Int. J. Open Problems Complex Analysis, Vol. 5, No. 3, November, 2013
ISSN 2074-2827; Copyright ©ICSRS Publication, 2013
www.i-csrs.org

Majorization for Certain Classes of Meromorphic Functions Defined by Integral Operator

Kirti Dhuria and Rachana Mathur

Department of Mathematics,
Govt. Dungar (P.G.) College, Bikaner, India
e-mail: rachnamathur@rediffmail.com
e-mail: kirti.dhuria@gmail.com

Abstract

Here we investigate a majorization problem involving starlike meromorphic function of complex order belonging to a certain subclass of meromorphic univalent function defined by an integral operator introduced recently by Lashin.

Keywords: *Meromorphic univalent functions, majorization property, starlike functions, integral operators.*
2000 Mathematical Subject Classification: Primary 30C45; Secondary 30C80.

1 Introduction and Preliminaries

Let $f(z)$ and $g(z)$ are analytic in the open unit disk

$$\triangle = \{z \in \mathbb{C} \text{ and } |\mathrm{z}| < 1\}. \tag{1.1}$$

For analytic functions $f(z)$ and $g(z)$ in Δ, we say that $f(z)$ is *majorized* by $g(z)$ in Δ (see [10]) and write

$$f(z) << g(z) \ \ (z \in \Delta), \tag{1.2}$$

if there exists a function $\phi(z)$, analytic in Δ such that $|\phi(z)| \leq 1$, and

$$f(z) = \phi(z)g(z) \ (z \in \Delta). \tag{1.3}$$

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.4}$$

which are analytic and univalent in the punctured unit disk

$$\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\} := \Delta \setminus \{0\} \tag{1.5}$$

with a simple pole at the origin.

For functions $f_j \in \Sigma$ given by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2; z \in \Delta^*), \tag{1.6}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{1.7}$$

Analogous to the operators defined by Jung, Kim and Srivastava [8] on the normalized analytic functions, Lashin [9] introduced the following integral operators

$$\mathcal{Q}_\beta^\alpha : \Sigma \longrightarrow \Sigma$$

defined by

$$\mathcal{Q}_\beta^\alpha = \mathcal{Q}_\beta^\alpha f(z) = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^\beta \left(1 - \frac{t}{z}\right)^{\alpha-1} f(t)dt \quad (\alpha, \beta > 0; z \in \Delta^*), \tag{1.8}$$

where $\Gamma(\alpha)$ is familiar Gamma function.

Using the integral representation of the Gamma function and (1.4), it can be easily shown that

$$\mathcal{Q}_\beta^\alpha f(z) = \frac{1}{z} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)} a_k z^k \quad (\alpha > 0, \beta > 0; z \in \Delta *). \tag{1.9}$$

Obviously

$$\mathcal{Q}_\beta^1 f(z) := \mathcal{J}_\beta. \tag{1.10}$$

The operator

$$\mathcal{J}_\beta : \Sigma \longrightarrow \Sigma$$

has also been studied by Lashin [9].
It is easy to verify that (see [9]),

$$z(\mathcal{Q}_\beta^\alpha f(z))' = (\alpha+\beta-1)\mathcal{Q}_\beta^{\alpha-1} f(z) - (\beta+\alpha)\mathcal{Q}_\beta^\alpha f(z). \tag{1.11}$$

Definition 1.3. *A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{S}_\beta^{\alpha,j}(\gamma)$ of meromorphic functions of complex order $\gamma \neq 0$ in Δ if and only if*

$$\Re\left\{1 - \frac{1}{\gamma}\left(\frac{z(\mathcal{Q}_\beta^\alpha f(z))^{(j+1)}}{(\mathcal{Q}_\beta^\alpha f(z))^{(j)}} + j + 1\right)\right\} > 0, \quad (z \in \Delta, j \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \alpha > 0, \beta > 0, \gamma \in \mathbb{C}\backslash\{0\}). \tag{1.12}$$

Clearly, we have the following relationships:

$$(i)\ \mathcal{S}_\beta^{0,0}(\gamma) = \mathcal{S}(\gamma) \quad (\gamma \in \mathbb{C}\setminus\{0\}),$$
$$(ii)\ \mathcal{S}_\beta^{0,0}(1-\eta) = \mathcal{S}^*(\eta) \quad (0 \le \eta < 1)$$

The classes $\mathcal{S}(\gamma)$ and $\mathcal{S}^*(\eta)$ are said to be classes of meromorphic starlike univalent functions of complex order $\gamma \neq 0$ and meromorphic starlike univalent functions of order η ($\eta \in \Re$ such that $0 \le \eta < 1$) in Δ^*.

An majorization problem for the normalized classes of starlike has been investigated by Altinas et al. [1] and MacGregor [10]. In the recent paper of Goyal and Goswami [3] generalized these results for the class of multivalent functions using fractional derivatives operators. Further, Goyal et al. [4], Goswami and Wang [5], Goswami and Aouf [6], Goswami et al. [7] studied majorization property for different - different classes. In this paper, we will study majorization properties for the class of meromorphic functions using integral operator $\mathcal{Q}_\beta^\alpha$.

2. Majorization problems for the class $\mathcal{S}_\beta^{\alpha,j}(\gamma)$

Theorem 2.1 *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{S}_\beta^{\alpha,j}(\gamma)$. If $(\mathcal{Q}_\beta^\alpha f(z))^{(j)}$ is majorized by $(\mathcal{Q}_\beta^\alpha g(z))^{(j)}$ in Δ^*, then*

$$|(\mathcal{Q}_\beta^{\alpha-1} f(z))^{(j)}| \le |(\mathcal{Q}_\beta^{\alpha-1} g(z))^{(j)}| \quad for\ |z| \le r_1(\alpha,\beta,\gamma), \tag{2.1}$$

where

$$r_1(\alpha,\beta,\gamma) = \frac{k_1 - \sqrt{{k_1}^2 - 4(\beta+\alpha-1)|\beta+\alpha-1+2\gamma|}}{2|\beta+\alpha-1+2\gamma|}$$

and

$$k_1 = (\beta+\alpha+1+|\beta+\alpha-1+2\gamma|, (\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C}\setminus\{0\}). \tag{2.2}$$

Proof. Since $g \in \mathcal{S}_\beta^{\alpha,j}(\gamma)$, we find from (2.1), if

$$h_1(z) = 1 - \frac{1}{\gamma}\left(\frac{z(\mathcal{Q}_\beta^\alpha g(z))^{(j+1)}}{(\mathcal{Q}_\beta^\alpha g(z))^{(j)}} + j + 1\right) (\alpha,\beta > 0, \gamma \in \mathbb{C}\setminus\{0\}, j \in \mathbb{N}_0\ ), \tag{2.3}$$

then $\Re\{h_1(z)\} > 0 \ \ (z \in \Delta)$ and

$$h_1(z) = \frac{1 + w(z)}{1 - w(z)} \quad (w \in \mathcal{Q}), \tag{2.4}$$

where $w(z) = c_1 z + c_2 z^2 + \ldots$ and $\mathcal{Q}$ denotes the well known class of bounded analytic functions in Δ and satisfies the conditions

$$w(0) = 0 \ \text{ and } \ |\mathrm{w(z)}| \leq |\mathrm{z}| \ (\mathrm{z} \in \Delta).$$

Making use of (2.3) and (2.4), we get

$$\frac{z(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j+1)}}{(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)}} = \frac{(1 + j - 2\gamma)w(z) - (1 + j)}{1 - w(z)}. \tag{2.5}$$

By principle of mathematical induction, and (1.11), we easily get

$$\begin{aligned} z(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j+1)} &= (\alpha + \beta - 1)(\mathcal{Q}^{\alpha-1}_{\beta} g(z))^{(j)} - (\alpha + \beta + j)(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)}, \\ &(\alpha > 1, \beta > 0; z \in \Delta^*). \end{aligned} \tag{2.6}$$

Now using (2.6) in (2.5), we find that

$$\begin{aligned} \frac{(\alpha + \beta - 1)(\mathcal{Q}^{\alpha-1}_{\beta} g(z))^{(j)}}{(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)}} &= (\beta + \alpha + j) + \frac{(1 + j - 2\gamma)w(z) - (1 + j)}{1 - w(z)} \\ &= \frac{(\alpha + \beta - 1) - (\alpha + \beta - 1 + 2\gamma)w(z)}{1 - w(z)} \end{aligned}$$

or

$$(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)} = \frac{(\alpha + \beta - 1)(1 - w(z))}{(\alpha + \beta - 1) - (\alpha + \beta - 1 + 2\gamma)w(z)} (\mathcal{Q}^{\alpha-1}_{\beta} g(z))^{(j)}. \tag{2.7}$$

Since $|w(z)| \leq |z| \ (z \in \Delta)$, therefore (2.6) yields

$$\left| (\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)} \right| \leq \frac{(\alpha + \beta - 1)[1 + |z|]}{\alpha + \beta - 1 - |\alpha + \beta - 1 + 2\gamma||z|} \left| (\mathcal{Q}^{\alpha-1}_{\beta} g(z))^{(j)} \right|. \tag{2.8}$$

Next since $(\mathcal{Q}^{\alpha}_{\beta} f(z))^{(j)}$ is majorized by $(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)}$ in the unit disk Δ^*, therefore from (1.3), we have

$$(\mathcal{Q}^{\alpha}_{\beta} f(z))^{(j)} = \phi(z)(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)}$$

Differentiating it with respect to 'z' and multiplying by 'z', we get

$$z(\mathcal{Q}^{\alpha}_{\beta} f(z))^{(j+1)} = z\varphi'(z)(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j)} + z\varphi(z)(\mathcal{Q}^{\alpha}_{\beta} g(z))^{(j+1)}$$

Using (2.6), in the above equation, it yields

$$(\mathcal{Q}_\beta^{\alpha-1}f(z))^{(j)} = \frac{z\varphi'(z)}{(\alpha+\beta-1)}(\mathcal{Q}_\beta^{\alpha}g(z))^{(j)} + \varphi(z)(\mathcal{Q}_\beta^{\alpha-1}g(z))^{(j)} \tag{2.9}$$

Thus, nothing that $\varphi \in \mathcal{Q}$ satisfies the inequality (see, e.g. Nehari [6])

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2} \tag{2.10}$$

and making use of (2.8) and (2.10) in (2.9), we get

$$\left|(\mathcal{Q}_\beta^{\alpha-1}f(z))^{(j)}\right| \le \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|}\frac{|z|}{[\alpha+\beta-1-|2\gamma+\beta+\alpha-1||z|]}\right)\left|(\mathcal{Q}_\beta^{\alpha-1}g(z))^{(j)}\right|, \tag{2.11}$$

which upon setting

$$|z| = r \text{ and } |\varphi(\text{z})| = \rho \quad (0 \le \rho \le 1),$$

leads us to the inequality

$$\left|\left((\mathcal{Q}_\beta^{\alpha-1}f(z))^{(j)}\right)\right| \le \frac{\Theta(\rho)}{(1-r)(\beta+\alpha-1-|2\gamma+\beta+\alpha-1|r)}\left|(\mathcal{Q}_\beta^{\alpha-1}g(z))^{(j)}\right|,$$

where

$$\Theta(\rho) = -r\rho^2 + (1-r)(\beta+\alpha-1-|2\gamma+\beta+\alpha-1|r)\rho + r \tag{2.12}$$

takes its maximum value at $\rho = 1$, with $r_2 = r_2(\alpha,\beta,\gamma)$ where $r_2(\alpha,\beta,\gamma)$ is given by equation (2.2). Furthermore, if $0 \le \rho \le r_2(\alpha,\beta,\gamma)$, then the function $\theta(\rho)$ defined by

$$\theta(\rho) = -\sigma\rho^2 + (1-\sigma)(\beta+\alpha-1-|2\gamma+\beta+\alpha-1|\sigma)\rho + \sigma \tag{2.13}$$

is seen to be increasing function on the interval $0 \le \rho \le 1$, so that

$$\begin{gathered}\theta(\rho) \le \theta(1) = (1-\sigma)(\beta+\alpha-1-|2\gamma+\beta+\alpha-1|\sigma),\\ (0 \le \rho \le 1; \quad 0 \le \sigma \le r_1(\alpha,\beta,\gamma)).\end{gathered} \tag{2.14}$$

Hence upon setting $\rho = 1$, in (2.14), we conclude that (2.1) of Theorem 2.1 holds true for $|z| \le r_1(\alpha,\beta,\gamma)$, where $r_1(\alpha,\beta,\gamma)$ is given by (2.2). This completes the Theorem 2.1.

Setting $\alpha = 1$, in Theorem 2.1, we get

Corollary 2.1. *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{S}_\beta^{1,j}(\gamma)$. If $(\mathcal{J}_\beta f(z))^{(j)}$ is majorized by $(\mathcal{J}_\beta g(z))^{(j)}$ in Δ^*, then*

$$|(f(z))^{(j)}| \le |(g(z))^{(j)}| \quad \textit{for } |z| \le r_2(\alpha,\beta,\gamma), \tag{2.15}$$

where

$$r_2(\beta,\gamma) = \frac{k_2 - \sqrt{k_2^2 - 4\beta|\beta+2\gamma|}}{2|\beta+2\gamma|}$$

and

$$(k_2 = (\beta + 2 + |\beta+2\gamma|), \beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C}\backslash\{0\}).$$

Further putting $\beta = 1$ and $\gamma = 1 - \eta$, $j = 0$ in Corollary 2.1, we get
Corollary 2.2. *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{S}_1^{1,0}(1-\eta)$. If $(\mathcal{J}_1 f(z))$ is majorized by $(\mathcal{J}_1 g(z))$ in Δ^*, then*

$$|f(z)| \leq |g(z)| \quad for\ |z| \leq r_3, \tag{2.16}$$

where

$$r_3 = \frac{3 - \eta - \sqrt{\eta^2 - 4\eta + 6}}{3 - \eta}.$$

For $\eta = 0$, the above corollary reduces to the following result :
Corollary 2.3. *Let the function $f(z) \in \Sigma$ and suppose that $g \in \mathcal{S}_1^{1,0}(1) := \mathcal{S}_1^{1,0}$. If $(\mathcal{J}_1 f(z))$ is majorized by $(\mathcal{J}_1 g(z))$ in Δ^*, then*

$$|f(z)| \leq |g(z)| \quad for\ |z| \leq \frac{3 - \sqrt{6}}{3}. \tag{2.17}$$

2 Open Problem

In this paper we studied majorization for the certain class of meromorphic analytic functions. If we define a class $f \in \Sigma_p$ such that

$$f(z) = z^{-p} + \sum_{0}^{\infty} a_{n+p} z^{n+p}, (z \in \Delta^*),$$

then we need to modify integral operator $\mathcal{Q}_\beta^\alpha$ for the class of meromorphic multivalent functions and further using this modified operator we have to find majorization conditions for modified integral operator.

References

[1] O. Altinas, Ö Özkan and H. M. Srivastava, Majorization by starlike functions of complex order, *Complex var. : Theory and applications*, **46** (2001), 207-218.

[2] S. P. Goyal and P. Goswami, Majorization for certain classes of analytic functions defined by fractional derivatives, Appl. Math. Lett., **22**(12)(2009), 1855-1858.

[3] S. P. Goyal and P. Goswami, Majorization for certain classes of analytic functions defined by integral operator, Annales UMCS, Mathematica, **66**(2)(2012), 57-62.

[4] S. P. Goyal, S. K. Bansal and P. Goswami, Majorization for certain classes of analytic functions defined by linear operator using differential subordination, J. Appl. Math. Stas. Informatics, **6**(2)(2010), 45–50.

[5] P. Goswami and Z.-G. Wang, Majorization for certain classes of analytic functions, Acta Universitatis Apulensis, **21**(2009), 97–104.

[6] P. Goswami and M. K. Aouf, Majorization properties for certain classes of analytic functions using the Salagean operator, Appl. Math. Letters, **23**(11) (2010), 1351–1354.

[7] P. Goswami, B. Sharma and T. Bulboaca,Majorization for certain classes of analytic functions using multiplier transformation, Appl. Math. Letters, **23**(10)(2010), 633–637.

[8] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operator, *J. Math. Anal. Appl.*, **176**(1)(1993), 138-147.

[9] A. Y. Lashin, On certain subclasses of meromorphic functions associated with certain integral operators, *Comp. Math. Appl.*, **59**(1) (2010), 524–531.

[10] T. H. McGreogor, Majorization by univalent functions, *Duke Math. J.*, **34** (1967), 95-102.

[11] Z. Nehari, Conformal Mapping, MacGraw-Hill Book Company, New York, Toronto and London (1955).