Int. J. Open Problems Complex Analysis, Vol. 6, No. 1, March 2014
ISSN 2074-2827; Copyright ©ICSRS Publication, 2014
www.i-csrs.org

Fekete-Szego Inequalities for Certain Class of Analytic Functions of Complex Order

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Abstract

In this paper, we obtain Fekete-Szegö inequalities for a certain class of analytic functions $f(z) \in \mathcal{A}$, ***for which*** $1+\frac{1}{b}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}-1\right] \prec \varphi(z)(b \neq 0,$ ***complex and*** $\alpha, \beta \geq 0)$. ***Sharp bounds for the Fekete-Szegö functional*** $|a_3 - \mu a_2^2|$ ***are obtained.***

Keywords and phrases: *Analytic functions, starlike and convex functions, close-to-convex functions, Fekete-Szegö problem, subordination.*
2000 Mathematics Subject Classification: 30C45, 30A20, 34A40.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in U). \tag{1.1}$$

Let $\mathcal{S}$ be the family of functions $f(z) \in \mathcal{A}$, which are univalent. A function $f(z) \in \mathcal{S}$ is said to be starlike of order ρ, denote by $\mathcal{S}^*(\rho)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \ (0 \leq \rho < 1; z \in U). \tag{1.2}$$

A function $f(z) \in \mathcal{S}$ is said to be convex of order ρ, denote by $\mathcal{K}(\rho)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho \ (0 \leq \rho < 1; z \in U). \tag{1.3}$$

The classes $\mathcal{S}^*(\rho)$ and $\mathcal{K}(\rho)$ were defined by Robertson [20]. From (1.2) and (1.3) it follows that

$$f(z) \in \mathcal{K}(\rho) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\rho). \tag{1.4}$$

A function $f(z) \in \mathcal{S}$ is said to be close-to-convex of order ρ, denote by $\mathcal{C}(\rho)$ if and only if

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \rho \ (0 \leq \rho < 1; g \in \mathcal{K}; z \in U). \tag{1.5}$$

where $\mathcal{C}(0) = \mathcal{C}$ (see Kaplan [6]).
We note that:

$$\mathcal{S}^*(0) = \mathcal{S}^* \text{ and } \mathcal{K}(0) = \mathcal{K}$$

and

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}.$$

A classical theorem of Fekete-Szegö [4] states that, for $f(z) \in \mathcal{S}$ given by (1.1),

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) \qquad \text{if } 0 \leq \mu \leq 1, \tag{1.6}$$

holds for any normalized univalent function of the form (1.1) in the open unit disc U and for $0 \leq \mu \leq 1$. This inequality is sharp for each μ(see [4]) . The coefficient functional

$$\phi_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\mu}{2}[f''(0)]\right), \tag{1.7}$$

on normalized analytic functions in U represents various geometric quantities for example, when $\mu = 1$, $\phi_1(f) = a_3 - a_2^2$, becomes $\mathcal{S}_f(0)/6$, where $\mathcal{S}_f$ denote the Schwarzain derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions in U. In literature, there exists a large number of results about inequalities for $\phi_\mu(f)$ corresponding to various subclass $\mathcal{S}$. The problem of maximising the absolute value of the functional is called the Fekete-Szegö problem (see [4] and [7]) , Koepf [8], solved the Fekete-Szegö problem for close-to-convex functions and the largest real number μ for which $\phi_\mu(f)$ is maximised by the Koebe function $z/(1-z)^2$ is $\mu = 1/3$ (see [8] and [13]), this result was generalized for functions that are close-to-convex.

Given two functions f and g, which are analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that $f(z) \prec g(z) \;\Rightarrow\; f(0) = g(0)$ and $f(U) \subset g(U)$.

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [15]).

Let $\varphi(z)$ be an analytic function with positive real part on U satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\mathcal{S}^*(\varphi)$ be the class of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in U), \tag{1.8}$$

and $\mathcal{K}(\varphi)$ be the class of functions $f(z) \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \ (z \in U). \tag{1.9}$$

The classes of $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ were introduced and studied by Ma and Minda [14]. The class $\mathcal{S}^*(\rho)$ of starlike functions of order ρ and the class $\mathcal{K}(\rho)$ of convex functions of order ρ $(0 \le \rho < 1)$ are the special cases of $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$, respectively, when $\varphi(z) = \dfrac{1 + (1-2\rho)z}{1-z}$ $(0 \le \rho < 1)$.

For $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. The class $\mathcal{P}^b_{\alpha,\beta}(\varphi)$ consists of all analytic functions $f(z) \in \mathcal{A}$ satisfying:

$$1 + \frac{1}{b}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} - 1\right] \prec \varphi(z). \tag{1.10}$$

We note that for suitable choices of α, β, b and $\varphi(z)$, we obtain the following subclasses:
(i) $\mathcal{P}^1_{1,0}(\varphi) = \mathcal{S}^*(\varphi)$ and $\mathcal{P}^1_{0,1}(\varphi) = \mathcal{K}(\varphi)$ (see Ma and Minda [14]);
(ii) $\mathcal{P}^b_{1,0}(\varphi) = \mathcal{S}^*_b(\varphi)(b \in \mathbb{C}^*)$ and $\mathcal{P}^b_{0,1}(\varphi) = \mathcal{C}_b(\varphi)(b \in \mathbb{C}^*)$ (see Ravichandran et al. [19]);

(iii) $\mathcal{P}_{1,0}^{1}(\frac{1+Az}{1+Bz}) = \mathcal{S}[A,B](-1 \leq B < A \leq 1)$ and $\mathcal{P}_{0,1}^{1}(\frac{1+Az}{1+Bz}) = \mathcal{K}[A,B](-1 \leq B < A \leq 1)$ (see Janowski [5]);
(iv) $\mathcal{P}_{1,0}^{1}\left(\frac{1+[B+(A-B)(1-\rho)]z}{1+Bz}\right) = \mathcal{S}^{*}(A,B,\rho)(0 \leq \rho < 1,\ -1 \leq B < A \leq 1)$ (see Aouf [1], with $p = 1$);
(v)$\mathcal{P}_{1,0}^{b}(\frac{1+z}{1-z}) = \mathcal{S}(b)(b \in \mathbb{C}^{*})$ (see Nasr and Aouf [17] and Aouf et al. [2]);
(vi) $\mathcal{P}_{0,1}^{b}(\frac{1+z}{1-z}) = \mathcal{C}(b)(b \in \mathbb{C}^{*})$ (see Waitrowski[22] and Nasr and Aouf [16]);
(vii) $\mathcal{P}_{1,0}^{1-\rho}(\frac{1+z}{1-z}) = \mathcal{S}^{*}(\rho)(0 \leq \rho < 1)$ and $\mathcal{P}_{0,1}^{1-\rho}(\frac{1+z}{1-z}) = \mathcal{K}(\rho)(0 \leq \rho < 1)$ (see Roberston [20]);
(iix) $\mathcal{P}_{\alpha,\beta}^{1}(\varphi) = \mathcal{M}_{\alpha,\beta}(\varphi)$ (see Ravichandran et al. [18]);
(ix) $\mathcal{P}_{1,0}^{\cos\lambda e^{-i\lambda}}(\frac{1+z}{1-z}) = \mathcal{S}^{\lambda}(|\lambda| < \frac{\pi}{2})$ (see Spacek [21]);
(x) $\mathcal{P}_{1,0}^{(1-\rho)\cos\lambda e^{-i\lambda}}(\frac{1+z}{1-z}) = \mathcal{S}^{\lambda}(\rho)(0 \leq \rho < 1,\ |\lambda| < \frac{\pi}{2})$ (see Keogh and Merkes [9] and Libera [10, 11]);
(xi) $\mathcal{P}_{0,1}^{(1-\rho)\cos\lambda e^{-i\lambda}}(\frac{1+z}{1-z}) = \mathcal{C}^{\lambda}(\rho)\ (0 \leq \rho < 1,\ |\lambda| < \frac{\pi}{2})$ (see Chichra [3]).

Also, we note that:
(i) Putting $\alpha = \beta = 1$ and $b = (1-\rho)\cos\lambda e^{-i\lambda}(0 \leq \rho < 1; |\lambda| < \frac{\pi}{2})$. Then, we have

$$\begin{aligned}\mathcal{P}_{1,1}^{(1-\rho)\cos\lambda e^{-i\lambda}}(\varphi) &= \mathcal{P}_{\rho}^{\lambda}(\varphi)\\ &= \left\{ f \in \mathcal{A} : \frac{e^{i\lambda}\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) - \rho\cos\lambda - i\sin\lambda}{(1-\rho)\cos\lambda} \prec \varphi(z)\right.\\ &\qquad \left.(0 \leq \rho < 1; |\lambda| < \frac{\pi}{2}; z \in U)\right\};\end{aligned}$$

(ii) Putting $b = (1-\rho)\cos\lambda e^{-i\lambda}(0 \leq \rho < 1; |\lambda| < \frac{\pi}{2})$. Then, we have

$$\begin{aligned}\mathcal{P}_{\alpha,\beta}^{(1-\rho)\cos\lambda e^{-i\lambda}}(\varphi) &= \mathcal{P}_{\alpha,\beta,\rho}^{\lambda}(\varphi)\\ &= \left\{ f \in \mathcal{A} : \frac{e^{i\lambda}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} - \rho\cos\lambda - i\sin\lambda}{(1-\rho)\cos\lambda} \prec \varphi(z)\right.\\ &\qquad \left.(0 \leq \alpha \leq 1; 0 \leq \beta \leq 1; 0 \leq \rho < 1; |\lambda| < \frac{\pi}{2}; z \in U)\right\};\end{aligned}$$

(iii) Putting $\beta = 0$ and $b = (1-\rho)\cos\lambda e^{-i\lambda}(0 \leq \rho < 1; |\lambda| < \frac{\pi}{2})$. Then, we have

$$\begin{aligned} \mathcal{P}_{\alpha,0}^{(1-\rho)\cos\lambda e^{-i\lambda}}(\varphi) &= \mathcal{S}_{\alpha,\rho}^{\lambda}(\varphi) \\ &= \left\{ f \in \mathcal{A} : \frac{e^{i\lambda}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} - \rho\cos\lambda - i\sin\lambda}{(1-\rho)\cos\lambda} \prec \varphi(z) \right. \\ &\qquad \left. (0 \leq \alpha \leq 1; 0 \leq \rho < 1; |\lambda| < \frac{\pi}{2}; z \in U) \right\}; \end{aligned}$$

(iv) Putting $\alpha = 0$ and $b = (1-\rho)\cos\lambda e^{-i\lambda}(0 \leq \rho < 1; |\lambda| < \frac{\pi}{2})$. Then, we have

$$\begin{aligned} \mathcal{P}_{0,\beta}^{(1-\rho)\cos\lambda e^{-i\lambda}}(\varphi) &= \mathcal{K}_{\beta,\rho}^{\lambda}(\varphi) \\ &= \left\{ f \in \mathcal{A} : \frac{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} - \rho\cos\lambda - i\sin\lambda}{(1-\rho)\cos\lambda} \prec \varphi(z) \right. \\ &\qquad \left. (0 \leq \beta \leq 1; 0 \leq \rho < 1; |\lambda| < \frac{\pi}{2}; z \in U) \right\}. \end{aligned}$$

In this paper, we obtain the Fekete-Szegö inequalities for functions in the class $\mathcal{P}_{\alpha,\beta}^{b}(\varphi)$.

2. Fekete-Szegö problem

To prove our results, we need the following lemmas.

Lemma 1 [9, 12]. *If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part in U and μ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leq 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 2 [14]. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part in U, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & if\ \nu \leq 0, \\ 2 & if\ 0 \leq \nu \leq 1, \\ 4\nu - 2 & if\ \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_2(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p_3(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z}\ (0 \leq \gamma \leq 1),$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if

$$\frac{1}{p_4(z)} = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z}\ (0 \leq \gamma \leq 1).$$

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \leq 2\ (0 < \nu \leq \frac{1}{2}),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \leq 2\ (\frac{1}{2} < \nu < 1).$$

Unless otherwise mentioned, we assume throughout this paper that $b \in \mathbb{C}^*$, $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$.

Theorem 1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...$, $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{P}_{\alpha,\beta}^b(\varphi)$ and μ is a complex number, then*

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \leq\ & \frac{B_1\left|b\right|}{2(\alpha+3\beta)}\max\left\{1, \left|-\frac{B_2}{B_1}\right.\right. \\ & \left.\left. + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta) + 4\mu(\alpha+3\beta)]b}{2(\alpha+2\beta)^2}B_1\right|\right\}.2.1 \quad (1) \end{aligned}$$

The result is sharp.

Proof. If $f(z) \in \mathcal{P}^b_{\alpha,\beta}(\varphi)$, then there is a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} - 1\right] = \varphi(w(z)). \tag{2.2}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \tag{2.3}$$

Since $w(z)$ is a Schwarz function, we see that *Re* $p_1(z) > 0$ and $p_1(0) = 1$. Define

$$p(z) = 1 + \frac{1}{b}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} - 1\right] = 1 + b_1 z + b_2 z^2 + \ldots. \tag{2.4}$$

In view of (2.2), (2.3) and (2.4), we have

$$p(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{2.5}$$

Since

$$\frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right) z^3 + \ldots\right].$$

Therefore, we have

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2} B_1 c_1 z + \left[\frac{1}{2} B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} B_2 c_1^2\right] z^2 + \ldots, \tag{2.6}$$

and from this equation and (2.4), we obtain

$$b_1 = \frac{1}{2} B_1 c_1,$$

and

$$b_2 = \frac{1}{2} B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} B_2 c_1^2.$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2) z^2 + \ldots,$$

and therefore we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1 + \alpha a_2 z + (2\alpha a_3 + \frac{\alpha^2 - 3\alpha}{2} a_2^2) z^2 + \ldots.$$

Similarly, we have

$$1+\frac{zf''(z)}{f'(z)}=1+2a_2z+(6a_3-4a_2^2)z^2+...,$$

and therefore we have

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}=1+2\beta a_2z+(6\beta a_3+2(\beta^2-3\beta)a_2^2z^2+....$$

Thus, we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}$$

$$=1+(\alpha+2\beta)a_2z+\left[2(\alpha+3\beta)a_3+\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}a_2^2\right]z^2+.... \quad (2.7)$$

Then, from (2.4), we see that

$$bb_1=(\alpha+2\beta)a_2, \quad (2.8)$$

and

$$bb_2=2(\alpha+3\beta)a_3+\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}a_2^2, \quad (2.9)$$

or, equivalently, we have

$$a_2=\frac{B_1c_1b}{2(\alpha+2\beta)},$$

and

$$a_3=\frac{b}{2(\alpha+3\beta)}\left[\frac{B_1c_2}{2}-\frac{1}{4}\left(B_1-B_2+\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]b}{2(\alpha+2\beta)^2}B_1^2\right)c_1^2\right].$$

Therefore

$$a_3-\mu a_2^2=\frac{B_1b}{4(\alpha+3\beta)}(c_2-\nu c_1^2), \quad (2.10)$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)+4\mu(\alpha+3\beta)]b}{2(\alpha+2\beta)^2}B_1\right]. \quad (2.11)$$

Our result now follows by an application of Lemma 1. The result is sharp for the functions

$$1+\frac{1}{b}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}-1\right]=\varphi(z^2), \quad (2.12)$$

and

$$1+\frac{1}{b}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}-1\right]=\varphi(z). \tag{2.13}$$

This completes the proof of Theorem 1.

Putting $b=1$ in Theorem 1, we improve the result obtained by Ravichandran et al. [18, Theorem 2.2], for the following corollary.

Corollary 1. *If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}_{\alpha,\beta}(\varphi)$ and μ is a complex number, then*

$$\begin{aligned}\left|a_3-\mu a_2^2\right| \leq\ & \frac{B_1}{2(\alpha+3\beta)}\max\left\{1,\left|-\frac{B_2}{B_1}\right.\right.\\ & \left.\left.+\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]+4\mu(\alpha+3\beta)}{2(\alpha+2\beta)^2}B_1\right|\right\}.\end{aligned}$$

The result is sharp.

By using Lemma 2, we can obtain the following theorem.

Theorem 2. *Let $\varphi(z)=1+B_1z+B_2z^2+...,(B_i>0, i\in\mathbb{N},\ b>0)$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{P}^{b}_{\alpha,\beta}$, then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases}\dfrac{b}{2(\alpha+3\beta)}\left\{B_2-\dfrac{\gamma}{2(\alpha+2\beta)^2}B_1^2\right\} & if\ \mu\leq\sigma_1,\\ \dfrac{B_1 b}{2(\alpha+3\beta)} & if\ \sigma_1\leq\mu\leq\sigma_2,\\ \dfrac{b}{2(\alpha+3\beta)}\left\{-B_2+\dfrac{\gamma}{2(\alpha+2\beta)^2}B_1^2\right\} & if\ \mu\geq\sigma_2,\end{cases} \tag{2.14}$$

where

$$\sigma_1 = \frac{2(\alpha+2\beta)^2(B_2-B_1)-[(\alpha+2\beta)^2-3(\alpha+4\beta)]bB_1^2}{4(\alpha+3\beta)bB_1^2}, 2.15 \tag{2}$$

$$\sigma_2 = \frac{2(\alpha+2\beta)^2(B_2+B_1)-[(\alpha+2\beta)^2-3(\alpha+4\beta)]bB_1^2}{4(\alpha+3\beta)bB_1^2}, 2.16 \tag{3}$$

and

$$\gamma=[(\alpha+2\beta)^2-3(\alpha+4\beta)+4\mu(\alpha+3\beta)]b. \tag{2.17}$$

The result is sharp.

To show that the bounds are sharp, we define the functions $K_{\varphi n}(z)$ $(n\geq 2)$ by

$$1+\frac{1}{b}\left[\left(\frac{zK'_{\varphi n}(z)}{K_{\varphi n}(z)}\right)^{\alpha}\left(1+\frac{zK''_{\varphi n}(z)}{K'_{\varphi n}(z)}\right)^{\beta}-1\right]=\varphi(z^{n-1}),\ K_{\varphi n}(0)=0=K'_{\varphi n}(0)-1,$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \le \eta \le 1)$ by

$$1+\frac{1}{b}\left[\left(\frac{zF_\eta'(z)}{F_\eta(z)}\right)^\alpha\left(1+\frac{zF_\eta''(z)}{F_\eta'(z)}\right)^\beta-1\right]=\varphi\left(\frac{z(z+\eta)}{1+\eta z}\right),\ F_\eta(0)=0=F_\eta'(0)-1,$$

and

$$1+\frac{1}{b}\left[\left(\frac{zG_\eta'(z)}{G_\eta(z)}\right)^\alpha\left(1+\frac{zG_\eta''(z)}{G_\eta'(z)}\right)^\beta-1\right]=\varphi\left(-\frac{z(z+\eta)}{1+\eta z}\right),\ G_\eta(0)=0=G_\eta'(0)-1.$$

Cleary the functions $K_{\varphi n}(z), F_\eta(z)$ and $G_\eta(z) \in \mathcal{P}^b_{\alpha,\beta}$. Also we write $K_\varphi = K_{\varphi 2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_η or one of its rotations. This completes the proof of Theorem 2.

Remark 1. *Putting $b = 1$ in Theorem 2, we obtain the result obtained by Ravichandran et al. [18, Theorem 2.1].*

Using arguments similar to those in the proof of Theorem 2, we obtain the following theorem.

Theorem 3. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + ...(B_i > 0, i \in \mathbb{N}, b > 0)$ and*

$$\sigma_3 = \frac{2(\alpha+2\beta)^2B_2 - \left[(\alpha+2\beta)^2 - 3(\alpha+4\beta)\right]bB_1^2}{4(\alpha+3\beta)bB_1^2}. \tag{2.18}$$

If $f(z)$ given by (1.1) belongs to the class $P^b_{\alpha,\beta}$, then we have

(i) If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{(\alpha+2\beta)^2}{4(\alpha+3\beta)bB_1^2}\left\{2(B_1-B_2)+\mu\frac{4(\alpha+3\beta)bB_1^2}{(\alpha+2\beta)^2}+\frac{\left[(\alpha+2\beta)^2-3(\alpha+4\beta)\right]bB_1^2}{(\alpha+2\beta)^2}\right\}|a_2|^2 \le \frac{bB_1}{2(\alpha+3\beta)}. \tag{2.19}$$

(ii) If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{(\alpha+2\beta)^2}{4(\alpha+3\beta)bB_1^2}\left\{2(B_1+B_2)-\mu\frac{4(\alpha+3\beta)bB_1^2}{(\alpha+2\beta)^2}-\frac{\left[(\alpha+2\beta)^2-3(\alpha+4\beta)\right]bB_1^2}{(\alpha+2\beta)^2}\right\}|a_2|^2 \le \frac{bB_1}{2(\alpha+3\beta)}. \tag{2.20}$$

where σ_1 and σ_2 given by (2.15) and (2.16).

Remark 2. *Putting $b = 1$ in Theorem 3, we obtaine the result obtained by Ravichandran et al. [18, Theorem 2.1].*

Remark 3. *Specializing the parameters α, β and b, we obtain results corresponding to the classes $\mathcal{P}_{\rho}^{\lambda}(\varphi)$, $\mathcal{P}_{\alpha,\beta,\rho}^{\lambda}(\varphi)$, $\mathcal{S}_{\alpha,\rho}^{\lambda}(\varphi)$ and $\mathcal{K}_{\beta,\rho}^{\lambda}(\varphi)$, mentioned in the introdution.*

3. Open problems

The authors suggest to introduce different operator on the function to define different classes and obtain different results.

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