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Intertwining operators associated with a singular integro-differential operator on the real line and certain of their applications

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Abstract

We consider a singular integro-differential operator Λ ***on the real line which includes as a particular case the Dunkl operator associated with the reflection group*** $\mathbb{Z}_2$ ***on*** $\mathbb{R}$***. We build intertwining operators of*** Λ ***and its dual*** $\widetilde{\Lambda}$ ***into the first derivative operator*** d/dx***. Using these intertwining operators, we firstly establish a Paley-Wiener theorem for the Fourier transform associated to*** Λ***, and secondly introduce a generalized convolution on*** $\mathbb{R}$ ***tied to*** Λ***.***

Keywords: *integro-differential operator, intertwining operators, Paley-Wiener theorem, generalized convolution.*
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1 Introduction

Consider the second-order singular differential operator on the real line

$$\Delta f(x) = \frac{d^2 f}{dx^2} + \frac{A'(x)}{A(x)} \frac{df}{dx} + q(x) f(x), \tag{1}$$

where q is a real-valued C^∞ even function on $\mathbb{R}$, and

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. Lions [7] has constructed an automorphism $\mathcal{X}$ of the space $\mathcal{E}_e(\mathbb{R})$ of C^∞ even functions on $\mathbb{R}$, satisfying

$$\mathcal{X} \frac{d^2}{dx^2} f = \Delta \mathcal{X} f \quad \text{and} \quad \mathcal{X} f(0) = f(0) \tag{2}$$

for all $f \in \mathcal{E}_e(\mathbb{R})$. The construction of the Lions operator $\mathcal{X}$ was aimed to allow the resolution of certain mixed value problems. Delsarte and Lions [2] have exploited the intertwining operator $\mathcal{X}$ to define in $\mathcal{E}_e(\mathbb{R})$ translation operators corresponding to the differential operator Δ. Later, Trimeche [12] investigated the dual operator ${}^t\mathcal{X}$ of $\mathcal{X}$, and obtained a Paley-Wiener theorem for the Fourier transform associated with Δ.

The main intention of this paper is to establish analogous results for the first-order singular integro-differential operator on $\mathbb{R}$

$$\Lambda f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)} \left(\frac{f(x) - f(-x)}{2} \right) + \frac{1}{A(x)} \int_0^x \left(\frac{f(t) + f(-t)}{2} \right) q(t) A(t) dt. \tag{3}$$

For $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, we regain the differential-difference operator

$$D_\alpha f = \frac{df}{dx} + (\alpha + \frac{1}{2}) \, \frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator with parameter $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl in connection with a generalization of the classical theory of spherical harmonics (see [3, 4, 11, 13] and the references therein). Besides its mathematical interest, the Dunkl operator D_α has quantum-mechanical applications; it is naturally involved in the study of one-dimensional harmonic oscillators governed by Wigner's commutation rules [5, 10, 15].

More precisely, we show in Section 2 that the solutions Φ_λ ($\lambda \in \mathbb{C}$) of the eigenvalue problem

$$\Lambda f(x) = i\lambda f(x), \quad f(0) = 1,$$

possess the Laplace type integral representation

$$\Phi_\lambda(x) = \int_{-|x|}^{|x|} K(x, y) e^{i\lambda y} dy, \quad x \neq 0.$$

Such a representation allows us to introduce in Section 3 a pair of integral transforms defined by

$$Vf(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y)f(y)dy & \text{if } x \neq 0, \\ f(0) & \text{if } x = 0, \end{cases}$$

$${}^tVf(y) = \int_{|x|\geq|y|} K(x,y)f(x)A(x)dx, \quad y \in \mathbb{R}.$$

Mainly, we show that V is the only automorphism of the space $\mathcal{E}(\mathbb{R})$ of $\mathcal{C}^\infty$ functions on $\mathbb{R}$, satisfying

$$V\frac{d}{dx}f = \Lambda Vf \quad \text{and} \quad Vf(0) = f(0)$$

for all $f \in \mathcal{E}(\mathbb{R})$. Moreover, we establish that ${}^tV$ is an automorphism of the space $\mathcal{D}(\mathbb{R})$ of $\mathcal{C}^\infty$ compactly supported functions on $\mathbb{R}$, satisfying

$$\frac{d}{dx}\,{}^tVf = {}^tV\widetilde{\Lambda}f, \quad f \in \mathcal{D}(\mathbb{R}),$$

$\widetilde{\Lambda}$ being the dual operator of Λ defined by

$$\widetilde{\Lambda}f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) + q(x)\int_{-\infty}^{x}\left(\frac{f(t)-f(-t)}{2}\right)dt. \quad (4)$$

The integral transform V (resp. ${}^tV$) is said to be an intertwining operator between Λ (resp. $\widetilde{\Lambda}$) and the first derivative operator d/dx.

Section 4 deals with the generalized Fourier transform defined on the space $\mathcal{E}'(\mathbb{R})$ of compactly supported distributions on $\mathbb{R}$ by

$$\mathcal{F}_\Lambda(S)(\lambda) = \langle S, \Phi_{-\lambda}\rangle, \quad \lambda \in \mathbb{C}.$$

This transform is factorized as the product of the usual Fourier transform on $\mathbb{R}$ and the intertwining operator ${}^tV$. Such a factorization gives rise to a Paley-Wiener theorem describing the spaces $\mathcal{D}(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$ through the properties of their generalized Fourier transforms. Furthermore, an inversion type formula for the generalized Fourier transform $\mathcal{F}_\Lambda$ is provided.

In Section 5, we exploit the intertwining operators V and ${}^tV$ to introduce a generalized convolution on $\mathbb{R}$ corresponding to the integro-differential operator Λ. Such a convolution is mapped firstly by the generalized Fourier transform $\mathcal{F}_\Lambda$ into the simple product, and secondly by the intertwining operator ${}^tV$ into the ordinary convolution on $\mathbb{R}$. The paper concludes with a Plancherel type formula for the generalized Fourier transform $\mathcal{F}_\Lambda$.

It is pointed out that all the results obtained in [9] may be recovered from those stated in the present article by simply taking $q = 0$. As for Lions operators [8], it is believed that our intertwining operators will be of great utility in the study of integro-differential problems, and will lead to generalizations of various analytic structures on the real line.

2 Laplace integral formula

In order to study the eigenfunctions of Λ, we need those of the differential operator Δ. Our basic reference about Δ will be the paper [12] from which we recall the following result.

Lemma 2.1 *(i) For each $\lambda \in \mathbb{C}$, the differential equation*

$$\Delta u = -\lambda^2 u, \quad u(0) = 1, \quad u'(0) = 0, \tag{5}$$

admits a unique even C^∞ solution on $\mathbb{R}$ denoted by φ_λ.

(ii) For every $x \in \mathbb{R}$, the function $\lambda \to \varphi_\lambda(x)$ is analytic.

(iii) For nonnegative x and complex λ, we have the majorization :

$$|\varphi_\lambda(x)| \leq \frac{Ce^{|\mathrm{Im}\lambda| x}}{\sqrt{B(x)}(1+|\lambda|x)^{\alpha+1/2}} \exp\left(\frac{Cx}{1+|\lambda|x}\int_0^x |\chi(t)|dt\right),$$

where C is a positive constant, and

$$\chi(x) = (2\alpha+1)\frac{B'(x)}{2xB(x)} + \frac{1}{2}\left(\frac{B'(x)}{B(x)}\right)' + \frac{1}{4}\left(\frac{B'(x)}{B(x)}\right)^2 - q(x). \tag{6}$$

Remark 2.2 *(i) For $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, the differential operator Δ is just the so-called Bessel operator. Furthermore,*

$$\varphi_\lambda(x) = j_\alpha(\lambda x),$$

where

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\,(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}) \tag{7}$$

is the normalized spherical Bessel function of index α (see [14]).

(ii) For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$ and $q(x) = (\alpha+\beta+1)^2$ with $\beta \geq -1/2$, the differential operator Δ reduces to the so-called Jacobi operator. Moreover,

$$\varphi_\lambda(x) = \psi_\lambda^{(\alpha,\beta)}(x),$$

where $\psi_\lambda^{(\alpha,\beta)}$ is the Jacobi function of index (α,β) given by

$$\psi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\Big(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}; \alpha+1; -\sinh^2 x\Big), \tag{8}$$

${}_2F_1$ being the Gauss hypergeometric function (see [6]).

Theorem 2.3 *For each $\lambda \in \mathbb{C}$, the integro-differential equation*

$$\Lambda u = i\lambda u, \qquad u(0) = 1, \tag{9}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ given by

$$\Phi_\lambda(x) = \varphi_\lambda(x) + \frac{i\lambda}{A(x)} \int_0^x \varphi_\lambda(t)A(t)dt. \tag{10}$$

Proof. Write $u = u_e + u_o$ with

$$u_e(x) = \frac{u(x) + u(-x)}{2} \quad \text{and} \quad u_o(x) = \frac{u(x) - u(-x)}{2}.$$

Then (9) is equivalent to the system

$$\begin{cases} (Au_o)' = i\lambda A u_e, \\ (Au_e')' + Aqu_e = -\lambda^2 A u_e, \\ u_e(0) = 1. \end{cases}$$

Identity (10) is now immediate. □

Remark 2.4 *(i) Assume that $q(x) = c$, where c is a given real number. Then (1) and (5) yield*

$$\frac{d}{dx}\varphi_\lambda(x) = -\frac{\lambda^2 + c}{A(x)} \int_0^x \varphi_\lambda(t)A(t)dt.$$

From this and (10) it follows that

$$\Phi_\lambda(x) = \begin{cases} \varphi_\lambda(x) - \dfrac{i\,\lambda}{\lambda^2 + c}\dfrac{d}{dx}\varphi_\lambda(x) & \text{if } \lambda^2 + c \neq 0, \\ 1 + \dfrac{i\lambda}{A(x)} \displaystyle\int_0^x A(t)dt & \text{if } \lambda^2 + c = 0. \end{cases}$$

(ii) If $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, then according to [4],

$$\Phi_\lambda(x) = e_\alpha(i\lambda x),$$

where e_α is the Dunkl kernel of index α given by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz) \quad (z \in \mathbb{C}),$$

j_α being as in (7).

(iii) Assume that $A(x) = (\sinh |x|)^{2\alpha+1} (\cosh x)^{2\beta+1}$ *and* $q(x) = (\alpha+\beta+1)^2$ *with* $\beta \geq -1/2$. *As by [6],*

$$\frac{d}{dx}\psi_\lambda^{(\alpha,\beta)}(x) = -\frac{\lambda^2+\rho^2}{4(\alpha+1)} \sinh(2x)\, \psi_\lambda^{(\alpha+1,\beta+1)}(x),$$

we deduce from (i) that

$$\Phi_\lambda(x) = \psi_\lambda^{(\alpha,\beta)}(x) + \frac{i\lambda}{4(\alpha+1)} \sinh(2x)\, \psi_\lambda^{(\alpha+1,\beta+1)}(x),$$

$\psi_\lambda^{(\alpha,\beta)}$ *being as in (8).*

Proposition 2.5 *(i) For every* $x \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$, *we have the estimate :*

$$|\Phi_\lambda(x)| \leq m(x)(1+|\lambda|)e^{|\mathrm{Im}\lambda||x|},$$

where m *is an even continuous function on* $\mathbb{R}$.

(ii) For every $x \in \mathbb{R}$, *the function* $\lambda \to \Phi_\lambda(x)$ *is analytic.*

(iii) For every $x \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$, *we have*

$$\Phi_{-\lambda}(x) = \Phi_\lambda(-x), \quad \overline{\Phi_\lambda(x)} = \Phi_{-\overline{\lambda}}(x). \tag{11}$$

Proof. Assertions (i) and (ii) follow directly from (10) and Lemma 2.1. Assertion (iii) is easily checked. □

Trimeche [12] obtained for $x \neq 0$ and $\lambda \in \mathbb{C}$ the following Mehler type representation for the eigenfunction $\varphi_\lambda(x)$ of the differential operator Δ :

$$\varphi_\lambda(x) = \int_0^{|x|} \mathcal{K}(x,y) \cos \lambda y \, dy, \tag{12}$$

where

$$\mathcal{K}(x,y) = H(x,y) + \frac{2\,\Gamma(\alpha+1)\,|x|^{-2\alpha}}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \sqrt{\frac{B(0)}{B(x)}}\,(x^2-y^2)_+^{\alpha-1/2}, \tag{13}$$

$H(x,\cdot)$ being an even continuous function on $\mathbb{R}$, with support in $[-|x|,|x|]$.

Lemma 2.6 *For* $x \in \mathbb{R}\setminus\{0\}$ *and* $\lambda \in \mathbb{C}$, *we have*

$$\frac{i\lambda}{A(x)} \int_0^{|x|} \varphi_\lambda(t)A(t)dt = -\frac{1}{2A(x)} \int_{-|x|}^{|x|} \frac{\partial}{\partial y} G_{\mathcal{K}}(x,y)\, e^{i\lambda y}\, dy, \tag{14}$$

where

$$G_{\mathcal{K}}(x,y) = \begin{cases} \displaystyle\int_{|y|}^{|x|} \mathcal{K}(t,y)A(t)dt & \text{if } |y| < |x|, \\ 0 & \text{otherwise .} \end{cases} \tag{15}$$

Proof. Recall first (see [9]) that the function $G_{\mathcal{K}}(x,\cdot)$ is continuous on $\mathbb{R}$, of class C^1 on $]-|x|,|x|[$, and $\frac{\partial}{\partial y}G_{\mathcal{K}}(x,\cdot)$ is integrable on $\mathbb{R}$. By using (12) and Fubini's theorem we obtain

$$\begin{aligned}\frac{i\lambda}{A(x)}\int_0^{|x|}\varphi_\lambda(t)A(t)dt &= \frac{i\lambda}{A(x)}\int_0^{|x|}\left(\int_0^t \mathcal{K}(t,y)\cos\lambda y\,dy\right)A(t)dt\\ &= \frac{i\lambda}{A(x)}\int_0^{|x|}\left(\int_y^{|x|}\mathcal{K}(t,y)A(t)dt\right)\cos\lambda y\,dy\\ &= \frac{i\lambda}{A(x)}\int_0^{|x|}G_{\mathcal{K}}(x,y)\cos\lambda y\,dy.\end{aligned}$$

An integration by parts yields

$$\frac{i\lambda}{A(x)}\int_0^{|x|}G_{\mathcal{K}}(x,y)\cos\lambda y\,dy = \frac{-i}{A(x)}\int_0^{|x|}\frac{\partial}{\partial y}G_{\mathcal{K}}(x,y)\sin\lambda y\,dy.$$

Identity (14) is now obvious. □

By combining (10), (12) and (14) we obtain the following result.

Theorem 2.7 *For $x\in\mathbb{R}\setminus\{0\}$ and $\lambda\in\mathbb{C}$, the eigenfunction $\Phi_\lambda(x)$ has the Laplace type integral representation*

$$\Phi_\lambda(x) = \int_{-|x|}^{|x|}K(x,y)\,e^{i\lambda y}\,dy, \tag{16}$$

where

$$K(x,y) = \frac{1}{2}\mathcal{K}(x,y) - \frac{sgn(x)}{2A(x)}\frac{\partial}{\partial y}G_{\mathcal{K}}(x,y). \tag{17}$$

3 Intertwining operators

Notation. We denote by $\mathcal{E}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. $\mathcal{E}'(\mathbb{R})$ stands for the space of distributions on $\mathbb{R}$ with compact support. Clearly Λ is a bounded linear operator from $\mathcal{E}(\mathbb{R})$ into itself. If $S\in\mathcal{E}'(\mathbb{R})$, we write ΛS for the compactly supported distribution on $\mathbb{R}$ defined by

$$\langle\Lambda S,f\rangle = -\langle S,\Lambda f\rangle,\quad f\in\mathcal{E}(\mathbb{R}).$$

Recall that each function f in $\mathcal{E}(\mathbb{R})$ may be decomposed uniquely into the sum $f=f_e+f_o$, where the even part f_e is defined by $f_e(x)=(f(x)+f(-x))/2$ and the odd part f_o by $f_o(x)=(f(x)-f(-x))/2$. $\mathcal{E}_e(\mathbb{R})$(resp. $\mathcal{E}_o(\mathbb{R})$) stands for the subspace of $\mathcal{E}(\mathbb{R})$ consisting of even (resp. odd) functions. For $a > 0$, $\mathcal{D}_a(\mathbb{R})$ designates the space of C^∞ functions on $\mathbb{R}$ supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$. Put $\mathcal{D}(\mathbb{R}) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology. $\mathcal{D}_e(\mathbb{R})$ (resp. $\mathcal{D}_o(\mathbb{R})$) denotes the subspace of $\mathcal{D}(\mathbb{R})$ consisting of even (resp. odd) functions. Let $\mathcal{I}$ (resp. $\mathcal{J}$) denotes the map defined on $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{D}(\mathbb{R})$) by $\mathcal{I}h(x) = \int_0^x h(t)dt$ (resp. $\mathcal{J}h(x) = \int_{-\infty}^x h(t)dt$).

Starting from the Laplace representation (16), we construct in this section a pair of integral transforms which turn out to be intertwining operators of Λ and its dual $\widetilde{\Lambda}$ into the first derivative operator d/dx.

Definition 3.1 *We define the integral transform V on $\mathcal{E}(\mathbb{R})$ by*

$$Vf(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y)f(y)dy & \text{if } x \neq 0, \\ \\ f(0) & \text{if } x = 0, \end{cases} \tag{18}$$

where $K(x, y)$ is given by (17).

Remark 3.2 *(i) It follows from (16) that*

$$\Phi_\lambda = V(e^{i\lambda\cdot}), \quad \textit{for all } \lambda \in \mathbb{C}. \tag{19}$$

(ii) If $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, then the integral transform V is given by

$$V(f)(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_{-1}^{1} f(tx)(1-t^2)^{\alpha-1/2}\,(1+t)\,dt,$$

and referred to as the Dunkl intertwining operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ (see [11]).

Trimeche [12] has proved that the Lions operator $\mathcal{X}$ may be written as

$$\mathcal{X}f(x) = \begin{cases} \displaystyle\int_0^{|x|} \mathcal{K}(x,y)f(y)dy & \text{if } x \neq 0, \\ \\ f(0) & \text{if } x = 0, \end{cases} \tag{20}$$

where $\mathcal{K}(x, y)$ is given by (13). For $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, the Lions operator $\mathcal{X}$ is just the Riemann-Liouville integral transform of order α defined by

$$R_\alpha(f)(x) = \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^1 f(tx)(1-t^2)^{\alpha-1/2}\,dt, \quad x \in \mathbb{R}.$$

Identity (20) will enable us to express the integral transform V in terms of $\mathcal{X}$. The following technical lemma, stated without proof, will be useful.

Lemma 3.3 *(i) The integral operator*

$$\mathcal{M}f(x) = \frac{1}{A(x)} \int_0^x f(t)A(t)dt \tag{21}$$

is an isomorphism from $\mathcal{E}_e(\mathbb{R})$ onto $\mathcal{E}_o(\mathbb{R})$, and its inverse operator is just Λ.

(ii) For all $f \in \mathcal{E}_e(\mathbb{R})$, we have the relations

$$\Lambda^2 f = \Delta f,$$

$$\Lambda f = \mathcal{M}\Delta f, \tag{22}$$

$$\mathcal{M}\mathcal{X}f = \frac{sgn(x)}{A(x)} \int_0^{|x|} G_{\mathcal{K}}(x,y)f(y)dy, \tag{23}$$

where $G_{\mathcal{K}}(x,y)$ is given by (15).

(iii) For any $f \in \mathcal{E}_e(\mathbb{R})$ and $g \in \mathcal{D}_o(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \mathcal{M}f(x)g(x)A(x)dx = -\int_{\mathbb{R}} f(x)\mathcal{J}g(x)A(x)dx. \tag{24}$$

Theorem 3.4 *For all $f \in \mathcal{E}(\mathbb{R})$ we have*

$$Vf = \mathcal{X}(f_e) + \mathcal{M}\mathcal{X}\frac{d}{dx}(f_o). \tag{25}$$

Proof. If $f \in \mathcal{E}_e(\mathbb{R})$, then (25) follows by combining (17), (18) and (20). If $f \in \mathcal{E}_o(\mathbb{R})$, then by (17), (18), (23) and an integration by parts we get

$$\begin{aligned} Vf(x) &= -\frac{\mathrm{sgn}(x)}{A(x)} \int_0^{|x|} \frac{\partial}{\partial y} G_{\mathcal{K}}(x,y)f(y)dy \\ &= \frac{\mathrm{sgn}(x)}{A(x)} \int_0^{|x|} G_{\mathcal{K}}(x,y)f'(y)dy \\ &= \mathcal{M}\mathcal{X}\frac{d}{dx}f(x). \end{aligned}$$

Therefore, identity (25) is true for every $f \in \mathcal{E}(\mathbb{R})$. $\square$

Remark 3.5 *(i) It follows from (2) and (22) that*

$$\mathcal{M}\mathcal{X}\frac{d}{dx}(f_o) = \mathcal{M}\mathcal{X}\frac{d^2}{dx^2}\mathcal{I}(f_o) = \mathcal{M}\Delta\mathcal{X}\mathcal{I}(f_o) = \Lambda\mathcal{X}\mathcal{I}(f_o).$$

So

$$Vf = \mathcal{X}(f_e) + \Lambda\mathcal{X}\mathcal{I}(f_o) \tag{26}$$

for all $f \in \mathcal{E}(\mathbb{R})$.

(ii) If $q(x) = 0$, then by (3) and (26),

$$Vf = \mathcal{X}(f_e) + \frac{d}{dx}\mathcal{X}\mathcal{I}(f_o) \tag{27}$$

for all $f \in \mathcal{E}(\mathbb{R})$.

Theorem 3.6 *The integral transform V is the only automorphism of $\mathcal{E}(\mathbb{R})$ satisfying*

$$V\frac{d}{dx}f = \Lambda V f \quad \text{and} \quad Vf(0) = f(0) \tag{28}$$

for all $f \in \mathcal{E}(\mathbb{R})$. The inverse transform V^{-1} is given by

$$V^{-1}f = \mathcal{X}^{-1}(f_e) + \mathcal{I}\mathcal{X}^{-1}\Lambda(f_o). \tag{29}$$

Proof. Notice that the first derivative operator d/dx is one-to-one from $\mathcal{E}_o(\mathbb{R})$ onto $\mathcal{E}_e(\mathbb{R})$, and $(d/dx)^{-1} = \mathcal{I}$. So according to (25) and Lemma 3.3(i), V is an automorphism of $\mathcal{E}(\mathbb{R})$. Let us check (28). If $f \in \mathcal{E}_e(\mathbb{R})$, then by (2), (22) and (25),

$$V\frac{d}{dx}f = \mathcal{M}\mathcal{X}\frac{d^2}{dx^2}f = \mathcal{M}\Delta\mathcal{X}f = \Lambda\mathcal{X}f = \Lambda V f.$$

If $f \in \mathcal{E}_o(\mathbb{R})$, then by (25) and Lemma 3.3(i),

$$\Lambda V f = \Lambda\mathcal{M}\mathcal{X}\frac{d}{dx}f = \mathcal{X}\frac{d}{dx}f = V\frac{d}{dx}f.$$

Finally, assume that $\widetilde{V}$ is another automorphism of $\mathcal{E}(\mathbb{R})$ satisfying (28). Set $\theta_\lambda(x) = \widetilde{V}(e^{i\lambda\cdot})(x)$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$. By (28) we have

$$\Lambda\theta_\lambda(x) = \Lambda\widetilde{V}(e^{i\lambda\cdot})(x) = \widetilde{V}\frac{d}{dx}(e^{i\lambda\cdot})(x) = i\lambda\widetilde{V}(e^{i\lambda\cdot})(x) = i\lambda\theta_\lambda(x),$$

$$\theta_\lambda(0) = \widetilde{V}(e^{i\lambda\cdot})(0) = 1.$$

From this and Theorem 2.3 we deduce that $\theta_\lambda(x) = \Phi_\lambda(x)$. As the functions $x \to e^{i\lambda x}$, $\lambda \in \mathbb{C}$, are dense in $\mathcal{E}(\mathbb{R})$, it follows that $\widetilde{V} = V$. This clearly achieves the proof. $\square$

Remark 3.7 *(i) By Theorem 3.6, V is an intertwining operator between Λ and d/dx on the space $\mathcal{E}(\mathbb{R})$.*

(ii) If $q(x) = 0$, it follows from (27) that

$$V^{-1}f = \mathcal{X}^{-1}(f_e) + \frac{d}{dx}\mathcal{X}^{-1}\mathcal{I}(f_o)$$

for all $f \in \mathcal{E}(\mathbb{R})$.

(iii) The inverse transform $\mathcal{X}^{-1}$ has been determined in [12, Theorem 5.3] in the form of an integro-differential operator.

(iv) For $a > 0$, let $\mathcal{E}_a(\mathbb{R})$ be the subspace of $\mathcal{E}(\mathbb{R})$ consisting of functions vanishing inside $[-a, a]$. Then from (18), (29) and [12, Theorem 5.3], it is not difficult to prove that the integral transform V is an automorphism of $\mathcal{E}_a(\mathbb{R})$.

Theorem 3.8 *(i) The dual transform ${}^tV$ of V, defined on $\mathcal{E}'(\mathbb{R})$ by*

$$\langle {}^tVS, f\rangle = \langle S, Vf\rangle, \quad f \in \mathcal{E}(\mathbb{R}),$$

is a bijection from $\mathcal{E}'(\mathbb{R})$ onto itself. More precisely, $\operatorname{supp} S \subset [-a, a]$ *if, and only if,* $\operatorname{supp} {}^tVS \subset [-a, a]$*. Moreover,*

$$\frac{d}{dx}\,{}^tVS = {}^tV\Lambda S, \quad \textit{for all } S \in \mathcal{E}'(\mathbb{R}). \tag{30}$$

(ii) If $f \in \mathcal{D}(\mathbb{R})$, then the distribution ${}^tV(Af)$ is given by the function

$${}^tVf(y) = \int_{|x|\geq|y|} K(x, y)f(x)A(x)dx, \quad y \in \mathbb{R}, \tag{31}$$

where $K(x, y)$ is given by (17).

Proof. Statement (ii) is obtained by using (18) and Fubini's theorem. Let us check (i). The fact that ${}^tV$ is one-to-one from $\mathcal{E}'(\mathbb{R})$ onto itself follows readily from Theorem 3.6. Identity (30) is a direct consequence of (28). Let $S \in \mathcal{E}'(\mathbb{R})$ be supported in $[-a, a]$. If $f \in \mathcal{E}(\mathbb{R})$ with support in $|x| > a$, then there is $\delta > 0$ such that $f = 0$ on $[-\delta - a, a + \delta]$, which implies that $Vf = 0$ on $[-\delta - a, a + \delta]$ by virtue of Remark 3.7(iv). Then $\operatorname{supp} Vf \subset |x| \geq a + \delta \subset |x| > a$, and consequently $\langle {}^tVS, f\rangle = \langle S, Vf\rangle = 0$. This proves that $|x| > a$ is a nullity open for ${}^tVS$, that is, $\operatorname{supp} {}^tVS \subset [-a, a]$. The same argument shows that $\operatorname{supp} {}^tV^{-1}S \subset [-a, a]$. □

Remark 3.9 *Theorem 3.8(ii) means that the integral transforms V and ${}^tV$, given respectively by (18) and (31), are transposed, i.e.,*

$$\int_{\mathbb{R}} Vf(x)g(x)A(x)dx = \int_{\mathbb{R}} f(y)\,{}^tVg(y)dy \tag{32}$$

for any $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$.

In order to study the integral transform ${}^tV$, we introduce an auxiliary integral transform ${}^t\mathcal{X}$ defined on $\mathcal{D}_e(\mathbb{R})$ by

$${}^t\mathcal{X}f(y) = \int_{|y|}^{\infty} \mathcal{K}(x, y)f(x)A(x)dx, \quad y \in \mathbb{R}, \tag{33}$$

where $\mathcal{K}(x, y)$ is given by (13). It was shown in [12] that ${}^t\mathcal{X}$ is an automorphism of $\mathcal{D}_e(\mathbb{R})$ satisfying the intertwining relation

$$\frac{d^2}{dx^2}\,{}^t\mathcal{X}f = {}^t\mathcal{X}\Delta f, \quad f \in \mathcal{D}_e(\mathbb{R}). \tag{34}$$

Moreover, $\mathcal{X}$ and ${}^t\mathcal{X}$ are dual in the sense of the relation

$$\int_{\mathbb{R}} \mathcal{X} f(x) g(x) A(x) dx = \int_{\mathbb{R}} f(y)\, {}^t\mathcal{X} g(y) dy, \tag{35}$$

which is valid for any $f \in \mathcal{E}_e(\mathbb{R})$ and $g \in \mathcal{D}_e(\mathbb{R})$. For $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, the intertwining operator ${}^t\mathcal{X}$ is exactly the Weyl integral transform of order α defined by

$$W_\alpha(f)(y) = \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_{|y|}^{\infty} f(x)\,(x^2-y^2)^{\alpha-1/2}\, x\, dx, \quad y \in \mathbb{R}.$$

Theorem 3.10 *For all $f \in \mathcal{D}(\mathbb{R})$,*

$${}^tV f = {}^t\mathcal{X}(f_e) + \frac{d}{dx}\, {}^t\mathcal{X}\mathcal{J}(f_o). \tag{36}$$

Proof. If $f \in \mathcal{D}_e(\mathbb{R})$, then (36) follows directly from (17), (31) and (33). Suppose $f \in \mathcal{D}_o(\mathbb{R})$. By (24), (25) and (35), we have

$$\begin{aligned}
\int_{\mathbb{R}} Vg(x) f(x) A(x) dx &= \int_{\mathbb{R}} \mathcal{M}\mathcal{X} \frac{d}{dx}(g_o)(x) f(x) A(x) dx \\
&= -\int_{\mathbb{R}} \mathcal{X} \frac{d}{dx}(g_o)(x) \mathcal{J} f(x) A(x) dx \\
&= -\int_{\mathbb{R}} \frac{d}{dx}(g_o)(x)\, {}^t\mathcal{X}\mathcal{J} f(x) dx \\
&= \int_{\mathbb{R}} g_o(x) \frac{d}{dx}\, {}^t\mathcal{X}\mathcal{J} f(x) dx \\
&= \int_{\mathbb{R}} g(x) \frac{d}{dx}\, {}^t\mathcal{X}\mathcal{J} f(x) dx
\end{aligned}$$

for any $g \in \mathcal{E}(\mathbb{R})$. This ends the proof in view of (32). □

Remark 3.11 *(i) It follows from (1), (4) and (34) that*

$$\frac{d}{dx}\, {}^t\mathcal{X}\mathcal{J}(f_o) = \mathcal{J} \frac{d^2}{dx^2}\, {}^t\mathcal{X}\mathcal{J}(f_o) = \mathcal{J}\, {}^t\mathcal{X} \Delta\, \mathcal{J}(f_o) = \mathcal{J}\, {}^t\mathcal{X}\, \widetilde{\Lambda}(f_o).$$

So

$${}^tV f = {}^t\mathcal{X}(f_e) + \mathcal{J}\, {}^t\mathcal{X}\, \widetilde{\Lambda}(f_o) \tag{37}$$

for all $f \in \mathcal{D}(\mathbb{R})$.

(ii) If $q(x) = 0$, then by (3), (4) and (37),

$${}^tV f = {}^t\mathcal{X}(f_e) + \mathcal{J}\, {}^t\mathcal{X}\, \Lambda(f_o)$$

for all $f \in \mathcal{D}(\mathbb{R})$.

A generalization of the classical integration by parts formula is as follows.

Lemma 3.12 *Let $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \Lambda f(x)g(x)A(x)dx = -\int_{\mathbb{R}} f(x)\widetilde{\Lambda}g(x)A(x)dx, \tag{38}$$

where $\widetilde{\Lambda}$ is given by (4).

Proof. We have

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda f(x)g(x)A(x)dx &= \int_{\mathbb{R}} (\Lambda f)_e(x)g_e(x)A(x)dx + \int_{\mathbb{R}} (\Lambda f)_o(x)g_o(x)A(x)dx \\
&= \kappa_1 + \kappa_2.
\end{aligned}$$

By (3), (4) and (24) we get

$$\begin{aligned}
\kappa_1 &= \int_{\mathbb{R}} \left(f_o'(x) + \frac{A'(x)}{A(x)} f_o(x) \right) g_e(x)A(x)dx \\
&= \int_{\mathbb{R}} (A(x)f_o(x))' g_e(x)dx \\
&= -\int_{\mathbb{R}} f_o(x)g_e'(x)A(x)dx \\
&= -\int_{\mathbb{R}} f_o(x)(\widetilde{\Lambda}g)_o(x)A(x)dx
\end{aligned}$$

and

$$\begin{aligned}
\kappa_2 &= \int_{\mathbb{R}} f_e'(x)g_o(x)A(x)dx + \int_{\mathbb{R}} \mathcal{M}(qf_e)(x)g_o(x)A(x)dx \\
&= -\int_{\mathbb{R}} f_e(x)(A(x)g_o(x))'dx - \int_{\mathbb{R}} q(x)f_e(x)\mathcal{J}(g_o)(x)A(x)dx \\
&= -\int_{\mathbb{R}} f_e(x) \left(g_o'(x) + \frac{A'(x)}{A(x)} g_o(x) + q(x)\mathcal{J}(g_o)(x) \right) A(x)dx \\
&= -\int_{\mathbb{R}} f_e(x)(\widetilde{\Lambda}g)_e(x)A(x)dx
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa_1 + \kappa_2 &= -\int_{\mathbb{R}} f_e(x)(\widetilde{\Lambda}g)_e(x)A(x)dx - \int_{\mathbb{R}} f_o(x)(\widetilde{\Lambda}g)_o(x)A(x)dx \\
&= -\int_{\mathbb{R}} f(x)\widetilde{\Lambda}g(x)A(x)dx.
\end{aligned}$$

This clearly yields the result. $\square$

Theorem 3.13 *(i) The integral transform* ${}^tV$ *is an automorphism of* $\mathcal{D}(\mathbb{R})$*. More precisely,* $f \in \mathcal{D}_a(\mathbb{R})$ *if, and only if,* ${}^tVf \in \mathcal{D}_a(\mathbb{R})$*.*

(ii) The inverse transform ${}^tV^{-1}$ *is given by*

$$ {}^tV^{-1}f = {}^t\mathcal{X}^{-1}(f_e) + \frac{d}{dx}\,{}^t\mathcal{X}^{-1}\mathcal{J}(f_o). $$

(iii) The transform ${}^tV$ *satisfies the permutation relation*

$$ \frac{d}{dx}\,{}^tVf = {}^tV\widetilde{\Lambda}f, \quad f \in \mathcal{D}(\mathbb{R}). \tag{39} $$

Proof. Observe that $\mathcal{J}$ is one-to-one from $\mathcal{D}_o(\mathbb{R})$ onto $\mathcal{D}_e(\mathbb{R})$, and $\mathcal{J}^{-1} = d/dx$. So (i) and (ii) follow from (36). Moreover, by (28), (32) and (38),

$$ \begin{aligned} \int_{\mathbb{R}} \frac{d}{dx}\,{}^tVf(x)g(x)dx &= -\int_{\mathbb{R}} {}^tVf(x)\frac{d}{dx}g(x)dx \\ &= -\int_{\mathbb{R}} f(x)\,V\frac{d}{dx}g(x)A(x)dx \\ &= -\int_{\mathbb{R}} f(x)\,\Lambda Vg(x)A(x)dx \\ &= \int_{\mathbb{R}} \widetilde{\Lambda}f(x)\,Vg(x)A(x)dx \\ &= \int_{\mathbb{R}} {}^tV\widetilde{\Lambda}f(x)\,g(x)dx \end{aligned} $$

for any $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$. This proves (39). □

Remark 3.14 *(i) From Theorem 3.13, we deduce that* ${}^tV$ *is an intertwining operator between* $\widetilde{\Lambda}$ *and* d/dx *on the space* $\mathcal{D}(\mathbb{R})$*.*

(ii) The inverse transform ${}^t\mathcal{X}^{-1}$ *has been expressed in [12, Theorem 6.3] in the form of an integro-differential operator.*

4 Generalized Fourier transform

Notation. We denote by

– $\mathbf{H}_a$, $a > 0$, the space of entire, rapidly decreasing functions of exponential type a; that is, $f \in \mathbf{H}_a$ if and only if , f is entire on $\mathbb{C}$ and for all $m = 0, 1, ...,$

$$ p_m(f) = \sup_{\lambda\in\mathbb{C}} \left|(1+\lambda)^m f(\lambda)\mathrm{e}^{-a|\mathrm{Im}\lambda|}\right| < \infty. $$

The topology of $\mathbf{H}_a$ is defined by the semi-norms p_m, $m = 0, 1, ...$.

– $\mathbf{H} = \cup_{a>0}\mathbf{H}_a$, endowed with the inductive limit topology.

– $\mathcal{H}_a$, $a > 0$, the space of entire, slowly increasing functions of exponential type a; that is, $f \in \mathcal{H}_a$, if and only if, f is entire on $\mathbb{C}$ and there is $m = 0, 1, ...$ such that,

$$\sup_{\lambda\in\mathbb{C}} \left|(1+|\lambda|)^{-m} f(\lambda) \mathrm{e}^{-a|\mathrm{Im}\lambda|}\right| < \infty.$$

– $\mathcal{H} = \cup_{a>0}\mathcal{H}_a$.

Definition 4.1 *(i) The generalized Fourier transform of a distribution $S \in \mathcal{E}'(\mathbb{R})$ is defined by*

$$\mathcal{F}_\Lambda(S)(\lambda) = \langle S, \Phi_{-\lambda}\rangle, \quad \lambda \in \mathbb{C}.$$

(ii) The generalized Fourier transform of a function $f \in \mathcal{D}(\mathbb{R})$ is defined by

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-\lambda}(x)A(x)dx, \quad \lambda \in \mathbb{C}.$$

Remark 4.2 *For $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, the transform $\mathcal{F}_\Lambda$ is exactly the Dunkl transform with parameter $\alpha+1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ (see [3]).*

Theorem 4.3 *(i) We have*

$$\mathcal{F}_\Lambda(S)(\lambda) = \mathcal{F}_u\left({}^tVS\right)(\lambda), \qquad \text{for all } S \in \mathcal{E}'(\mathbb{R}), \tag{40}$$

$$\mathcal{F}_\Lambda(f)(\lambda) = \mathcal{F}_u\left({}^tVf\right)(\lambda), \qquad \text{for all } f \in \mathcal{D}(\mathbb{R}), \tag{41}$$

where $\mathcal{F}_u$ denotes the usual Fourier transform on $\mathbb{R}$ given by

$$\mathcal{F}_u(S)(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x}\, dS(x), \qquad S \in \mathcal{E}'(\mathbb{R}).$$

(ii) For all $f \in \mathcal{D}(\mathbb{R})$ and $\lambda \in \mathbb{C}$,

$$\mathcal{F}_\Lambda\left(\overline{f}\right)(\lambda) = \overline{\mathcal{F}_\Lambda(f)\left(-\overline{\lambda}\right)}, \quad \mathcal{F}_\Lambda(f^-)(\lambda) = \mathcal{F}_\Lambda(f)(-\lambda),$$

where $f^-(x) = f(-x)$, $x \in \mathbb{R}$.

(iii) We have

$$\mathcal{F}_\Lambda(\Lambda S)(\lambda) = i\lambda\, \mathcal{F}_\Lambda(S)(\lambda), \qquad \text{for all } S \in \mathcal{E}'(\mathbb{R}),$$

$$\mathcal{F}_\Lambda(\widetilde{\Lambda} f)(\lambda) = i\lambda\, \mathcal{F}_\Lambda(f)(\lambda), \qquad \text{for all } f \in \mathcal{D}(\mathbb{R}). \tag{42}$$

(iv) For all $f \in \mathcal{D}(\mathbb{R})$,

$$\mathcal{F}_\Lambda(f)(\lambda) = \mathcal{F}_\Delta(f_e)(\lambda) + i\lambda\, \mathcal{F}_\Delta \mathcal{J}(f_o)(\lambda), \tag{43}$$

where $\mathcal{F}_\Delta$ stands for the Fourier transform related to the differential operator Δ, defined (see [12]) on $\mathcal{D}_e(\mathbb{R})$ by

$$\mathcal{F}_\Delta(h)(\lambda) = \int_{\mathbb{R}} h(x)\varphi_\lambda(x)A(x)dx, \quad \lambda \in \mathbb{C},$$

φ_λ being the eigenfunction of Δ defined by Lemma 2.1.

Proof. Assertion (i) follows directly from (19), Theorem 3.8 and Definition 4.1. Assertion (ii) is a consequence of (11). Assertion (iii) follows by applying the usual Fourier transform to both sides of (30) (resp. (39)) and by using (40) (resp. (41)). Assertion (iv) follows by combining (36), (41) and the identity

$$\mathcal{F}_\Delta(h)(\lambda) = \mathcal{F}_u\left({}^t\mathcal{X}h\right)(\lambda), \quad h \in \mathcal{D}_e(\mathbb{R}),$$

(see [12]). □

Remark 4.4 *(i) For $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, the transform $\mathcal{F}_\Delta$ is just the Fourier-Bessel transform of order α (see [14]).*

(ii) For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$ and $q(x) = (\alpha+\beta+1)^2$ with $\beta \geq -1/2$, the transform $\mathcal{F}_\Delta$ coincides with the Jacobi transform of order (α,β) (see [6]).

We can now state the main result of this section.

Theorem 4.5 ***(Paley-Wiener)*** *(i) The generalized Fourier transform $\mathcal{F}_\Lambda$ is a bijection from $\mathcal{E}'(\mathbb{R})$ onto $\mathcal{H}$. More precisely, S has its support in $[-a,a]$ if, and only if, $\mathcal{F}_\Lambda(S) \in \mathcal{H}_a$.*

(ii) The generalized Fourier transform $\mathcal{F}_\Lambda$ is an isomorphism from $\mathcal{D}(\mathbb{R})$ onto $\mathbf{H}$. More precisely, $f \in \mathcal{D}_a(\mathbb{R})$ if, and only if, $\mathcal{F}(f) \in \mathbf{H}_a$.

Proof. The result follows by combining Theorems 3.8 and 3.13, identities (40) and (41), and the classical Paley-Wiener theorem. □

Trimeche [12] has obtained for the transform $\mathcal{F}_\Delta$ the following inversion result.

Theorem 4.6 *For all $f \in \mathcal{D}_e(\mathbb{R})$,*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Delta(f)(\lambda)\varphi_\lambda(x)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Delta(f)(i\lambda)\varphi_{i\lambda}(x)d\mu_2(\lambda), \tag{44}$$

where μ_1 is an even positive tempered measure on $\mathbb{R}$, and μ_2 is an even positive measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} e^{a|y|}d\mu_2(y) < \infty, \quad \text{for all } a > 0.$$

Remark 4.7 *(i) The pair (μ_1, μ_2) is called the spectral measure associated with the differential operator Δ.*

(ii) When $q(x) = 0$, it was shown in [9] that $\mu_2 = 0$. Furthermore, if the function χ defined by (6), is integrable at infinity, then according to Bracco [1],

$$d\mu_1(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2},$$

where $c(s)$ is a continuous function on $]0, \infty[$ such that $c(s) \sim k/s^{\alpha+1/2}$, as $s \to \infty$, for some $k \in \mathbb{C}$.

(iii) If $A(x) = |x|^{2\alpha+1}$ and $q(x) = 0$, then by [12],

$$d\mu_1(\lambda) = \frac{1}{2^{2\alpha+2}\,(\Gamma(\alpha+1))^2}\,|\lambda|^{2\alpha+1} d\lambda \quad \text{and} \quad \mu_2 = 0.$$

(iv) For $A(x) = (\sinh|x|)^{2\alpha+1}\,(\cosh x)^{2\beta+1}$ and $q(x) = (\alpha+\beta+1)^2$ with $\alpha \geq \beta > -1/2$, we know by [6] that

$$d\mu_1(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2} \quad \text{and} \quad \mu_2 = 0,$$

where

$$c(s) = \frac{\sqrt{\pi}\, 2^{\alpha+\beta+2-is}\,\Gamma(is)\,\Gamma(\alpha+1)}{\Gamma[(\alpha+\beta+1+is)/2]\,\Gamma[(\alpha-\beta+1+is)/2]}, \quad s > 0.$$

A direct consequence of Theorem 4.6 is as follows.

Theorem 4.8 *For all $f \in \mathcal{D}(\mathbb{R})$,*

$$f(x) + \mathcal{M}(q\,\mathcal{J} f_o)(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Phi_\lambda(x)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(i\lambda)\Phi_{i\lambda}(x)d\mu_2(\lambda), \tag{45}$$

μ_1 and μ_2 being as in Theorem 4.6.

Proof. By (5), (10), (21) and (43),

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Phi_\lambda(x)d\mu_1(\lambda) \;&=\; \int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(\lambda)\varphi_\lambda(x)d\mu_1(\lambda) \\
&-\; \int_{\mathbb{R}} \lambda^2\,\mathcal{F}_\Delta \mathcal{J}(f_o)(\lambda)\mathcal{M}(\varphi_\lambda)(x)d\mu_1(\lambda) \\
&=\; \int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(\lambda)\varphi_\lambda(x)d\mu_1(\lambda) \\
&+\; \mathcal{M}\Delta\left(\int_{\mathbb{R}} \mathcal{F}_\Delta \mathcal{J}(f_o)(\lambda)\varphi_\lambda(\cdot)d\mu_1(\lambda)\right)(x)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(i\lambda)\Phi_{i\lambda}(x)d\mu_2(\lambda) &= \int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(i\lambda)\varphi_{i\lambda}(x)d\mu_2(\lambda) \\
&+ \int_{\mathbb{R}} \lambda^2\, \mathcal{F}_\Delta\mathcal{J}(f_o)(i\lambda)\mathcal{M}(\varphi_{i\lambda}(x)d\mu_2(\lambda) \\
&= \int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(i\lambda)\varphi_{i\lambda}(x)d\mu_2(\lambda) \\
&+ \mathcal{M}\Delta\left(\int_{\mathbb{R}} \mathcal{F}_\Delta\mathcal{J}(f_o)(i\lambda)\varphi_{i\lambda}(\cdot)d\mu_2(\lambda)\right)(x).
\end{aligned}$$

But by (3), (22) and (44),

$$\int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(\lambda)\varphi_\lambda(x)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(i\lambda)\varphi_{i\lambda}(x)d\mu_2(\lambda) = f_e(x)$$

and

$$\begin{aligned}
&\mathcal{M}\Delta\Big(\int_{\mathbb{R}} \mathcal{F}_\Delta J(f_o)(\lambda)\varphi_\lambda(\cdot)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Delta\mathcal{J}(f_o)(i\lambda)\varphi_{i\lambda}(\cdot)d\mu_2(\lambda)\Big)(x) = \\
&= \mathcal{M}\Delta\mathcal{J}(f_o)(x) \\
&= \Lambda\mathcal{J}(f_o)(x) \\
&= f_o(x) + \mathcal{M}(q\,\mathcal{J}f_o)(x),
\end{aligned}$$

which concludes the proof. □

5 Generalized translation

With the help of the intertwining operators studied in Section 3, we introduce in $\mathcal{E}(\mathbb{R})$ translation operators corresponding to the integro-differential operator Λ, and which generalize the usual translation operators on the real line :

$$f \to \tau^x f(y) = f(x+y).$$

Definition 5.1 *We define the generalized translation operators T^x, $x \in \mathbb{R}$, on $\mathcal{E}(\mathbb{R})$ by*

$$T^x f(y) = V_x V_y\left[V^{-1} f(x+y)\right], \quad y \in \mathbb{R}.$$

This generalized translation shares several properties with the ordinary translation on $\mathbb{R}$.

Theorem 5.2 *(i) For all $x \in \mathbb{R}$, T^x is a linear bounded operator from $\mathcal{E}(\mathbb{R})$ into itself; the function $x \mapsto T^x$ is C^∞.*

(ii) We have

$$T^0 = \text{identity}, \quad T^x T^y = T^y T^x, \quad \Lambda T^x = T^x \Lambda.$$

(iii) For all $f \in \mathcal{E}(\mathbb{R})$,

$$T^x f(y) = T^y f(x).$$

(iv) For each $\lambda \in \mathbb{C}$, the eigenfunction Φ_λ satisfies the product formula :

$$T^x(\Phi_\lambda)(y) = \Phi_\lambda(x)\Phi_\lambda(y).$$

(v) For $f \in \mathcal{E}(\mathbb{R})$, the function $u(x,y) = T^x f(y)$ is the unique solution of the problem

$$\begin{cases} \Lambda_x u(x,y) = \Lambda_y u(x,y), \\ \\ \quad u(0,y) = f(y). \end{cases}$$

In order to construct a convolution product tied to the integro-differential operator Λ, we need to compute the transposed operators of T^x, $x \in \mathbb{R}$.

Theorem 5.3 *For all $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} T^x f(y) g(y) A(y) dy = \int_{\mathbb{R}} f(y)\, {}^t T^x g(y) A(y) dy,$$

where

$${}^t T^x g(y) = V_x \big({}^t V^{-1}\big)_y \big[{}^t V g(y-x)\big], \quad y \in \mathbb{R}.$$

Proof. By (32) and Definition 5.1,

$$\begin{aligned}
\int_{\mathbb{R}} T^x f(y) g(y) A(y) dy &= \int_{\mathbb{R}} V_x V_y \left[V^{-1} f(x+y)\right] g(y) A(y) dy \\
&= V_x \left(\int_{\mathbb{R}} V_y \left[V^{-1} f(x+y)\right] g(y) A(y) dy \right) \\
&= V_x \left(\int_{\mathbb{R}} V^{-1} f(x+y)\, {}^t V g(y) dy \right) \\
&= V_x \left(\int_{\mathbb{R}} V^{-1} f(z)\, {}^t V g(z-x) dz \right) \\
&= V_x \left(\int_{\mathbb{R}} f(z) \left({}^t V^{-1}\right)_z \left[{}^t V g(z-x)\right] A(z) dz \right) \\
&= \int_{\mathbb{R}} f(z)\, V_x \left({}^t V^{-1}\right)_z \left[{}^t V g(z-x)\right] A(z) dz \\
&= \int_{\mathbb{R}} f(z)\, {}^t T^x g(z) A(z) dz,
\end{aligned}$$

which is the desired result. $\square$

Remark 5.4 *(i) According to the proof of Theorem 5.3, the operators* ${}^tT^x$, $x \in \mathbb{R}$, *may also be formulated as*

$$ {}^tT^x f(y) = \left({}^tV^{-1}\right)_y V_x \left[{}^tV f(y-x)\right], \quad y \in \mathbb{R}. $$

(ii) If $q(x) = 0$, *then by [9],* ${}^tT^x = T^{-x}$, *for all* $x \in \mathbb{R}$.

Theorem 5.5 *Let* f *be in* $\mathcal{D}_a(\mathbb{R})$, $a > 0$. *Then for all* $x \in \mathbb{R}$, ${}^tT^x f$ *is an element of* $\mathcal{D}_{a+|x|}(\mathbb{R})$ *and*

$$ \mathcal{F}\left({}^tT^x f\right)(\lambda) = \Phi_{-\lambda}(x)\,\mathcal{F}f(\lambda), \quad \lambda \in \mathbb{C}. \tag{46} $$

Proof. Set

$$ K_x = \begin{cases} K(x,\cdot) & \text{if } x \neq 0, \\ \delta_0 & \text{if } x = 0, \end{cases} $$

where $K(x,\cdot)$ is given by (17), and δ_0 is the Dirac measure at the point $x = 0$. It is not hard to see that

$$ {}^tT^x f = {}^tV^{-1}\left[K_x * {}^tV f\right], $$

where $*$ stands for the usual convolution on $\mathbb{R}$. The result is now a consequence of (16), (41) and Theorem 3.13. $\square$

Remark 5.6 *Let* $f \in \mathcal{D}(\mathbb{R})$. *From (11), (45) and (46) we get*

$$ \begin{aligned} {}^tT^x f(0) &= \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Phi_\lambda(-x)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(i\lambda)\Phi_{i\lambda}(-x)d\mu_2(\lambda) \\ &= f(-x) - \mathcal{M}(q\,\mathcal{J}f_o)(x). \end{aligned} $$

Delsarte and Lions [2] have defined in $\mathcal{E}_e(\mathbb{R})$ translation operators S^x, $x \in \mathbb{R}$, related to the differential operator Δ, and which generalize the usual symmetric translation operators on the real line :

$$ f \to \sigma^x f(y) = \frac{f(x+y) + f(x-y)}{2}. $$

More explicitly,

$$ S^x f(y) = \mathcal{X}_x\,\mathcal{X}_y\left[\sigma^x \mathcal{X}^{-1} f(y)\right], \quad y \in \mathbb{R}. $$

The S^x, $x \in \mathbb{R}$, are linear bounded operator from $\mathcal{E}_e(\mathbb{R})$ into itself, and possess the following fundamental properties :

$$ S^0 = \text{identity}, \quad S^x f(y) = S^y f(x) \quad \text{and} \quad \Delta S^x = S^x \Delta. \tag{47} $$

Trimeche [12] pointed out that the S^x map $\mathcal{D}_e(\mathbb{R})$ into itself, and satisfy the relation

$$ \mathcal{F}_\Delta\left(S^x f\right)(\lambda) = \varphi_\lambda(x)\,\mathcal{F}_\Delta(f)(\lambda), \qquad f \in \mathcal{D}_e(\mathbb{R}). \tag{48} $$

In the following theorem, the operators ${}^tT^x$ are expressed in terms of S^x.

Theorem 5.7 *For all $f \in \mathcal{D}(\mathbb{R})$,*

$$ {}^tT^x f(y) = S^x f_e(y) - \Lambda S^y \mathcal{J} f_o(x) + \frac{\partial}{\partial y} S^x \mathcal{J} f_o(y) - \frac{\partial}{\partial y} \mathcal{M}(S^y f_e)(x). $$

Remark 5.8 *Notice that for $f \in \mathcal{D}_e(\mathbb{R})$,*

$$ S^x f(y) = \frac{{}^tT^x f(y) + {}^tT^x f(-y)}{2}. $$

In order to simplify the proof of Theorem 5.7, we first establish the following technical lemma.

Lemma 5.9 *For all $f \in \mathcal{D}(\mathbb{R})$,*

$$ \int_{\mathbb{R}} \Lambda S^y \mathcal{J} f_o(x) \varphi_\lambda(y) A(y) dy = -\lambda^2 \mathcal{M}\varphi_\lambda(x) \mathcal{F}_\Delta(\mathcal{J} f_o)(\lambda), $$

$$ \int_{\mathbb{R}} \mathcal{M} S^y f_e(x) \varphi_\lambda(y) A(y) dy = \mathcal{M}\varphi_\lambda(x) \mathcal{F}_\Delta(f_e)(\lambda). $$

Proof. By use of (21), (22), (47), (48) and the identity

$$ \mathcal{F}_\Delta(\Delta h)(\lambda) = -\lambda^2 \, \mathcal{F}_\Delta(h)(\lambda), \quad h \in \mathcal{D}_e(\mathbb{R}), $$

(see [12]), we obtain

$$ \begin{aligned} \int_{\mathbb{R}} \Lambda S^y \mathcal{J} f_o(x) \varphi_\lambda(y) A(y) dy &= \int_{\mathbb{R}} \mathcal{M} \Delta S^y \mathcal{J} f_o(x) \varphi_\lambda(y) A(y) dy \\ &= \int_{\mathbb{R}} \left(\frac{1}{A(x)} \int_0^x \Delta S^y \mathcal{J} f_o(t) A(t) dt \right) \times \\ &\times \ \varphi_\lambda(y) A(y) dy \\ &= \int_{\mathbb{R}} \left(\frac{1}{A(x)} \int_0^x S^t \Delta \mathcal{J} f_o(y) A(t) dt \right) \times \\ &\times \ \varphi_\lambda(y) A(y) dy \\ &= \frac{1}{A(x)} \int_0^x \mathcal{F}_\Delta\big(S^t \Delta \mathcal{J} f_o\big)(\lambda) A(t) dt \\ &= \frac{1}{A(x)} \int_0^x \varphi_\lambda(t) A(t) dt \; \mathcal{F}_\Delta(\Delta \mathcal{J} f_o)(\lambda) \\ &= -\lambda^2 \mathcal{M}\varphi_\lambda(x) \; \mathcal{F}_\Delta(\mathcal{J} f_o)(\lambda) \end{aligned} $$

and

$$ \int_{\mathbb{R}} \mathcal{M} S^y f_e(x) \varphi_\lambda(y) A(y) dy = \int_{\mathbb{R}} \left(\frac{1}{A(x)} \int_0^x S^y f_e(t) A(t) dt \right) \varphi_\lambda(y) A(y) dy $$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \left(\frac{1}{A(x)} \int_0^x S^t f_e(y) A(t) dt \right) \varphi_\lambda(y) A(y) dy \\
&= \frac{1}{A(x)} \int_0^x \mathcal{F}_\Delta \big(S^t f_e \big)(\lambda) A(t) dt \\
&= \frac{1}{A(x)} \int_0^x \varphi_\lambda(t) A(t) dt \; \mathcal{F}_\Delta(f_e)(\lambda) \\
&= \mathcal{M}\varphi_\lambda(x) \; \mathcal{F}_\Delta(f_e)(\lambda).
\end{aligned}
$$

This ends the proof. □

Proof of Theorem 5.7. Let $f \in \mathcal{D}(\mathbb{R})$. Put

$$
u_x(y) = \frac{{}^tT^x f(y) + {}^tT^x f(-y)}{2} \quad \text{and} \quad v_x(y) = \frac{{}^tT^x f(y) - {}^tT^x f(-y)}{2}.
$$

A combination of (10), (43) and (46) yields

$$
\mathcal{F}_\Delta(u_x)(\lambda) = \varphi_\lambda(x) \mathcal{F}_\Delta(f_e)(\lambda) + \lambda^2 \mathcal{M}\varphi_\lambda(x) \mathcal{F}_\Delta(\mathcal{J} f_o)(\lambda) \tag{49}
$$

and

$$
\mathcal{F}_\Delta(\mathcal{J} v_x)(\lambda) = \varphi_\lambda(x) \mathcal{F}_\Delta(\mathcal{J} f_o)(\lambda) - \mathcal{M}\varphi_\lambda(x) \mathcal{F}_\Delta(f_e)(\lambda). \tag{50}
$$

As by (48),

$$
\varphi_\lambda(x) \mathcal{F}_\Delta(f_e)(\lambda) = \mathcal{F}_\Delta\left(S^x f_e\right)(\lambda) \quad \text{and} \quad \varphi_\lambda(x) \mathcal{F}_\Delta(\mathcal{J} f_o)(\lambda) = \mathcal{F}_\Delta(S^x \mathcal{J} f_o)(\lambda),
$$

it follows from (49), (50) and Lemma 5.9 that

$$
u_x(y) = S^x f_e(y) - \Lambda S^y \mathcal{J} f_o(x)
$$

and

$$
\mathcal{J} v_x(y) = S^x \mathcal{J} f_o(y) - \mathcal{M}\left(S^y f_e\right)(x).
$$

This clearly yields the result. □

Definition 5.10 *For $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$, the generalized convolution product $f \# g$ is defined by*

$$
f \# g(x) = \int_{\mathbb{R}} {}^tT^y f(x) g(y) A(y) dy, \quad x \in \mathbb{R}.
$$

Theorem 5.11 *(i) Let $f \in \mathcal{D}_a(\mathbb{R})$ and $g \in \mathcal{D}_b(\mathbb{R})$. Then $f \# g \in \mathcal{D}_{a+b}(\mathbb{R})$ and*

$$
\mathcal{F}_\Lambda(f \# g)(\lambda) = \mathcal{F}_\Lambda(f)(\lambda) \mathcal{F}_\Lambda(g)(\lambda), \quad \lambda \in \mathbb{C}. \tag{51}
$$

(ii) For all $f, g \in \mathcal{D}(\mathbb{R})$,

$$
{}^tV(f \# g) = {}^tV f \, * \, {}^tV g. \tag{52}
$$

Proof. Assertion (i) follows from Theorem 4.5(ii) and formula (46). Identity (52) follows by applying the usual Fourier transform to both its sides and by using formulas (41) and (51). $\square$

We conclude the paper by a Plancherel type formula for the generalized Fourier transform $\mathcal{F}_\Lambda$.

Theorem 5.12 *For all $f, g \in \mathcal{D}(\mathbb{R})$,*

$$\int_{\mathbb{R}} f(y)g(-y)A(y)dy + \int_{\mathbb{R}} q(y)\mathcal{J}f_o(y)\mathcal{J}g_o(y)A(y)dy =$$
$$= \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(i\lambda)\mathcal{F}_\Lambda(g)(i\lambda)d\mu_2(\lambda),$$

μ_1 and μ_2 being as in Theorem 4.6.

Proof. By (24) and Remark 5.6,

$$\begin{aligned} f\#g(0) &= \int_{\mathbb{R}} {}^tT^y f(0)g(y)A(y)dy \\ &= \int_{\mathbb{R}} \left[f(-y) - \mathcal{M}(q\,\mathcal{J}f_o)(y)\right] g(y)A(y)dy \\ &= \int_{\mathbb{R}} f(-y)g(y)A(y)dy - \int_{\mathbb{R}} \mathcal{M}(q\,\mathcal{J}f_o)(y)g_o(y)A(y)dy \\ &= \int_{\mathbb{R}} f(y)g(-y)A(y)dy + \int_{\mathbb{R}} q(y)\mathcal{J}f_o(y)\mathcal{J}g_o(y)A(y)dy. \end{aligned}$$

Moreover, it follows from (45) and (51), that

$$f\#g(0) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\mu_1(\lambda) + \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(i\lambda)\mathcal{F}_\Lambda(g)(i\lambda)d\mu_2(\lambda).$$

This completes the proof. $\square$

6 Open Problems

The most important open questions about the integro-differential operator Λ are as follows :

1. Is-it possible to determine sufficient conditions on the function A, in order to ensure the positivity of the kernel $K(x, y)$ in the Laplace integral formula (16).

2. Is-it possible to obtain a product type formula for the eigenfunctions $\Phi_\lambda(x)$, namely,

$$\Phi_\lambda(x)\Phi_\lambda(y) = \int_{\mathbb{R}} \Phi_\lambda(z)\, d\Omega_{x,y}(z),$$

where $\Omega_{x,y}$ are finite Borel measures on $\mathbb{R}$ satisfying

$$\|\Omega_{x,y}\| \leq C \quad \text{for all } x, y \in \mathbb{R},$$

for some constant $C > 0$.

The resolution of these problems will certainly allow us to extend many mathematical theories on the real line to the integro-differential operator Λ.

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