(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]\ c_{n,p}^{m}(\lambda,\ell)}z^n\quad(n\geq k). \tag{2.6}$$

3 Growth and Distortion Theorems

Theorem 2. Let the function f defined by (1.10) be in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$. Then for $|z|=r<1$, we have

$$|f(z)|\geq r^p-\frac{2\xi\beta(p-\gamma)}{\left[(k-p)(1-\beta)+2\beta\xi(k-\gamma)\right]\ c_{k,p}^{m}(\lambda,\ell)}r^k \tag{3.1}$$

and

$$|f(z)| \leq r^p + \frac{2\xi\beta\,(p-\gamma)}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell)}r^k. \tag{3.2}$$

The equalities in (3.1) and (3.2) are attained for the function f given by

$$f(z) = z^p - \frac{2\xi\beta\,(p-\gamma)}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell)}z^k, \tag{3.3}$$

at $z = r$ and $z = re^{i(2n+1)\pi}\,(n \geq k)$.

Proof. Since for $n \geq k$,

$$[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell)\sum_{n=k}^{\infty}a_n$$

$$\leq \sum_{n=k}^{\infty}[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^{m}\,(\lambda,\ell)\,a_n \leq 2\xi\beta\,(p-\gamma)\,,$$

then

$$\sum_{n=k}^{\infty}a_n \leq \frac{2\xi\beta\,(p-\gamma)}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell))}. \tag{3.4}$$

From (1.10) and (3.4) , we have

$$|f(z)| \geq r^p - r^k\sum_{n=k}^{\infty}a_n \ \geq r^p - \frac{2\xi\beta\,(p-\gamma)}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell))}r^k$$

and

$$|f(z)| \leq r^p + r^k\sum_{n=k}^{\infty}a_n \ \leq r^p + \ \frac{2\xi\beta\,(p-\gamma)}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell))}r^k.$$

This completes the proof of Theorem 2.

Theorem 3. Let the function f defined by (1.10) be in the class $T_p^m\,(\lambda,\ell,\alpha,\beta,\xi)$. Then for $|z| = r < 1$,we have

$$\left|f^{'}(z)\right| \geq pr^{p-1} - \frac{2k\xi\beta\,(p-\gamma)}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]\ \ c_{k,p}^{m}\,(\lambda,\ell))}r^{k-1} \tag{3.5}$$

and

$$\left|f^{'}(z)\right| \leq pr^{p-1} + \frac{2k\xi\beta\,(p-\gamma)}{\left[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)\right]\;c_{k,p}^{m}\,(\lambda,\ell))}r^{k-1}. \tag{3.6}$$

The equalities in (3.5) and (3.6) are attained for the function f given by (3.3).
Proof. For $n \geq k$,

$$\left|f^{'}(z)\right| \leq pr^{p-1} - r^{k-1}\sum_{n=k}^{\infty} na_n, \tag{3.7}$$

and by Theorem 1, we have

$$\sum_{n=k}^{\infty} na_n \leq \frac{2k\xi\beta\,(p-\gamma)}{\left[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)\right]\;c_{k,p}^{m}\,(\lambda,\ell))}. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\left|f^{'}(z)\right| \geq pr^{p-1} - r^{k-1}\sum_{n=k}^{\infty} na_n \geq pr^{p-1} - \frac{2k\xi\beta\,(p-\gamma)}{\left[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)\right]\;c_{k,p}^{m}\,(\lambda,\ell))}r^{k-1}$$

and

$$\left|f^{'}(z)\right| \leq pr^{p-1} + r^{k-1}\sum_{n=k}^{\infty} na_n \leq pr^{p-1} + \frac{2k\xi\beta\,(p-\gamma)}{\left[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)\right]\;c_{k,p}^{m}\,(\lambda,\ell))}r^{k-1}.$$

This completes the proof of Theorem 3.

4 Radii of Starlikeness, Convexity and Close-to-Convexity.

In this section we obtain the radii of p-valent starlikeness, p-valent convexity and p-valent close-to-convexity for functions in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$.
Theorem 4. Let the function f defined by (1.10) be in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$. Then f is p-valent starlike of order $\delta, 0 \leq \delta < p$ in disc $|z| < R_1$ where

$$R_1 = \inf_{n\geq k}\left\{\frac{(p-\delta)\left[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)\right]\;c_{n,p}^{m}\,(\lambda,\ell)}{2\beta\xi\,(p-\gamma)\,(n-\delta)}\right\}^{\frac{1}{n-p}}. \tag{4.1}$$

The result is sharp, with the externmal function f given by (2.6).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq p-\delta \text{ for } |z|<R_1, \tag{4.2}$$

where R_1 is given by (4.1). Indeed we find, again from (1.10) that

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq \frac{\sum\limits_{n=k}^{\infty}(n-p)\,a_n\,|z|^{n-p}}{1-\sum\limits_{n=k}^{\infty}a_n\,|z|^{n-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq p-\delta,$$

if

$$\sum_{n=k}^{\infty}\frac{(n-\delta)}{(p-\delta)}a_n\,|z|^{n-p} \leq 1. \tag{4.3}$$

But, by Theorem 1, (4.3) will be true if

$$\frac{(n-\delta)}{(p-\delta)}\,|z|^{n-p} \leq \frac{[(n-p)(1-\beta)+2\beta\xi\,(n-\gamma)]\; c_{n,p}^{m}(\lambda,\ell)}{2\beta\xi\,(p-\gamma)}, \tag{4.4}$$

that is, if

$$R_1=|z|\leq\left\{\frac{(p-\delta)\,[(n-p)(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^{m}(\lambda,\ell)}{2\beta\xi\,(p-\gamma)\,(n-\delta)}\right\}^{\frac{1}{n-p}} \quad (n\geq k). \tag{4.5}$$

Theorem 4 follows easily from (4.5).

Theorem 5. Let the function f defined by (1.10) be in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$. Then f is p-valent convex of order $\delta,(0\leq\delta<p)$ in the disc $|z|<R_2$, where

$$R_2=\inf_{n\geq k}\left\{\frac{p\,(p-\delta)\,[(n-p)(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^{m}(\lambda,\ell)}{2\beta\xi n\,(p-\gamma)\,(n-\delta)}\right\}^{\frac{1}{n-p}}. \tag{4.6}$$

The result is sharp for the function f given by (2.6).

Proof. We must show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq p-\delta, \text{ for } |z|<R_2,$$

where R_2 is given by (4.6). Indeed we find from (1.10) that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq \frac{\sum\limits_{n=k}^{\infty} n\,(n-p)\,a_n\,|z|^{n-p}}{p-\sum\limits_{n=2}^{\infty} na_n\,|z|^{n-p}}.$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}+(1-p)\right| \leq p-\delta,$$

if

$$\sum_{n=p}^{\infty}\frac{n\,(n-\delta)}{p\,(p-\delta)}a_n\,|z|^{n-p} \leq 1. \tag{4.7}$$

But, by Theorem 1, (4.7) will be true if

$$\frac{(n-\delta)}{p\,(p-\delta)}\,|z|^{n-p} \leq \frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\ c_{n,p}^{m}\,(\lambda,\ell)}{2\beta\xi n\,(p-\gamma)},$$

that is, if

$$|z| \leq \left\{\frac{p\,(p-\delta)\,[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^{m}\,(\lambda,\ell)}{2\beta\xi n\,(p-\gamma)\,(n-\delta)}\right\}^{\frac{1}{n-p}},\ n \geq k. \tag{4.8}$$

Theorem 5 follows easily from (4.8).

Corrolary 2. Let the function f defined by (1.10) be in the class$T_p^m\,(\lambda,\ell,\gamma,\beta,\xi)$. Then f is p-valent close-to-convex of order $\delta, (0 \leq \delta < p)$ in the disc $|z| < R_3$, where

$$R_3 = \inf_{n\geq 2}\left\{\frac{(p-\delta)\,[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^{m}\,(\lambda,\ell)}{2\beta\xi n\,(p-\gamma)}\right\}^{\frac{1}{n-p}}. \tag{4.9}$$

The result is sharp, with the extermal function f given by (2.6).

5 Closure Theorems

Theorem 6. Let $\mu_j \geq 0$ for $j = 1, 2, ..., m$, and $\sum\limits_{j=1}^{m}\mu_j \leq 1$. If the functions f_j defined by

$$f_j\,(z) = z^p - \sum_{n=k}^{\infty} a_{n,j}z^n\,(a_{n,j} \geq 0; j = 1, 2, ..., m)\,, \tag{5.1}$$

are in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$, for every $j=1,2,...,m$.Then the function $h(z)$ defined by

$$h(z)=z^p-\sum_{n=k}^{\infty}\left(\sum_{j=1}^{m}\mu_j a_{n,j}\right)z^n,$$

is in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$.

Proof. Since $f_j\in T_p^m(\lambda,\ell,\gamma,\beta,\xi)$, it follows from Theorem 1, that

$$\sum_{n=k}^{\infty}[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)\,a_{n,j}\le 2\xi\beta(p-\gamma),$$

for every $j=1,2,...,m$. Hence

$$\sum_{n=k}^{\infty}[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)\left(\sum_{j=1}^{m}\mu_j a_{n,j}\right)$$

$$=\sum_{j=1}^{m}\mu_j\left(\sum_{n=k-1}^{\infty}[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)\,a_{n,j}\right)$$

$$\le 2\beta\xi(p-\gamma)\sum_{j=1}^{m}\mu_j\le 2\beta\xi(p-\gamma).$$

By Theorem 1, it follows that $h(z)\in T_p^m(\lambda,\ell,\gamma,\beta,\xi)$, and so the proof of Theorem 6 is completed.

Theorem 7. Let $f_{k-1}(z)=z^p$ and

$$f_n(z)=z^p-\frac{2\xi\beta(p-\gamma)}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)}z^n\ (n\ge k). \qquad (5.2)$$

Then$f$ is in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$, if and only if it can be expressed in the form:

$$f(z)=\sum_{n=k-1}^{\infty}\mu_n f_n(z), \qquad (5.3)$$

where $\mu_n\ge 0\,(n\ge k-1)$ and $\sum\limits_{n=k-1}^{\infty}\mu_n=1$.

Proof. Assume that

$$f(z)=\sum_{n=k-1}^{\infty}\mu_n f_n(z)$$

$$=z^p-\sum_{n=k}^{\infty}\frac{2\xi\beta(p-\gamma)}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\ c_{n,p}^m(\lambda,\ell)}\mu_n z^n.$$

Then it follows that

$$\sum_{n=k}^{\infty} \frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right] c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)} \cdot$$

$$\cdot \frac{2\xi\beta(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right] c_{n,p}^{m}(\lambda,\ell)} \mu_n$$

$$= \sum_{n=k}^{\infty} \mu_n = 1-\mu_{k-1} \leq 1.$$

So, by Theorem 1, $f \in T_p^m(\lambda,\ell,\gamma,\beta,\xi)$.
Conversely, assume that the functions f defined by (1.10) belongs to the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$. Then

$$a_n \leq \frac{2\xi\beta(p-\gamma)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right] c_{n,p}^{m}(\lambda,\ell)} (n \geq k).$$

Setting

$$\mu_n = \frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right] c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)} a_n (n \geq k),$$

and

$$\mu_{k-1} = 1-\sum_{n=k}^{\infty} \mu_n.$$

We can see that f can by expressed in the form (5.3). This completes the proof of Theorem 7.
Corrolary 3. The extreme points of the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$ are the functions $f_{k-1}=z^p$ and f_n $(n \geq k)$ given by (5.2).

6 Modified Hadamard Product

For the functions

$$f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n \left(a_{n,j} \geq 0; j=1,2; p<k; p,k \in \mathbb{N}\right), \qquad (6.1)$$

we denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functionsf_1 andf_2,that is,

$$(f_1 * f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{n,1} a_{n,2} z^n. \qquad (6.2)$$

Theorem 8. Let the functions $f_j\,(j=1,2)$, defined by (6.1) be in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$. Then $(f_1*f_2)\in T_p^m(\lambda,\ell,\mu,\beta,\xi)$, where

$$\mu=p-\frac{2\beta\xi\,(p-\gamma)^2\,(k-p)\,[(1-\beta)+2\beta\xi]}{[(k-p)\,(1-\beta)+2\beta\xi\,(k-\gamma)]^2\,c_{n,p}^m(\lambda,\ell)-4\beta^2\xi^2\,(p-\gamma)^2}. \tag{6.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest μ such that

$$\sum_{n=k}^{\infty}\frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\mu)]\;c_{n,p}^m(\lambda,\ell)}{2\xi\beta\,(p-\mu)}a_{n,1}a_{n,2}\le 1. \tag{6.4}$$

Since $f_j(z)\in T_p^m(\lambda,\ell,\gamma,\beta,\xi)\,(j=1,2)$, we readily see that

$$\sum_{n=k}^{\infty}\frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)}{2\xi\beta\,(p-\gamma)}a_{n,1}\le 1, \tag{6.5}$$

and

$$\sum_{n=k}^{\infty}\frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\;c_{n,p}^m(\lambda,\ell)}{2\xi\beta\,(p-\gamma)}a_{n,2}\le 1, \tag{6.6}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=k}^{\infty}\frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\;c_{n,p}^m(\lambda,\ell)}{2\xi\beta\,(p-\gamma)}\sqrt{a_{n,1}a_{n,2}}\le 1. \tag{6.7}$$

Thus it is sufficient to show that

$$\begin{aligned}&\frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\mu)]\;c_{n,p}^m(\lambda,\ell)}{2\xi\beta\,(p-\mu)}a_{n,1}a_{n,2}\le\\ &\frac{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)}{2\xi\beta\,(p-\gamma)}\sqrt{a_{n,1}a_{n,2}}\;\;(n\ge k),\end{aligned} \tag{6.8}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}}\le\frac{(p-\mu)\,[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]}{(p-\gamma)\,[(n-p)\,(1-\beta)+2\beta\xi\,(n-\mu)]}\quad(n\ge k). \tag{6.9}$$

From (6.7) we have

$$\sqrt{a_{n,1}a_{n,2}}\le\frac{2\xi\beta\,(p-\gamma)}{[(n-p)\,(1-\beta)+2\beta\xi\,(n-\gamma)]\,c_{n,p}^m(\lambda,\ell)}\quad(n\ge k). \tag{6.10}$$

Consequently, we need only to prove that

$$\frac{2\xi\beta(p-\gamma)}{[(n\ -p)(1-\beta)+2\beta\xi(n\ -\gamma)]c_{n,p}^{m}(\lambda,\ell)} \leq \frac{(p-\mu)[(n\ -p)(1-\beta)+2\beta\xi(n\ -\gamma)]}{(p-\gamma)[(n\ -p)(1-\beta)+2\beta\xi(n\ -\mu)]}, \tag{6.11}$$

or, equivalently, that

$$\mu \leq p - \frac{2\beta\xi\left(p-\gamma\right)^{2}\left(n-p\right)\left[\left(1-\beta\right)+2\beta\xi\right]}{\left[\left(n\ -p\right)\left(1-\beta\right)+2\beta\xi\left(n\ -\gamma\right)\right]^{2}c_{n,p}^{m}\left(\lambda,\ell\right)-4\beta^{2}\xi^{2}\left(p-\gamma\right)^{2}}\ \ (n\geq k). \tag{6.12}$$

Since

$$\Phi\left(n\right) = p - \frac{2\beta\xi\left(p-\gamma\right)^{2}\left(n-p\right)\left[\left(1-\beta\right)+2\beta\xi\right]}{\left[\left(n\ -p\right)\left(1-\beta\right)+2\beta\xi\left(n\ -\gamma\right)\right]^{2}c_{n,p}^{m}\left(\lambda,\ell\right)-4\beta^{2}\xi^{2}\left(p-\gamma\right)^{2}}, \tag{6.13}$$

is an increasing function of n $(n \geq k)$, letting$n = k$ in (6.13), we obtain

$$\mu \leq \Phi\left(k\right) = p - \frac{2\beta\xi\left(p-\gamma\right)^{2}\left(k-p\right)\left[\left(1-\beta\right)+2\beta\xi\right]}{\left[\left(k-p\right)\left(1-\beta\right)+2\beta\xi\left(k-\gamma\right)\right]^{2}c_{n,p}^{m}\left(\lambda,\ell\right)-4\beta^{2}\xi^{2}\left(p-\gamma\right)^{2}}, \tag{6.14}$$

which proves the main assertion of Theorem 8.
Finally, by taking the functions $f_j\,(j=1,2)$ given by

$$f_j(z) = z^p - \frac{2\xi\beta\left(p-\gamma\right)}{\left[\left(k-p\right)\left(1-\beta\right)+2\beta\xi\left(k-\gamma\right)\right]\ c_{k,p}^{m}\left(\lambda,\ell\right)}z^k,\ \ (j=1,2)\,, \tag{6.15}$$

we can see that the result is sharp.
Theorem 9. Let the function $f_j\,(j=1,2)$ defined by (6.1) , $f_1 \in T_p^m\left(\lambda,\ell,\mu_1,\beta,\xi\right)$ and $f_2 \in T_p^m\left(\lambda,\ell,\mu_2,\beta,\xi\right)$. Then $\left(f_1 * f_2\right) \in T_p^m\left(\lambda,\ell,\mu,\beta,\xi\right)$, where

$$\mu = p - \frac{2\xi\beta\left(p-\mu_1\right)\left(p-\mu_2\right)\left(k-p\right)\left[\left(1-\beta\right)+2\beta\xi\right]}{A_1\left(\mu_1,p,\beta,\xi,k\right).A_2\left(\mu_2,p,\beta,\xi,k\right)c_{k,p}^{m}\left(\lambda,\ell\right)-4\xi^2\beta^2\left(p-\mu_1\right)\left(p-\mu_2\right)} \tag{6.16}$$

and

$$\begin{aligned} A_1\left(\mu_1,p,\beta,\xi,k\right) &= \left[\left(k-p\right)\left(1-\beta\right)+2\beta\xi\left(k-\mu_1\right)\right] \\ A_2\left(\mu_2,p,\beta,\xi,k\right) &= \left[\left(k-p\right)\left(1-\beta\right)+2\beta\xi\left(k-\mu_2\right)\right]. \end{aligned} \tag{6.17}$$

Proof. We need to find the largest μ such that

$$\sum_{n=k}^{\infty}\frac{\left[\left(n\ -p\right)\left(1-\beta\right)+2\beta\xi\left(n\ -\mu\right)\right]\ c_{n,p}^{m}\left(\lambda,\ell\right)}{2\xi\beta\left(p-\mu\right)}a_{n,1}a_{n,2} \leq 1,$$

Since

$$\left(f_1\ \in T_p^m\left(\lambda,\ell,\mu_1,\beta,\xi\right)\ \ and\ f_2 \in T_p^m\left(\lambda,\ell,\mu_2,\beta,\xi\right)\right)$$

Then

$$\sum_{n=k}^{\infty}\frac{\left[(n\ -p)\,(1-\beta)+2\beta\xi\,(n\ -\mu_1)\right]\ c_{n,p}^{m}\,(\lambda,\ell)}{2\xi\beta\,(p-\mu_1)}a_{n,1}\leq 1$$

and

$$\sum_{n=k}^{\infty}\frac{\left[(n\ -p)\,(1-\beta)+2\beta\xi\,(n\ -\mu_2)\right]\ c_{n,p}^{m}\,(\lambda,\ell)}{2\xi\beta\,(p-\mu_2)}a_{n,2}\leq 1$$

Therefore, by the Cauchy's-Schwarz inequality, we obtain

$$\sum_{n=k}^{\infty}\frac{\left[A_1\,(\mu_1,p,\beta,\xi,n)\right]^{\frac{1}{2}}\left[A_2\,(\mu_2,p,\beta,\xi,n)\right]^{\frac{1}{2}}c_{n,p}^{m}\,(\lambda,\ell)}{2\xi\beta\sqrt{(p-\mu_1)\,(p-\mu_2)}}\sqrt{a_{n,1}a_{n,2}}\leq 1, \tag{6.18}$$

where

$$A_1\,(\mu_1,p,\beta,\xi,n)=\left[(n-p)\,(1-\beta)+2\beta\xi\,(n-\mu_1)\right]$$
$$A_2\,(\mu_2,p,\beta,\xi,n)=\left[(n-p)\,(1-\beta)+2\beta\xi\,(n-\mu_2)\right].$$

Thus we only need to show that find largest μ such that

$$\sum_{n=k}^{\infty}\frac{\left[(n\ -p)\,(1-\beta)+2\beta\xi\,(n\ -\mu)\right]\ c_{n,p}^{m}\,(\lambda,\ell)}{2\xi\beta\,(p-\mu)}a_{n,1}a_{n,2}$$

$$\leq\sum_{n=k}^{\infty}\frac{\left[A_1\,(\mu_1,p,\beta,\xi,n)\right]^{\frac{1}{2}}\left[A_2\,(\mu_2,p,\beta,\xi,n)\right]^{\frac{1}{2}}c_{n,p}^{m}\,(\lambda,\ell)}{2\xi\beta\sqrt{(p-\mu_1)\,(p-\mu_2)}}\sqrt{a_{n,1}a_{n,2}}$$

or, equivalently, that

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{p-\mu}{\sqrt{(p-\mu_1)\,(p-\mu_2)}}\frac{\left[A_1\,(\mu_1,p,\beta,\xi,n)\right]^{\frac{1}{2}}\left[A_2\,(\mu_2,p,\beta,\xi,n)\right]^{\frac{1}{2}}}{\left[(n\ -p)\,(1-\beta)+2\beta\xi\,(n\ -\mu)\right]}\qquad(n\geq k)\,.$$

Hence, in light of inequality (6.18), it is sufficient to prove that

$$\frac{2\xi\beta\sqrt{(p-\mu_1)\,(p-\mu_2)}}{\left[A_1\,(\mu_1,p,\beta,\xi,n)\right]^{\frac{1}{2}}\left[A_2\,(\mu_2,p,\beta,\xi,n)\right]^{\frac{1}{2}}c_{n,p}^{m}\,(\lambda,\ell)}$$

$$\leq\frac{p-\mu}{\sqrt{(p-\mu_1)\,(p-\mu_2)}}\frac{\left[A_1\,(\mu_1,p,\beta,\xi,n)\right]^{\frac{1}{2}}\left[A_2\,(\mu_2,p,\beta,\xi,n)\right]^{\frac{1}{2}}}{\left[(n\ -p)\,(1-\beta)+2\beta\xi\,(n\ -\mu)\right]}. \tag{6.19}$$

It follows from (6.19) that

$$\mu=p-\frac{2\xi\beta\,(p-\mu_1)\,(p-\mu_2)\,(n-p)\left[(1-\beta)+2\beta\xi\right]}{A_1\,(\mu_1,p,\beta,\xi,n)\,.A_2\,(\mu_2,p,\beta,\xi,n)\,c_{k,p}^{m}\,(\lambda,\ell)-4\xi^2\beta^2\,(p-\mu_1)\,(p-\mu_2)}.$$

Now, defining the function $\Phi(n)$ by

$$\Phi(n) = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(n-p)[(1-\beta)+2\beta\xi]}{A_1(\mu_1,p,\beta,\xi,n).A_2(\mu_2,p,\beta,\xi,n)\,c_{k,p}^{m}(\lambda,\ell) - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)}.$$

We see that $\Phi(n)$ is an increasing function of n $(n \geq k)$. Therefore, we conclued that

$$\mu = \Phi(k) = p - \frac{2\xi\beta(p-\mu_1)(p-\mu_2)(k-p)[(1-\beta)+2\beta\xi]}{A_1(\mu_1,p,\beta,\xi,k).A_2(\mu_2,p,\beta,\xi,k)\,c_{k,p}^{m}(\lambda,\ell) - 4\xi^2\beta^2(p-\mu_1)(p-\mu_2)},$$

where $A_1(\mu_1,p,\beta,\xi,k)$ and $A_2(\mu_2,p,\beta,\xi,k)$ are given by (6.17), which evidently completes the proof of Theorem 9.

Theorem 10. Let the functions $f_j\,(j=1,2)$ defined by (6.1) be in the class $T_p^m(\lambda,\ell,\gamma,\beta,\xi)$.Then the function

$$h(z) = z^p - \sum_{n=k}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n \tag{6.20}$$

belongs to the class $T_p^m(\lambda,\ell,\tau,\beta,\xi)$, where

$$\tau \leq p - \frac{4\beta\xi(p-\gamma)^2(n-p)(1-\beta+2\beta\xi)}{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]^2\,c_{n,p}^{m}(\lambda,\ell) - 8\beta^2\xi^2(p-\gamma)^2} \tag{6.21}$$

The result is sharp for the functions $f_j\,(j=1,2)$ defined by (6.15).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{n=k}^{\infty}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\;c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)}\right]^2 a_{n,1}^2 \leq$$

$$\left[\sum_{n=k}^{\infty}\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\;c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)}a_{n,1}\right]^2 \leq 1 \tag{6.22}$$

and

$$\sum_{n=k}^{\infty}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\;c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)}\right]^2 a_{n,2}^2 \leq$$

$$\left[\sum_{n=k}^{\infty}\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\;c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)}a_{n,2}\right]^2 \leq 1. \tag{6.23}$$

It follows from (6.22) and (6.23) that

$$\sum_{n=k}^{\infty}\frac{1}{2}\left[\frac{[(n-p)(1-\beta)+2\beta\xi(n-\gamma)]\,c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)}\right]^2\left(a_{n,1}^2 + a_{n,2}^2\right) \leq 1. \tag{6.24}$$

Therefore, we need to find the largest τ such that

$$\frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\tau)\right]c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\tau)}\leq$$
$$\frac{1}{2}\left[\frac{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]c_{n,p}^{m}(\lambda,\ell)}{2\xi\beta(p-\gamma)}\right]^{2}\ (n\geq k)\,, \tag{6.25}$$

that is, that

$$\tau\leq p-\frac{4\beta\xi(p-\gamma)^{2}(n-p)(1-\beta+2\beta\xi)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]^{2}c_{n,p}^{m}(\lambda,\ell)-8\beta^{2}\xi^{2}(p-\gamma)^{2}}. \tag{6.26}$$

Since

$$D(n)=p-\frac{4\beta\xi(p-\gamma)^{2}(n-p)(1-\beta+2\beta\xi)}{\left[(n-p)(1-\beta)+2\beta\xi(n-\gamma)\right]^{2}c_{n,p}^{m}(\lambda,\ell)-8\beta^{2}\xi^{2}(p-\gamma)^{2}},$$

is an increasing function of n $(n\geq k)$, we readily have

$$\tau\leq D(k)=p-\frac{4\beta\xi(p-\gamma)^{2}(k-p)(1-\beta+2\beta\xi)}{\left[(k-p)(1-\beta)+2\beta\xi(k-\gamma)\right]^{2}c_{k,p}^{m}(\lambda,\ell)-8\beta^{2}\xi^{2}(p-\gamma)^{2}},$$

and Theorem 9 follows at once.

Remark. Specializing the parameters m,λ,ℓ and p in the above results, we obtained results corresponding to the subclasses maintain in the introduction.

7 Open problem

The authors suggest to study:

Neighbourhood problems partial sums for the class $T_{p}^{m}(\lambda,\ell,\gamma,\beta,\xi)$.

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