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General univalence conditions related to a new integral operator

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Abstract

In this paper, we introduce a new integral operator for analytic functions f and g in the open unit disk $\mathbb{U}$. The aim of this paper is to obtain new conditions for univalence of this integral operator. Several corollaries are also considered.

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1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$, $\mathbb{C}$ being the set of complex numbers.
Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.
The following univalence condition was derived by Ozaki and Nunokawa [2]

Theorem 1.1. (see [2]) *Let $f \in \mathcal{A}$ satisfies the following inequality:*

$$\left| \frac{z^2 f'(z)}{[f\,(z)]^2} - 1 \right| \leq |z|^2\,, \qquad z \in \mathbb{U}. \tag{1}$$

Then the function f is in the class $\mathcal{S}$.

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [6] and the references cited therein).
Here, in our present investigation, we study the univalence conditions for the following integral operator:

$$F_{\alpha,\beta\,,\gamma}\,(z) = \left(\beta \int_0^z t^{\beta-\alpha-1} [f(t)]^{\alpha} (e^{g(t)})^{\gamma} dt \right)^{\frac{1}{\beta}} \tag{2}$$

for some α, β, γ be complex numbers, $\beta \neq 0$ and $f, g \in \mathcal{A}$.

Remark 1.2. *For $\beta = \lambda$, $\alpha = 0$ and $g(z) = q(z)$ the integral operator in (2) would obviously reduce to the integral operator*

$$Q_\lambda(z) = \left(\lambda \int_0^z t^{\lambda-1} \left(e^{q(t)}\right)^{\lambda} dt \right)^{\frac{1}{\lambda}}$$

which was studied by Pescar in [4].

In the proof of our main result (Theorem 2.1 below), we need each of the following univalence criteria. The first univalence criterion, asserted by Theorem 1.3 below, is a generalization of the Ozaki-Nunokawa criterion (1); it was obtained by Răducanu in [5]. The second univalence criterion asserted by Theorem 1.4 below was proven by Pascu [3].

Theorem 1.3. (see [5]) *Let $f \in \mathcal{A}$ and $m > 0$ be so constrained that*

$$\left| \left(\frac{z^2 f'(z)}{[f(z)]^2} - 1 \right) - \frac{m-1}{2} |z|^{m+1} \right| \leq \frac{m+1}{2} |z|^{m+1}\,, \qquad z \in \mathbb{U}. \tag{3}$$

Then the function f is in the class $\mathcal{S}$.

Theorem 1.4. (see [3]) *Let $f \in \mathcal{A}$ and $\beta \in \mathbb{C}$. If* $\mathrm{Re}\beta > 0$ *and*

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad z \in \mathbb{U}$$

then the function $F_\beta(z)$ given by

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt \right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [1]).

Lemma 1.5. (see [1]) *Let the function f be regular in the disk*

$$\mathbb{U}_R = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < R \quad (R > 0)\}$$

with $|f(z)| < M$ for a fixed number $M > 0$. If the function f has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m , \qquad z \in \mathbb{U}_R. \tag{4}$$

The equality (4) can hold true only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is a real constant.

2 Main Results

Theorem 2.1. *Let the functions $f, g \in \mathcal{A}$, where f satisfies the hypothesis (3) of Theorem 1.3. Suppose that*

$$M \geq 1, \quad m > 0, \quad N \geq 1 \quad \text{and} \quad |g(z)| \leq N.$$

If

$$\mathrm{Re}\beta \geq \left[|\alpha| \left((m+1) M + 1 \right) + 2 |\gamma| N \right] >, \quad \alpha, \beta, \gamma \in \mathbb{C} \tag{5}$$

and

$$|f(z)| \leq M \quad z \in \mathbb{U}, \qquad \left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1 \quad z \in \mathbb{U} \tag{6}$$

then the function $F_{\alpha,\beta,\gamma}(z)$ defined by (2) is in the class $\mathcal{S}$.

Proof. We begin by observing that the integral operator $F_{\alpha,\beta,\gamma}(z)$ in (2) can be rewritten as follows:

$$F_{\alpha,\beta,\gamma}(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{\alpha} \left(e^{g(t)} \right)^{\gamma} dt \right)^{\frac{1}{\beta}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} \left(e^{g(t)} \right)^{\gamma} dt. \tag{7}$$

The function $h(z)$ is indeed regular in $\mathbb{U}$ and satisfies the following usual normalization condition $h(0) = h'(0) - 1 = 0$. From (7), we have

$$h'(z) = \left(\frac{f(z)}{z}\right)^{\alpha} \left(e^{g(z)}\right)^{\gamma}. \tag{8}$$

Differentiating (8) logarithmically and multiplying by z, we obtain

$$\begin{aligned} \frac{zh''(z)}{h'(z)} &= \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \gamma z g'(z) \\ &= \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \gamma \frac{zg'(z)}{g(z)} g(z) \end{aligned}$$

which readily shows that

$$\begin{aligned} \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| &= \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \gamma\frac{zg'(z)}{g(z)}g(z)\right| \\ &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[|\alpha|\left(\left|\frac{z^2 f'(z)}{[f(z)]^2}\right|\left|\frac{f(z)}{z}\right| + 1\right) + |\gamma|\left|\frac{zg'(z)}{g(z)}\right||g(z)|\right] \\ &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[|\alpha|\left(\left|\frac{z^2 f'(z)}{[f(z)]^2}\right|\left|\frac{f(z)}{z}\right| + 1\right) + \right. \\ &\left. + |\gamma|\left(\left|\frac{zg'(z)}{g(z)} - 1\right| + 1\right)|g(z)|\right]. \end{aligned} \tag{9}$$

Furthermore, from the hypothesis (6) of Theorem 2.1, we have

$$|f(z)| \leq M, \qquad z \in \mathbb{U} \quad \text{and} \quad \left|\frac{zg'(z)}{g(z)} - 1\right| \leq 1, \qquad z \in \mathbb{U}.$$

By applying the General Schwarz Lemma, we thus obtain

$$|f(z)| \leq M\,|z|, \qquad z \in \mathbb{U}.$$

Next, by making use of (9), we have

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[|\alpha|\left(\left|\frac{z^2f'(z)}{[f(z)]^2}\right|M+1\right)+2|\gamma|N\right]$$
$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[|\alpha|\left[\left|\left(\frac{z^2f'(z)}{[f(z)]^2}-1\right)-\frac{m-1}{2}|z|^{m+1}\right|M+\right.\right.$$
$$\left.\left.+\left(1+\frac{m-1}{2}|z|^{m+1}\right)M+1\right]+2|\gamma|N\right]$$
$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[|\alpha|\left[\frac{m+1}{2}|z|^{m+1}M+\right.\right.$$
$$\left.\left.+\left(1+\frac{m-1}{2}|z|^{m+1}\right)M+1\right]+2|\gamma|N\right]$$
$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left[|\alpha|\left((m+1)M+1\right)+2|\gamma|N\right]$$
$$\leq \frac{1}{\mathrm{Re}\beta}\left[|\alpha|\left((m+1)M+1\right)+2|\gamma|N\right]$$
$$\leq 1,$$

where we have also used the hypothesis (5) of Theorem 2.1.

Finally, by applying Theorem 1.4, we conclude that the function $F_{\alpha,\beta,\gamma}(z)$ defined by (2) is in the class $\mathcal{S}$. This evidently completes the proof of Theorem 2.1. □

First of all, upon setting $m=1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

Corollary 2.2. *Let the functions $f,g\in\mathcal{A}$, where f satisfies the hypothesis (3) of Theorem 1.3. Suppose that*

$$M\geq 1,\quad N\geq 1\quad \text{and}\quad |g(z)|\leq N.$$

If

$$\mathrm{Re}\beta\geq\left[|\alpha|(2M+1)+2|\gamma|N\right]>0,\quad \alpha,\beta,\gamma\in\mathbb{C}$$

and

$$|f(z)|\leq M\quad z\in\mathbb{U},\qquad \left|\frac{zg'(z)}{g(z)}-1\right|\leq 1\quad z\in\mathbb{U}$$

then the function $F_{\alpha,\beta,\gamma}(z)$ defined by (2) is in the class $\mathcal{S}$.

We set $g(z)=0$ in Theorem 2.1. We thus obtain the following interesting consequence of Theorem 2.1.

Corollary 2.3. *Let the function $f \in \mathcal{A}$ satisfies the hypothesis (3) of Theorem 1.3 and suppose that $M \geq 1$, $m > 0$. If*

$$\mathrm{Re}\beta \geq |\alpha| \left[(m+1) M + 1 \right] > 0, \quad \alpha, \beta, \gamma \in \mathbb{C}$$

and

$$|f(z)| \leq M, \quad z \in \mathbb{U}$$

then the function

$$F_{\alpha,\beta}(z) = \left(\beta \int_0^z t^{\beta-\alpha-1} [f(t)]^\alpha dt \right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Finally, upon setting $m = 1$ and $g(z) = 0$ in Theorem 2.1, we obtain the following consequence of Theorem 2.1.

Corollary 2.4. *Let the functions $f \in \mathcal{A}$ satisfies the hypothesis (3) of Theorem 1.3 and suppose that $M \geq 1$. If*

$$\mathrm{Re}\beta \geq |\alpha| (2M + 1) > 0, \quad \alpha, \beta \in \mathbb{C}$$

and

$$|f(z)| \leq M, \quad z \in \mathbb{U}$$

then the function

$$F_{\alpha,\beta}(z) = \left(\beta \int_0^z t^{\beta-\alpha-1} [f(t)]^\alpha dt \right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

3 Open Problem

New results can be obtained by using the integral operator defined in (2) for other classes of analytic functions.

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