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A note on certain classes of multivalent analytic functions

Lifeng Guo[1], Yi Ling[2] and Gejun Bao[3]

1.School of Mathematical Science and Technology
Northeast Petroleum University, Daqing 163318, China.
(e-mail: hitglf@yahoo.com.cn)
2. Department of Mathematics Delaware State University Dover
DE 19901, U. S. A (e-mail: yiling@desu.edu)
3. Department of Mathematics, Harbin Institute of Technology.
Harbin 150001, China (e-mail: baogj@hit.edu.cn)

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Abstract

A theorem involving multivalent analytic functions is considered and then its certain consequences are given.

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1 Introduction

Let $\mathcal{H}_n(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad (n, p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1}$$

which are analytic and multivalent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{H}_n(p)$ is said to be in the class $\mathcal{S}_n^*(p,\alpha)$ of multivalent starlike functions of order α in $\mathcal{U}$ if it satisfies the following inequality:

$$\mathfrak{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad 0 \le \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}. \tag{2}$$

On the other hand, a function $f \in \mathcal{H}_n(p)$ is said to be in the class $\mathcal{C}_n(p,\alpha)$ of multivalent close-to-convex functions of order α in $\mathcal{U}$ if it satisfies the following inequality:

$$\mathfrak{Re}(z^{1-p}f'(z)) > \alpha, \quad 0 \le \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}. \tag{3}$$

We observe that $\mathcal{S}_1^*(1,\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}_1(1,\alpha) = \mathcal{C}(\alpha)$, where $\mathcal{S}^*(\alpha)$ are the usual subclasses of $\mathcal{H}_1(1)$ consisting of functions which are starlike of order $\alpha(0 \le \alpha < 1)$ and close-to-convex of order $\alpha(0 \le \alpha < 1)$ in $\mathcal{U}$, respectively (see, for details, [1, 2]).

Recently Frasin (see [3]) introduced and studied the following class of analytic and multivalent functions defined as follows (see also [4]).

Definition 1.1. *A function $f \in \mathcal{H}_n(p)$ is said to be a member of the class $\mathcal{B}_n(p,\mu,\alpha)$ if and only if*

$$\Big|\Big(\frac{z^p}{f(z)}\Big)^{\mu-1} z^{1-p}f'(z) - p\Big| < p - \alpha, \ (p \in \mathbb{N}) \tag{4}$$

for some $\mu \ge 0, \alpha(0 \le \alpha < p), z \in \mathbb{U}$.

Note that condition (1.4) implies that

$$\mathfrak{Re}\Big(\Big(\frac{z^p}{f(z)}\Big)^{\mu-1} z^{1-p}f'(z)\Big) > \alpha, \ z \in \mathcal{U}. \tag{5}$$

The class $\mathcal{B}_1(1,1,\alpha) = \mathcal{B}(\alpha)$ is the class which has been introduced and studied by Frasin and Darus [5] (see also [6]).

To prove our main result, we need the following Lemma:

Lemma 1.1 (see [7]). *Let the function $w(z)$ be(nonconstant) analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attsts its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0 w'(z_0) = kw(z_0), \tag{6}$$

where $k \ge 1$ is a real number.

2 Main results and their consequences

Theorem 2.1. *Let $f \in \mathcal{H}_n(p), w \in \mathbb{C} \setminus \{0\}, \mu \ge 0, 0 \le \alpha < p, p \in \mathbb{N}$, and also let the function $\mathcal{H}$ be defined by*

$$\mathcal{H}(z) = \Bigg(\frac{(\frac{z^p}{f(z)})^{\mu-1} z^{1-p}f'(z)}{(\frac{z^p}{f(z)})^{\mu-1} z^{1-p}f'(z) - p}\Bigg)\Big(1 + \frac{zf''(z)}{f'(z)} + (1-\mu)\frac{zf'(z)}{f(z)} + (\mu-2)p\Big). \tag{7}$$

If $\mathcal{H}(z)$ satisfies one of the following conditions:

$$\mathfrak{Re}\{\mathcal{H}(z)\} = \begin{cases} < |w|^{-2}\mathfrak{Re}\{w\} & \text{if } \mathfrak{Re}\{w\} > 0, \\ \neq 0 & \text{if } \mathfrak{Re}\{w\} = 0, \\ > |w|^{-2}\mathfrak{Re}\{w\} & \text{if } \mathfrak{Re}\{w\} < 0. \end{cases} \tag{8}$$

or

$$\mathfrak{Im}\{\mathcal{H}(z)\} = \begin{cases} < |w|^{-2}\mathfrak{Im}\{\bar{w}\} & \text{if } \mathfrak{Im}\{\bar{w}\} > 0, \\ \neq 0 & \text{if } \mathfrak{Im}\{\bar{w}\} = 0, \\ > |w|^{-2}\mathfrak{Im}\{\bar{w}\} & \text{if } \mathfrak{Im}\{\bar{w}\} < 0. \end{cases} \tag{9}$$

then

$$\left|\Big((\frac{z^p}{f(z)})^{\mu-1} z^{1-p} f'(z) - p\Big)^w\right| < p - \alpha, \tag{10}$$

where the value of complex power in (10) is taken to be as its principal value.

Proof. We define the function Ω by

$$\Big((\frac{z^p}{f(z)})^{\mu-1} z^{1-p} f'(z) - p\Big)^w = (p-\alpha)\Omega(z), \tag{11}$$

where $w \in \mathbb{C} \setminus \{0\}, \mu \geq 0, 0 \leq \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}, f \in \mathcal{H}_n(p)$.

We see clearly that the function Ω is regular in $\mathcal{U}$ and $\Omega(0) = 0$.Making use of the logarithmic differentiation of both sides of (11) with respect to the known complex variable z, and if we make use of equality (11) once again, then we find that

$$wz\Big((\frac{z^p}{f(z)})^{\mu-1} z^{1-p} f'(z) - p\Big)^{-1}\Big((\frac{z^p}{f(z)})^{\mu-1} z^{1-p} f'(z) - p\Big)' = \frac{z\Omega'(z)}{\Omega(z)}, \tag{12}$$

which yields

$$\mathcal{H}(z) = \frac{\bar{w}}{|w|^2}\frac{z\Omega'(z)}{\Omega(z)}, w \in \mathbb{C} \setminus \{0\}, z \in \mathcal{U}. \tag{13}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|<|z_0|} |\Omega(z)| = |\Omega(z_0)| = 1, z \in \mathcal{U}. \tag{14}$$

Applying Lemma 1.1, we can then write

$$z_0\Omega'(z_0) = c\Omega(z_0), c \geq 1. \tag{15}$$

Then (13) yields

$$\mathfrak{Re}\{\mathcal{H}(z_0)\} = \mathfrak{Re}\Big\{\frac{\bar{w}}{|w|^2}\frac{z_0\Omega'(z_0)}{\Omega(z_0)}\Big\} = \mathfrak{Re}\{c\bar{w}|w|^{-2}\}, \tag{16}$$

so that

$$\mathfrak{Re}\{\mathcal{H}(z_0)\} = \begin{cases} \geq |w|^{-2}\mathfrak{Re}\{w\} & \text{if } \mathfrak{Re}\{w\} > 0, \\ = 0 & \text{if } \mathfrak{Re}\{w\} = 0, \\ \leq |w|^{-2}\mathfrak{Re}\{w\} & \text{if } \mathfrak{Re}\{w\} < 0, \end{cases} \tag{17}$$

or

$$\mathfrak{Im}\{\mathcal{H}(z_0)\} = \begin{cases} \geq |w|^{-2}\mathfrak{Im}\{\bar{w}\} & \text{if } \mathfrak{Im}\{\bar{w}\} > 0, \\ = 0 & \text{if } \mathfrak{Im}\{\bar{w}\} = 0, \\ \leq |w|^{-2}\mathfrak{Im}\{\bar{w}\} & \text{if } \mathfrak{Im}\{\bar{w}\} < 0. \end{cases} \tag{18}$$

But the inequalities in (17) and (18) contradict, respectively, the inequalities in (8) and (9). Hence, we conclude that $|\Omega(z)| < 1$ for all $z \in \mathcal{U}$. Consequently, it follows from (11) that

$$\left|\Big((\frac{z^p}{f(z)})^{\mu-1} z^{1-p} f'(z) - p \Big)^{w}\right| = (p-\alpha)|\Omega(z)| < p - \alpha. \tag{19}$$

Therefore, the desired proof is completed. $\square$

This theorem has many interesting and important consequences in analytic function theory and geometric function theory. We give some of these with their corresponding geometric properties.

First, if we choose the value of the parameter w as a real number with $w = \delta \in \mathbb{R} \setminus \{0\}$ in the theorem, then we obtain the following corollary.

Corollary 2.2. *Let $f \in \mathcal{H}_n(p), \delta \in \mathbb{R} \setminus \{0\}, \mu \geq 0, 0 \leq \alpha < p, p \in \mathbb{N}$, and let the function $\mathcal{H}$ be defined by (7). Also, if $\mathcal{H}(z)$ satisfies the following conditions:*

$$\mathfrak{Re}\{\mathcal{H}(z)\} = \begin{cases} < \frac{1}{\delta} & \text{if } \delta > 0, \\ > -\frac{1}{\delta} & \text{if } \delta < 0, \end{cases} \quad \text{or} \quad \mathfrak{Im}\{\mathcal{H}(z)\} \neq 0, \tag{20}$$

then

$$\mathfrak{Re}\Big\{ (\frac{z^p}{f(z)})^{\mu-1} z^{1-p} f'(z) \Big\} > p - (p-\alpha)^{1/\delta}. \tag{21}$$

Putting $w = 1$ in the theorem, we get the following corollary.

Corollary 2.3. *Let $f \in \mathcal{H}_n(p), 0 \leq \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}$, and let the function $\mathcal{H}$ be defined by (7). Also, if $\mathcal{H}(z)$ satisfies the following conditions:*

$$\mathfrak{Re}\{\mathcal{H}(z)\} < 1 \quad \text{or} \quad \mathfrak{Im}\{\mathcal{H}(z)\} \neq 0, \tag{22}$$

then $f \in \mathcal{B}_n(p, \mu, \alpha)$.

Setting $w = 1$ and $\mu = 1$ in the theorem, we have the following corollary.

Corollary 2.4. *Let $f \in \mathcal{H}_n(p), 0 \leq \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}$, and let the function $\mathcal{H}$ be defined by*

$$\mathcal{H}(z) = \Big(\frac{z^{1-p}f'(z)}{z^{1-p}f'(z) - p}\Big)\Big(1 + \frac{zf''(z)}{f'(z)} - p\Big). \tag{23}$$

If $\mathcal{H}(z)$ satisfies the following conditions:

$$\mathfrak{Re}\{\mathcal{H}(z)\} < 1 \quad or \quad \mathfrak{Im}\{\mathcal{H}(z)\} \neq 0, \tag{24}$$

then $\mathfrak{Re}\{zf'(z)/f(z)\} > \alpha$, that is, f is multivalent starlike of order α in $\mathcal{U}$.

Setting $w = 1$ and $\mu = 2$ in the theorem, we have the following corollary.

Corollary 2.5. *Let $f \in \mathcal{H}_n(p), 0 \leq \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}$, and let the function $\mathcal{H}$ be defined by*

$$\mathcal{H}(z) = \Big(\frac{zf'(z)}{zf'(z) - pf(z)}\Big)\Big(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big). \tag{25}$$

If $\mathcal{H}(z)$ satisfies the following conditions:

$$\mathfrak{Re}\{\mathcal{H}(z)\} < 1 \quad or \quad \mathfrak{Im}\{\mathcal{H}(z)\} \neq 0, \tag{26}$$

then $\mathfrak{Re}\{z^{1-p}f'(z)\} > \alpha$, that is, f is multivalent close-to-convex of order α in $\mathcal{U}$.

Lastly, if we take $p = 1$ in Corollaries 2.4 and 2.5, then we easily obtain the three important results involving starlike functions of order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$, and close-to-convex functions of order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$, respectively, (see, e.g., [8,9]).

3 Open Problem

With regards to the problems solved, the this work can also be applied to other classes. For example, can the same problem be applied for following classes.

Definition 3.1.(see [10]) *A function $f \in \mathcal{H}_n(p)$ is said to be a member of the class $\mathcal{B}_p(\alpha, \mu, \lambda)$ if and only if*

$$\Big|(1-\alpha)\Big(\frac{z}{f(z)}\Big)^{\mu} + \alpha\frac{zf'(z)}{pf(z)}\Big(\frac{z}{f(z)}\Big)^{\mu} - 1\Big| < 1 - \lambda, \ (p \in \mathbb{N}, z \in \mathbb{U}) \tag{27}$$

for some $\mu \geq 0, 0 \leq \lambda < 1, \alpha \in \mathbb{C}$.

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