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Subclasses Of Analytic Functions Defined By Convolution

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Abstract

In this paper, we define a generalized class of starlike functions defined by convolution with negative coefficients and obtain some geometric properties for this class such as coefficient estimates, distortion theorems, radii of close-to-convexity, starlikeness and convexity, closure theorems and extreme points. Further we obtain modified Hadamard produt, for functions belonging to this class.

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of S consisting of all functions which are, respectively, starlike and convex of order α $(0 \le \alpha < 1)$. Thus,

$$S^*(\alpha) = \left\{ f \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ (0 \le \alpha < 1; z \in U \right\}$$

and

$$K(\alpha) = \left\{ f \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ (0 \le \alpha < 1;\ z \in U) \right\}.$$

The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Reberston [11].
Let $f \in S$ be given by (1.1) and $g \in S$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (b_k > 0). \tag{1.2}$$

We define the Hadmard product (or convolution) of f and g as follows:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

We denote by $S_{A,B}(f, g, \alpha, \beta, \gamma)$ $(-1 \le A < B \le 1,\ 0 < B \le 1)$ the subclass of S, where f and g are given by (1.1) and (1.2), respectively and satisfies:

$$\left| \frac{\frac{z(f*g)'(z)}{(f*g)(z)} - 1}{2(B-A)\gamma\left(\frac{z(f*g)'(z)}{(f*g)(z)} - \alpha\right) - B\left(\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right)} \right| < \beta \ (z \in U; 0 \le \alpha < 1; 0 < \beta \le 1), \tag{1.4}$$

where $(f * g)(z)$ is given by (1.3) and $\frac{B}{2(B-A)} < \gamma \le \begin{cases} \frac{B}{2(B-A)\alpha} & ,\alpha \ne 0 \\ 1 & ,\alpha = 0. \end{cases}$

We also let

$$T_{A,B}(f, g, \alpha, \beta, \gamma) = S_{A,B}(f, g, \alpha, \beta, \gamma) \cap T,$$

where

$$T = \left\{ f \in S : f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k ; z \in U \right\}. \tag{1.5}$$

We note that:

(i) $T_{A,B}(f, S_\delta, \alpha, \beta, \gamma)$ $\left(S_\delta = \frac{z}{(1-z)^{2(1-\delta)}}, 0 \le \delta < 1\right)$

$$= \left\{ f \in T : \left| \frac{\frac{z(f*S_\delta)'(z)}{(f*S_\delta)(z)} - 1}{2(B-A)\gamma\left(\frac{z(f*S_\delta)'(z)}{(f*S_\delta)(z)} - \alpha\right) - B\left(\frac{z(f*S_\delta)'(z)}{(f*S_\delta)(z)} - 1\right)} \right| < \beta \ \ (z \in U) \right\},$$

(see Magesh et al. [6 with $m=0$]);

(ii) $T_{A,B}\left(f, z+\sum\limits_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\cdot\frac{\lambda!\,(k+n-2)!}{(n-1)!\,(k+\lambda-1)!}\right|z^k,\alpha,\beta,\gamma\right)$
$(n\geq 2;\lambda>-1;\mu\in\mathbb{C};b\in\mathbb{C}\backslash\{\mathbb{Z}_0^-=0,-1,-2,...\})$
$=\left\{f\in T:\left|\frac{\frac{z\left(J_{\mu,b}^{\lambda,n}f(z)\right)'}{J_{\mu,b}^{\lambda,n}f(z)}-1}{2\,(B-A)\,\gamma\left(\frac{z\left(J_{\mu,b}^{\lambda,n}f(z)\right)'}{J_{\mu,b}^{\lambda,n}f(z)}-\alpha\right)-B\left(\frac{z\left(J_{\mu,b}^{\lambda,n}f(z)\right)'}{J_{\mu,b}^{\lambda,n}f(z)}-1\right)}\right|<\beta\ \ (z\in U)\right\},$
(see Owa et al. [9]);
(iii) $T_{A,B}\left(f,z+\sum\limits_{k=2}^{\infty}\frac{\left(\Gamma\left(n+1\right)\right)^2\Gamma\left(2+\eta-\mu\right)\Gamma\left(2-\eta\right)}{\Gamma\left(n+\eta-\mu+1\right)\Gamma\left(n-\eta+1\right)}z^k,\alpha,\beta,\gamma\right)$
$(\eta-1<\mu<\eta<2)$
$=\left\{f\in T:\left|\frac{\frac{z\left(\Im_{\mu}^{\eta}f(z)\right)'}{\Im_{\mu}^{\eta}f(z)}-1}{2\,(B-A)\,\gamma\left(\frac{z\left(\Im_{\mu}^{\eta}f(z)\right)'}{\Im_{\mu}^{\eta}f(z)}-\alpha\right)-B\left(\frac{z\left(\Im_{\mu}^{\eta}f(z)\right)'}{\Im_{\mu}^{\eta}f(z)}-1\right)}\right|<\beta\,(z\in U)\right\},$
(see Murugusndramoorthy and Thilagvathi [7]);
(iv) $T_{A,B}\left(f,z+\sum\limits_{k=2}^{\infty}\frac{\Omega\Gamma\left(p_1+A_1\left(n-1\right)\right)...\Gamma\left(p_\ell+A_\ell\left(n-1\right)\right)}{(n-1)!\Gamma q_1+B_1\left(n-1\right))...\Gamma q_m+B_m\left(n-1\right))}z^k,\alpha,\beta,\gamma\right)$
$(\ell\leq m+1;m\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,...\};\Omega=\{\prod\limits_{t=0}^{l}\Gamma(p_t)\}^{-1}\{\prod\limits_{t=0}^{m}\Gamma(q_t)\})$
$=\left\{f\in T:\left|\frac{\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}-1}{2\,(B-A)\,\gamma\left(\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}-\alpha\right)-B\left(\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}-1\right)}\right|<\beta\ \ (z\in U)\right\},$
(see Murugusndramoorthy and Magesh [8]).
Also we note that:
(i) $T_{A,B}(f,\frac{z}{1-z},\alpha,\beta,\gamma)$
$=T_{A,B}\left(\alpha,\beta,\gamma\right)=\left\{f\in T:\left|\frac{\frac{zf'(z)}{f(z)}-1}{2\,(B-A)\,\gamma\left(\frac{zf'(z)}{f(z)}-\alpha\right)-B\left(\frac{zf'(z)}{f(z)}-1\right)}\right|<\beta\ \ (z\in U)\right\};$
(ii) $T_{A,B}\left(f,\frac{z}{(1-z)^2},\alpha,\beta,\gamma\right)$
$=K_{A,B}\left(\alpha,\beta,\gamma\right)=\left\{f\in T:\left|\frac{\frac{zf''(z)}{f'(z)}}{2\,(B-A)\,\gamma\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)-B\frac{zf''(z)}{f'(z)}}\right|<\beta\ \ (z\in U)\right\};$
(iii) $T_{A,B}\left(f,z+\sum\limits_{k=2}^{\infty}k^nz^k,\alpha,\beta,\gamma\right)\left(n\in\mathbb{N}_0\right)$
$=S_{A,B}\left(n,\alpha,\beta,\gamma\right)=\left\{f\in T:\left|\frac{\frac{D^{n+1}f(z)}{D^nf(z)}-1}{2\,(B-A)\,\gamma\left(\frac{D^{n+1}f(z)}{D^nf(z)}-\alpha\right)-B\left(\frac{D^{n+1}f(z)}{D^nf(z)}-1\right)}\right|<\beta\ \ (z\in U)\right\},$

where D^n is the Salagean operator (see [13]);
(iv) $T_{A,B}\left(f, z+\sum\limits_{k=2}^{\infty}\left[1+\lambda\left(k-1\right)\right]^n z^k, \alpha, \beta, \gamma\right)$ $(\lambda>0; n\in\mathbb{N}_0)$
$= S_{A,B}\left(n,\lambda,\alpha,\beta,\gamma\right)=\left\{f\in T:\left|\frac{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-1}{2(B-A)\gamma\left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-\alpha\right)-B\left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}-1\right)}\right|<\beta\ \ (z\in U)\right\},$
where D_λ^n is the Al-Oboudi operator (see [2]);
(v) $T_{A,B}\left(f, z+\sum\limits_{k=2}^{\infty}\binom{k+\lambda-1}{\lambda}z^k, \alpha, \beta, \gamma\right)$ $(\lambda>-1)$
$= S_{A,B}\left(\lambda,\alpha,\beta,\gamma\right)=\left\{f\in T:\left|\frac{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-1}{2\left(B-A\right)\gamma\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-\alpha\right)-B\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-1\right)}\right|<\beta\ (z\in U)\right\},$
where D^λ is the λ-th order Ruscheweyh derivative of $f(z)\in S$ (see [1], [12]);
(vi) $T_{A,B}\left(f, z+\sum\limits_{k=2}^{\infty}\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k, \alpha, \beta, \gamma\right)$
$(\lambda\geq 0; \ell>-1; m\in\mathbb{Z}=\{0,\pm 1,...\})$
$=\left\{f\in T:\left|\frac{\frac{z(J^m(\lambda,\ell)f(z))'}{J^m(\lambda,\ell)f(z)}-1}{2\left(B-A\right)\gamma\left(\frac{z(J^m(\lambda,\ell)f(z))'}{J^m(\lambda,\ell)f(z)}-\alpha\right)-B\left(\frac{z(J^m(\lambda,\ell)f(z))'}{J^m(\lambda,\ell)f(z)}-1\right)}\right|<\beta\left(z\in U\right)\right\},$
where $J^m\left(\lambda,\ell\right)$ is the Prajapat operator (see [10], [3], [5], with $p=1$);
(vii) $T_{A,B}\left(f, z+\sum\limits_{k=2}^{\infty}\frac{\left(\alpha_1\right)_{k-1}\cdots\left(\alpha_q\right)_{k-1}}{\left(\beta_1\right)_{k-1}\cdots\left(\beta_s\right)_{k-1}}\cdot\frac{1}{(k-1)!}z^k, \alpha, \beta, \gamma\right)$
$(\alpha_i>0, i=1,...,q; \beta_j>0, j=1,...,s;\ q\leq s+1; q,s\in\mathbb{N}_0)$
$=\left\{f\in T:\left|\frac{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}-1}{2\left(B-A\right)\gamma\left(\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}-\alpha\right)-B\left(\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}-1\right)}\right|<\beta\left(z\in U\right)\right\},$
where $H_{q,s}\left(\alpha_1\right)f\left(z\right)$ is the Dzoik-Srivastava operator (see [4]).

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $-1\leq A<B\leq 1$, $0<B\leq 1$, $0\leq\alpha<1$, $0<\beta\leq 1$, $z\in U$ and g is given by (1.2).

Theorem 1. *Let the function $f\left(z\right)$ be defined by (1.5). Then $f(z)$ is in the class $T_{A,B}\left(f,g,\alpha,\beta,\gamma\right)$ if and only if*

$$\sum_{k=2}^{\infty}\left[2\left(B-A\right)\beta\gamma\left(k-\alpha\right)+\left(1-\beta B\right)\left(k-1\right)\right]b_k\left|a_k\right|\leq 2\left(B-A\right)\beta\gamma\left(1-\alpha\right). \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds true, we find from (1.5) and

(2.1) that

$$\left|z(f*g)'(z)-(f*g)(z)\right|-$$
$$\beta\left|2(B-A)\gamma\left[z(f*g)'(z)-\alpha(f*g)(z)\right]-B\left[z(f*g)'(z)-(f*g)(z)\right]\right|$$
$$=\left|\sum_{k=2}^{\infty}(k-1)b_ka_kz^k\right|-$$
$$\beta\left|2(B-A)\gamma(1-\alpha)z+\sum_{k=2}^{\infty}[2(B-A)\gamma(k-\alpha)-B(k-1)]b_ka_kz^k\right|$$
$$\leq\sum_{k=2}^{\infty}(k-1)b_ka_kr^k-2(B-A)\beta\gamma(1-\alpha)r+\beta\sum_{k=2}^{\infty}[2(B-A)\gamma(k-\alpha)-B(k-1)]b_ka_kr^k$$
$$=\sum_{k=2}^{\infty}[(k-1)+\beta[2(B-A)\gamma(k-\alpha)-B(k-1)]b_ka_kr^k-2(B-A)\beta\gamma(1-\alpha)r$$
$$\leq\sum_{k=2}^{\infty}[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]b_ka_k-2(B-A)\beta\gamma(1-\alpha)\leq 0\ \ (z\in U).$$

Hence, by the maximum modulus theorem, we have $f(z)\in T_{A,B}(f,g,\alpha,\beta,\gamma)$.
Conversely, Let

$$\left|\frac{\frac{z(f*g)'(z)}{(f*g)(z)}-1}{2(B-A)\gamma\left(\frac{z(f*g)'(z)}{(f*g)(z)}-\alpha\right)-B\left(\frac{z(f*g)'(z)}{(f*g)(z)}-1\right)}\right|$$
$$=\left|\frac{\sum_{k=2}^{\infty}(k-1)b_ka_kz^k}{2(B-A)\gamma(1-\alpha)z-\sum_{k=2}^{\infty}[2(B-A)\gamma(k-\alpha)+B(k-1)]b_ka_kz^k}\right|<\beta\ \ (z\in U).$$

Now since $\operatorname{Re}\{z\}\leq|z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{\sum_{k=2}^{\infty}(k-1)b_ka_kz^k}{2(B-A)\gamma(1-\alpha)z-\sum_{k=2}^{\infty}[2(B-A)\gamma(\alpha-k)-B(k-1)]b_ka_kz^k}\right\}<\beta. \tag{2.2}$$

Choos values of z on the real axis so that $\frac{z(f*g)'(z)}{(f*g)(z)}$ is real. Then upon clearing the denominator in (2.2) and letting $z\to 1^-$through real values, we have

$$\sum_{k=2}^{\infty}[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]b_k|a_k|-2(B-A)\beta\gamma(1-\alpha)\leq 0.$$

This given the required condition.

Corollary 1. *Let the function $f(z)$ defined by(1.5) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then we have*

$$a_k \leq \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_k} \quad (k \geq 2), \tag{2.3}$$

the result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_k} z^k \quad (k \geq 2). \tag{2.4}$$

3. Distortion theorem

Theorem 2. *Let the function $f(z)$ defined by (1.5) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \geq r - \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(2-\alpha)+(1-\beta B)]\,b_2} r^2 \tag{3.1}$$

and

$$|f(z)| \leq r + \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(2-\alpha)+(1-\beta B)]\,b_2} r^2, \tag{3.2}$$

provided that $b_k \geq b_2$ $(k \geq 2)$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(2-\alpha)+(1-\beta B)]\,b_2} z^2, \tag{3.3}$$

at $z = r$ and $z = re^{i(2k+1)\pi}$ $(k \geq 2)$.

Proof. Since for $k \geq 2$,

$$\begin{aligned} & [2(B-A)\beta\gamma(2-\alpha)+(1-\beta B)]\,b_2 \sum_{k=2}^{\infty} a_k \\ \leq & \sum_{k=2}^{\infty} [2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_k a_k \\ \leq & \ 2(B-A)\beta\gamma(1-\alpha) \end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty} a_k \leq \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(2-\alpha)+(1-\beta B)]\,b_2}. \tag{3.4}$$

From (1.5) and (3.4), we have

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_2} r^2$$

and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_2} r^2.$$

This completes the proof of Theorem 2.

4. Radii of close-to-convexity, starlikeness and convexity

In this section we obtain the radii of close-to-convexity, starlikness and convexity for the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$.
Theorem 3. *Let the function $f(z)$ defined by (1.5) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $f(z)$ is close-to-convex of order η $(0 \leq \eta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{k\geq 2} \left[\frac{(1-\eta)[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_k}{2(B-A)\beta\gamma k(1-\alpha)} \right]^{\frac{1}{k-1}}. \tag{4.1}$$

The result is sharp for the function $f(z)$ given by (2.4).
Proof. We must show that

$$|f'(z)-1| \leq 1-\eta, \text{ for } |z| < r_1,$$

where r_1 is given by (4.1). Indeed we find from the definition (1.5) that

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|f'(z)-1| \leq 1-\eta,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\eta}\right) a_k |z|^{k-1} \leq 1 \tag{4.2}$$

But, by Theorem 1, (4.2) will be true if

$$\left(\frac{k}{1-\eta}\right)|z|^{k-1} \leq \frac{[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)]\,b_k}{2(B-A)\beta\gamma(1-\alpha)},$$

that is, if

$$|z| \leq \left[\frac{(1-\eta)\left[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)\right] b_k}{2(B-A)\beta\gamma k(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.3}$$

Theorem 3 follwos easily from (4.3).

Theorem 4. *Let $f(z)$ be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $f(z)$ is starlike of order η $(0 \leq \eta < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{k \geq 2} \left[\left(\frac{1-\eta}{k-\eta} \right) \frac{\left[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)\right] b_k}{2(B-A)\beta\gamma(1-\alpha)} \right]^{\frac{1}{k-1}}. \tag{4.4}$$

The result is sharp, with the extermal function $f(z)$ given by (2.4).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1-\eta \text{ for } |z| < r_2, \tag{4.5}$$

where r_2 is given by (4.4). Indeed we find, again from the definition (1.5) that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1-\eta,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k-\eta}{1-\eta} \right) a_k |z|^{k-1} \leq 1. \tag{4.6}$$

But, by Theorem 1, (4.6) will be true if

$$\left(\frac{k-\eta}{1-\eta} \right) |z|^{k-1} \leq \frac{\left[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)\right] b_k}{2(B-A)\beta\gamma(1-\alpha)},$$

that is, if

$$r_2 = |z| \leq \left[\left(\frac{1-\eta}{k-\eta} \right) \frac{\left[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)\right] b_k}{2(B-A)\beta\gamma(1-\alpha)} \right]^{\frac{1}{k-1}}. \tag{4.7}$$

Theorem 4 follwos easily from (4.7).

Corrolary 2. *Let the function $f(z)$ defined by (1.5) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $f(z)$ is convex of order η $(0 \leq \eta < 1)$ in the disc $|z| < r_3$, where*

$$r_3 = \inf_{k \geq 2} \left[\left(\frac{1-\eta}{k(k-\eta)} \right) \frac{\left[2(B-A)\beta\gamma(k-\alpha)+(1-\beta B)(k-1)\right] b_k}{2(B-A)\beta\gamma(1-\alpha)} \right]^{\frac{1}{k-1}}.$$

The result is sharp, with the extermal function $f(z)$ given by (2.4).

5. Closure theorems

Theorem 5. *Let $\mu_j \geq 0$ for $j = 1, 2, ...m$, and $\sum_{j=1}^{m} \mu_j \leq 1$. If the functions $f_j(z)$ defined by*

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k (a_{k,j} \geq 0; j = 1, 2, ...m), \tag{5.1}$$

are in the class $T_{A,B}(f, g, \alpha, \beta, \gamma)$, for every $j = 1, 2, ..., m$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} \mu_j a_{k,j} \right) z^k,$$

is in the class $T_{A,B}(f, g, \alpha, \beta, \gamma)$.

Proof. Since $f_j(z) \in T_{A,B}(f, g, \alpha, \beta, \gamma)$, it follows from Theorem 1, that

$$\sum_{k=2}^{m} [2(B-A)\beta\gamma(k-\alpha) + (1-\beta B)(k-1)] b_k a_{k,j} \leq 2(B-A)\beta\gamma(1-\alpha),$$

for every $j = 1, 2, ..., m$. Hence

$$\begin{aligned} & \sum_{k=2}^{\infty} [2(B-A)\beta\gamma(k-\alpha) + (1-\beta B)(k-1)] \left(\sum_{j=1}^{m} \mu_j a_{k,j} \right) b_k \\ = & \sum_{j=1}^{m} \mu_j \left(\sum_{j=1}^{\infty} [2(B-A)\beta\gamma(k-\alpha) + (1-\beta B)(k-1)] b_k a_{k,j} \right) \\ \leq & 2(B-A)\beta\gamma(1-\alpha) \sum_{j=1}^{m} \mu_j \leq 2(B-A)\beta\gamma(1-\alpha). \end{aligned}$$

By Theorem 1, it follows that $h(z) \in T_{A,B}(f, g, \alpha, \beta, \gamma)$, and so the proof of Theorem 5 is completed.

Theorem 6. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(k-\alpha) + (1-\beta B)(k-1)] b_k} z^k (k \geq 2). \tag{5.2}$$

Then $f(z)$ is in the class $T_{A,B}(f, g, \alpha, \beta, \gamma)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{5.3}$$

where $\mu_k \geq 0\,(k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \mu_k f_k(z) \\ &= z - \sum_{k=2}^{\infty} \frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\left[2\,(B-A)\,\beta\gamma\,(k-\alpha)+(1-\beta B)\,(k-1)\right] b_k}\mu_k z^k. \end{aligned}$$

Then it follows that.

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\left[2\,(B-A)\,\beta\gamma\,(k-\alpha)+(1-\beta B)\,(k-1)\right] b_k}{2\,(B-A)\,\beta\gamma\,(1-\alpha)} \cdot \\ &\cdot \frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\left[2\,(B-A)\,\beta\gamma\,(k-\alpha)+(1-\beta B)\,(k-1)\right] b_k}\mu_k \\ = &\sum_{k=2}^{\infty} \mu_k = 1-\mu_1 \le 1. \end{aligned}$$

So, by Theorem 1, $f(z) \in T_{A,B}(f,g,\alpha,\beta,\gamma)$
Conversely, assume that the function $f(z)$ defined by (1.5) belongs to the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then

$$a_k \le \frac{2\,(B-A)\,\beta\gamma\,(1-\alpha)}{\left[2\,(B-A)\,\beta\gamma\,(k-\alpha)+(1-\beta B)\,(k-1)\right] b_k} \quad (k \ge 2).$$

Setting

$$\mu_k = \frac{\left[2\,(B-A)\,\beta\gamma\,(k-\alpha)+(1-\beta B)\,(k-1)\right] b_k}{2\,(B-A)\,\beta\gamma\,(1-\alpha)} a_k \quad (k \ge 2),$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_{k.}$$

We can see that $f(z)$ can be expressed in the form (5.3). This completes the proof of Theorem 6.
Corrolary 3. *The extreme points of the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$ are the functions $f_1(z) = z$ and $f_k(z)$ given by (5.2).*

6. Modified Hadamard products

For the functions

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \left(a_{k,j} \ge 0; j = 1,2\right), \tag{6.1}$$

we denote by $(f_1 * f_2)(z)$ the modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ that is

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{6.2}$$

Theorem 7. *Let the function $f_1(z)$ defined by (6.1), be in the class $T_{A,B}(f, g, \alpha, \beta, \gamma)$. Suppose also that a function $f_2(z)$ defined by (6.1), be in the class $T_{A,B}(f, g, \delta, \beta, \gamma)$. Then $(f_1 * f_2) \in T_{A,B}(f, g, \xi, \beta, \gamma)$, where*

$$\xi = 1 - \frac{2(B-A)\beta\gamma(1-\alpha)(1-\delta)(1+2(B-A)\beta\gamma - \beta B)}{\Lambda_1(\alpha,\beta,\gamma,A,B,2)\Lambda_2(\delta,\beta,\gamma,A,B,2)b_2 - 4(B-A)^2\beta^2\gamma^2(1-\alpha)(1-\delta)}, \tag{6.3}$$

and

$$\begin{aligned} \Lambda_1(\alpha,\beta,\gamma,A,B,2) &= 2(B-A)\beta\gamma(2-\alpha) + (1-\beta B), \\ \Lambda_2(\delta,\beta,\gamma,A,B,2) &= 2(B-A)\beta\gamma(2-\delta) + (1-\beta B). \end{aligned} \tag{6.4}$$

This result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_1(z) = z - \frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(2-\alpha) + (1-\beta B)]b_2} z^2$$

and

$$f_2(z) = z - \frac{2(B-A)\beta\gamma(1-\delta)}{[2(B-A)\beta\gamma(2-\delta) + (1-\beta B)]b_2} z^2.$$

Proof. Employing the technique used earlier by Shild and Silverman [14], we need to find the largest ξ such that

$$\sum_{k=2}^{\infty} \frac{2(B-A)\beta\gamma(k-\xi) + (1-\beta B)(k-1)]b_k}{2(B-A)\beta\gamma(1-\xi)} a_{k,1}a_{k,2} \le 1, \ (0 \le \xi < 1).$$
$$(f_1(z) \in T_{A,B}(f, g, \alpha, \beta, \gamma) \text{ and } f_2(z) \in T_{A,B}(f, g, \delta, \beta, \gamma)). \tag{6.5}$$

Therefore, by the Cauchy's-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty} \frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,k)]^{\frac{1}{2}} [\Lambda_2(\delta,\beta,\gamma,A,B,k)]^{\frac{1}{2}} b_k}{\sqrt{(1-\alpha)(1-\delta)}} \sqrt{a_{k,1}a_{k,2}} \le 1, \tag{6.6}$$

where

$$\Lambda_1(\alpha,\beta,\gamma,A,B,k) = 2(B-A)\beta\gamma(k-\alpha) + (1-\beta B) \tag{6.7}$$

and

$$\Lambda_2(\delta,\beta,\gamma,A,B,k) = 2(B-A)\beta\gamma(k-\delta) + (1-\beta B). \tag{6.8}$$

Thus we only need to show thatfind largest ξ such that

$$\sum_{k=2}^{\infty}\frac{\left[2\left(B-A\right)\beta\gamma\left(k-\xi\right)+\left(1-\beta B\right)\left(k-1\right)\right]b_k}{2\left(B-A\right)\beta\gamma\left(1-\xi\right)}a_{k,1}a_{k,2}$$

$$\leq \sum_{k=2}^{\infty}\frac{\left[\Lambda_1\left(\alpha,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}\left[\Lambda_2\left(\delta,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}b_k}{2\left(B-A\right)\beta\gamma\sqrt{\left(1-\alpha\right)\left(1-\delta\right)}}\sqrt{a_{k,1}a_{k,2}} \quad (6.9)$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{1-\xi}{\sqrt{\left(1-\alpha\right)\left(1-\mu\right)}}\frac{\left[\Lambda_1\left(\alpha,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}\left[\Lambda_2\left(\delta,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}}{2\left(B-A\right)\beta\gamma\left(k-\xi\right)+\left(1-\beta B\right)\left(k-1\right)} \quad (k\geq 2). \quad (6.10)$$

Hence, in light of inequality (6.6), it is sufficient to prove that

$$\frac{2\left(B-A\right)\beta\gamma\sqrt{\left(1-\alpha\right)\left(1-\delta\right)}\left(b_k\right)^{-1}}{\left[\Lambda_1\left(\alpha,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}\left[\Lambda_2\left(\delta,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}}$$

$$\leq \frac{1-\xi}{\sqrt{\left(1-\alpha\right)\left(1-\delta\right)}}\frac{\left[\Lambda_1\left(\alpha,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}\left[\Lambda_2\left(\delta,\beta,\gamma,A,B,k\right)\right]^{\frac{1}{2}}}{2\left(B-A\right)\beta\gamma\left(k-\xi\right)+\left(1-\beta B\right)\left(k-1\right)}. \quad (6.11)$$

It follows from (6.11) that

$$\xi\leq 1-\frac{2\left(B-A\right)\beta\gamma\left(1-\alpha\right)\left(1-\delta\right)\left(k-1\right)\left(1+2\left(B-A\right)\beta\gamma-\beta B\right)}{\Lambda_1\left(\alpha,\beta,\gamma,A,B,k\right)\Lambda_2\left(\delta,\beta,\gamma,A,B,k\right)b_k-4\left(B-A\right)^2\beta^2\gamma^2\left(1-\alpha\right)\left(1-\delta\right)}. \quad (6.12)$$

Now, defining the function $\Psi(k)$ by

$$\Psi(k)=1-\frac{2\left(B-A\right)\beta\gamma\left(1-\alpha\right)\left(1-\delta\right)\left(k-1\right)\left(1+2\left(B-A\right)\beta\gamma-\beta B\right)}{\Lambda_1\left(\alpha,\beta,\gamma,A,B,k\right)\Lambda_2\left(\delta,\beta,\gamma,A,B,k\right)b_k-4\left(B-A\right)^2\beta^2\gamma^2\left(1-\alpha\right)\left(1-\delta\right)}. \quad (6.13)$$

We see that $\Psi(k)$ is an increasing function of k $(k\geq 2)$. Therefore, we conclued that

$$\xi=\Psi(2)=1-\frac{2\left(B-A\right)\beta\gamma\left(1-\alpha\right)\left(1-\delta\right)\left(1+2\left(B-A\right)\beta\gamma-\beta B\right)}{\Lambda_1\left(\alpha,\beta,\gamma,A,B,2\right)\Lambda_2\left(\delta,\beta,\gamma,A,B,2\right)b_2-4\left(B-A\right)^2\beta^2\gamma^2\left(1-\alpha\right)\left(1-\delta\right)}, \quad (6.14)$$

where $\Lambda_1\left(\alpha,\beta,\gamma,A,B,2\right)$ and $\Lambda_2\left(\delta,\beta,\gamma,A,B,2\right)$ are given by (6.4), which evidently completes the proof of Theorem 7.
Using arguments similar to those in proof of Theorem 7, we obtain the following results.

Corollary 4. *Let the functions $f_j(z)(j=1,2)$ defined by (6.1) are in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $(f_1 * f_2)\in T_{A,B}(f,g,\xi,\beta,\gamma)$, where*

$$\xi=1-\frac{2\left(B-A\right)\beta\gamma\left(1-\alpha\right)^2\left(1+2\left(B-A\right)\beta\gamma-\beta B\right)}{\Lambda_1\left(\alpha,\beta,\gamma,A,B,2\right)^2b_2-4\left(B-A\right)^2\beta^2\gamma^2\left(1-\alpha\right)^2}. \quad (6.15)$$

The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_j(z)=z-\frac{2(B-A)\beta\gamma(1-\alpha)}{[2(B-A)\beta\gamma(2-\alpha)+(1-\beta B)]\,b_2}z^2 \ (j=1,2). \tag{6.16}$$

Theorem 9. *Let the functions $f_j(z)(j=1,2)$ defined by (6.1) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by*

$$h(z)=z-\sum_{k=2}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)z^k \tag{6.17}$$

is in the class$T_{A,B}(f,g,\xi,\beta,\gamma)$, where

$$\xi=1-\frac{4(B-A)\beta\gamma(1-\alpha)^2(1+2(B-A)\beta\gamma-\beta B)}{[(1+2(B-A)\beta\gamma(2-\alpha)-B\beta)]^2 b_2-8(B-A)^2\beta^2\gamma^2(1-\alpha)^2}. \tag{6.18}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ defined by (6.16).
Theorem 10. *Let the function $f(z)$ defined by (1.5) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Also let $g(z)=z-\sum_{k=2}^{\infty}b_kz^k$ for $|b_k|\le 1$. Then $(f*g)(z)\in T_{A,B}(f,g,\alpha,\beta,\gamma)$.*

7. Open problems

The authors suggest to study :
(1) Factor sequence problem for the class $S_{A,B}(f,g,\alpha,\beta,\gamma)$;
(2) Neighbourhood problems for the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$;
(3) For $f(z)$, $g(z)\in\sum$ the class of univalent meromorphic functions, construct the analogous class $\sum_{A,B}(f,g,\alpha,\beta,\gamma)$ as follows:

$$\left|\frac{\frac{z(f*g)'(z)}{(f*g)(z)}+1}{2(B-A)\gamma\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\alpha\right)-B\left(\frac{z(f*g)'(z)}{(f*g)(z)}+1\right)}\right|<\beta,\ z\in U^*.$$

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