General univalence conditions related to a new integral operator

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Abstract

In this paper, we introduce a new integral operator for analytic functions \( f \) and \( g \) in the open unit disk \( U \). The aim of this paper is to obtain new conditions for univalence of this integral operator. Several corollaries are also considered.

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1 Introduction and preliminaries

Let \( A \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disk

\[
U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}
\]

and satisfy the following usual normalization condition \( f(0) = f'(0) - 1 = 0 \), \( \mathbb{C} \) being the set of complex numbers.

Let \( S \) denote the subclass of \( A \) consisting of functions \( f(z) \) which are univalent in \( U \).

The following univalence condition was derived by Ozaki and Nunokawa [2]...
Theorem 1.1. (see [2]) Let \( f \in A \) satisfies the following inequality:

\[
\left| \frac{z^2 f'(z)}{|f(z)|^2} - 1 \right| \leq |z|^2, \quad z \in U.
\]  

Then the function \( f \) is in the class \( S \).

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [6] and the references cited therein).

Here, in our present investigation, we study the univalence conditions for the following integral operator:

\[
F_{\alpha, \beta, \gamma}(z) = \left( \beta \int_0^z t^{\beta - \alpha - 1} [f(t)]^\alpha (e^{g(t)})^\gamma dt \right)^{\frac{1}{\beta}}
\]  

for some \( \alpha, \beta, \gamma \) be complex numbers, \( \beta \neq 0 \) and \( f, g \in A \).

Remark 1.2. For \( \beta = \lambda, \alpha = 0 \) and \( g(z) = q(z) \) the integral operator in (2) would obviously reduce to the integral operator

\[
Q_{\lambda}(z) = \left( \lambda \int_0^z t^{\lambda - 1} (e^{g(t)})^\lambda dt \right)^{\frac{1}{\lambda}}
\]  

which was studied by Pescar in [4].

In the proof of our main result (Theorem 2.1 below), we need each of the following univalence criteria. The first univalence criterion, asserted by Theorem 1.3 below, is a generalization of the Ozaki-Nunokawa criterion (1); it was obtained by Răducanu in [5]. The second univalence criterion asserted by Theorem 1.4 below was proven by Pascu [3].

Theorem 1.3. (see [5]) Let \( f \in A \) and \( m > 0 \) be so constrained that

\[
\left| \frac{z^2 f'(z)}{|f(z)|^2} - 1 \right| - \frac{m - 1}{2} |z|^{m+1} \leq \frac{m + 1}{2} |z|^{m+1}, \quad z \in U.
\]  

Then the function \( f \) is in the class \( S \).

Theorem 1.4. (see [3]) Let \( f \in A \) and \( \beta \in \mathbb{C} \). If \( \text{Re}\beta > 0 \) and

\[
\frac{1 - |z|^{2\text{Re}\beta}}{\text{Re}\beta} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U
\]

then the function \( F_\beta(z) \) given by

\[
F_\beta(z) = \left( \beta \int_0^z t^{\beta - 1} f'(t) dt \right)^{\frac{1}{\beta}}
\]  

is in the class \( S \).
Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [1]).

**Lemma 1.5.** (see [1]) Let the function $f$ be regular in the disk

$$\mathbb{U}_R = \{ z : z \in \mathbb{C} \quad \text{and} \quad |z| < R \ (R > 0) \}$$

with $|f(z)| < M$ for a fixed number $M > 0$. If the function $f$ has one zero with multiplicity order bigger than a positive integer $m$ for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathbb{U}_R. \quad (4)$$

The equality $(4)$ can hold true only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where $\theta$ is a real constant.

## 2 Main Results

**Theorem 2.1.** Let the functions $f, g \in \mathcal{A}$, where $f$ satisfies the hypothesis (3) of Theorem 1.3. Suppose that

$$M \geq 1, \quad m > 0, \quad N \geq 1 \quad \text{and} \quad |g(z)| \leq N.$$

If

$$\text{Re}\beta \geq \left[ |\alpha| ((m + 1) M + 1) + 2 |\gamma| N \right], \quad \alpha, \beta, \gamma \in \mathbb{C} \quad (5)$$

and

$$|f(z)| \leq M \quad z \in \mathbb{U}, \quad \left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1 \quad z \in \mathbb{U} \quad (6)$$

then the function $F_{\alpha,\beta,\gamma}(z)$ defined by (2) is in the class $\mathcal{S}$.

**Proof.** We begin by observing that the integral operator $F_{\alpha,\beta,\gamma}(z)$ in (2) can be rewritten as follows:

$$F_{\alpha,\beta,\gamma}(z) = \left( \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} \right)^\alpha (e^{g(t)})^\gamma dt \right)^\frac{1}{\beta}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha (e^{g(t)})^\gamma dt. \quad (7)$$
The function \( h(z) \) is indeed regular in \( U \) and satisfies the following usual normalization condition \( h(0) = h'(0) - 1 = 0 \). From (7), we have

\[
h'(z) = \left( \frac{f(z)}{z} \right)^\alpha (e^{g(z)})^\gamma. \tag{8}
\]

Differentiating (8) logarithmically and multiplying by \( z \), we obtain

\[
\frac{zh''(z)}{h'(z)} = \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \gamma zg'(z)
\]

\[
= \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \gamma \frac{zg'(z)}{g(z)}g(z)
\]

which readily shows that

\[
1 - |z|^{2\text{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| = 1 - |z|^{2\text{Re}\beta} \left| \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \gamma \frac{zg'(z)}{g(z)}g(z) \right|
\]

\[
\leq 1 - |z|^{2\text{Re}\beta} \left[ \left| \alpha \right| \left( \left| z^2 f'(z) \right| \left| \frac{f(z)}{z} \right| + 1 \right) + \left| \gamma \right| \left| \frac{zg'(z)}{g(z)} \right| \left| g(z) \right| \right]
\]

\[
\leq 1 - |z|^{2\text{Re}\beta} \left[ \left| \alpha \right| \left( \left| z^2 f'(z) \right| \left| \frac{f(z)}{z} \right| + 1 \right) + \left| \gamma \right| \left( \frac{zg'(z)}{g(z)} - 1 \right) + 1 \right] |g(z)| \]. \tag{9}
\]

Furthermore, from the hypothesis (6) of Theorem 2.1, we have

\[
|f(z)| \leq M, \quad z \in U \quad \text{and} \quad \left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1, \quad z \in U.
\]

By applying the General Schwarz Lemma, we thus obtain

\[
|f(z)| \leq M |z|, \quad z \in U.
\]
Next, by making use of (9), we have

\[
1 - \frac{|z|^{2 \Re \beta}}{\Re \beta} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2 \Re \beta}}{\Re \beta} \left[ |\alpha| \left( \left( \frac{z^2 f'(z)}{|f(z)|^2} \right) M + 1 \right) + 2 |\gamma| N \right]
\]

\[
\leq \frac{1 - |z|^{2 \Re \beta}}{\Re \beta} \left[ |\alpha| \left( \left( \frac{z^2 f'(z)}{|f(z)|^2} - 1 \right) - \frac{m - 1}{2} |z|^{m+1} \right) M + 
\right.

\[
+ \left( 1 + \frac{m - 1}{2} |z|^{m+1} \right) M + 1 \right] + 2 |\gamma| N \right]
\]

\[
\leq \frac{1 - |z|^{2 \Re \beta}}{\Re \beta} \left[ |\alpha| \left( \left( \frac{m + 1}{2} |z|^{m+1} M + 
\right. \right.
\]

\[
+ \left( 1 + \frac{m - 1}{2} |z|^{m+1} \right) M + 1 \right] + 2 |\gamma| N \right]
\]

\[
\leq \frac{1 - |z|^{2 \Re \beta}}{\Re \beta} \left[ |\alpha| \left( (m + 1) M + 1 \right) + 2 |\gamma| N \right]
\]

\[
\leq \frac{1}{\Re \beta} \left[ |\alpha| \left( (m + 1) M + 1 \right) + 2 |\gamma| N \right]
\]

\[
\leq 1,
\]

where we have also used the hypothesis (5) of Theorem 2.1.

Finally, by applying Theorem 1.4, we conclude that the function \( F_{\alpha, \beta, \gamma} (z) \) defined by (2) is in the class \( S \). This evidently completes the proof of Theorem 2.1.

\[\square\]

First of all, upon setting \( m = 1 \) in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

**Corollary 2.2.** Let the functions \( f, g \in A \), where \( f \) satisfies the hypothesis (3) of Theorem 1.3. Suppose that

\[ M \geq 1, \quad N \geq 1 \quad \text{and} \quad |g(z)| \leq N. \]

If

\[ \Re \beta \geq \left[ |\alpha| \left( 2M + 1 \right) + 2 |\gamma| N \right] > 0, \quad \alpha, \beta, \gamma \in \mathbb{C} \]

and

\[ |f(z)| \leq M, \quad z \in \mathbb{U}, \quad \left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1, \quad z \in \mathbb{U} \]

then the function \( F_{\alpha, \beta, \gamma} (z) \) defined by (2) is in the class \( S \).

We set \( g(z) = 0 \) in Theorem 2.1. We thus obtain the following interesting consequence of Theorem 2.1.
Corollary 2.3. Let the function \( f \in A \) satisfies the hypothesis (3) of Theorem 1.3 and suppose that \( M \geq 1, \ m > 0 \). If

\[ \Re \beta \geq |\alpha| \left[ (m+1)M + 1 \right] > 0, \ \alpha, \beta, \gamma \in \mathbb{C} \]

and

\[ |f(z)| \leq M, \ z \in U \]

then the function

\[ F_{\alpha,\beta}(z) = \left( \beta \int_0^z t^{\beta-\alpha-1}[f(t)]^\alpha \, dt \right)^\frac{1}{\beta} \]

is in the class \( S \).

Finally, upon setting \( m = 1 \) and \( g(z) = 0 \) in Theorem 2.1, we obtain the following consequence of Theorem 2.1.

Corollary 2.4. Let the functions \( f \in A \) satisfies the hypothesis (3) of Theorem 1.3 and suppose that \( M \geq 1 \). If

\[ \Re \beta \geq |\alpha| (2M + 1) > 0, \ \alpha, \beta \in \mathbb{C} \]

and

\[ |f(z)| \leq M, \ z \in U \]

then the function

\[ F_{\alpha,\beta}(z) = \left( \beta \int_0^z t^{\beta-\alpha-1}[f(t)]^\alpha \, dt \right)^\frac{1}{\beta} \]

is in the class \( S \).

3 Open Problem

New results can be obtained by using the integral operator defined in (2) for other classes of analytic functions.

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