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Sandwich results of normalized analytic functions defined by generalized Salagean integral operator

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Abstract

In this paper we obtain some applications of first order differential subordination and superordination results involving generalized Sălăgean integral operator for certain normalized analytic functions. our results generalize previously known results.

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1 Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} ... \ (a \in C). \tag{1}$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

If f, $g \in H(U)$, we say that f is subordinate to g or f is superordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such that $f(z) = g(\omega(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[3], [6] and [7]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \ and \ f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp^{'}(z); z\right) \prec h(z), \tag{3}$$

then $p(z)$ is a solution of the differential subordination (3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp^{'}(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp^{'}(z); z\right), \tag{4}$$

then $p(z)$ is a solution of the differential superordination (4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (4) is called the best subordinant. Using the results of Miller and Mocanu [7], Bulboaca [3] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [1], have used the results of Bulboaca [2] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [12] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [11] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [11] also obtained results for functions defined by using Carlson-Shaffer operator [4], Ruscheweyh derivative [9] and Sălăgean operator [10].

For functions f given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, 3, ...\}, \lambda > 0$ and $f \in A$, Patel [8] considered the integral operator defined as follows:

$$\begin{aligned} I_\lambda^0 f(z) &= f(z), \\ I_\lambda^1 f(z) &= \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} f(t)\, dt \ = z + \sum_{k=2}^{\infty} \left[\frac{1}{1+\lambda(k-1)} \right] a_k z^k, \\ I_\lambda^2 f(z) &= \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^1 f(t)\, dt = z + \sum_{k=2}^{\infty} \left[\frac{1}{1+\lambda(k-1)} \right]^2 a_k z^k, \end{aligned}$$

and (in general)

$$\begin{aligned} I_\lambda^n f(z) &= \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^{n-1} f(t)\, dt \\ &= z + \sum_{k=2}^{\infty} \left[\frac{1}{1+\lambda(k-1)} \right]^n a_k z^k \qquad (5) \\ &= \underbrace{I_\lambda^1 \left(\frac{z}{1-z} \right) * I_\lambda^1 \left(\frac{z}{1-z} \right) * ... * I_\lambda^1 \left(\frac{z}{1-z} \right)}_{n-times} * f(z) \end{aligned}$$

then from (5), we can easily deduced that

$$\lambda z \left(I_\lambda^n f(z) \right)' = I_\lambda^{n-1} f(z) - (1-\lambda) I_\lambda^n f(z) \quad (\lambda > 0; n \in \mathbb{N}). \qquad (6)$$

We note that $I_1^n f(z) = I^n f(z)$, where I^n is Salagean integral operator [10].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator I_λ^n.

2 Main Results

In order to prove our results, we need the following definition and lemmas.

Definition 1 [7]. Denote by Q, the set of all functions f that are analytic and injective on $U\backslash E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f\left(z\right)=\infty\right\},$$

and are such that $f^{'}\left(\zeta\right)\neq 0$ for $\zeta\in\partial U\backslash E\left(f\right)$.

Lemma 1 [11]. Let $q\left(z\right)$ be univalent in U with $q(0)=1$. Let $\alpha\in\mathbb{C}$; $\gamma\in\mathbb{C}^{*}$, further assume that

$$\Re\left\{1+\frac{zq^{''}\left(z\right)}{q^{'}\left(z\right)}\right\}>\max\left\{0,-\Re\left(\frac{\alpha}{\gamma}\right)\right\}.$$

If $p\left(z\right)$ is analytic in U, and

$$\alpha p\left(z\right)+\gamma zp^{'}\left(z\right)\prec\alpha q\left(z\right)+\gamma zq^{'}\left(z\right),$$

then $p\left(z\right)\prec q\left(z\right)$ and $q\left(z\right)$ is the best dominant.

Lemma 2 [11]. Let $q\left(z\right)$ be convex univalent in U, $q(0)=1$. Let $\alpha\in\mathbb{C}$; $\gamma\in\mathbb{C}^{*}$ and $\Re\left(\frac{\alpha}{\gamma}\right)>0$. If $p(z)\in H[q(0),1]\cap Q$, $\alpha p\left(z\right)+\gamma zp^{'}\left(z\right)$ is univalent in U and

$$\alpha q\left(z\right)+\gamma zq^{'}\left(z\right)\prec\alpha p\left(z\right)+\gamma zp^{'}\left(z\right),$$

then $q\left(z\right)\prec p\left(z\right)$ and $q\left(z\right)$ is the best subordinant

Unless otherwise mentioned, we assume throughout this paper that $\lambda>0$ and $n\in\mathbb{N}$.

Theorem 1. Let $q\left(z\right)$ be univalent in U with $q(0)=1$, and $\gamma\in\mathbb{C}^{*}$. Further, assume that

$$\Re\left\{1+\frac{zq^{''}\left(z\right)}{q^{'}\left(z\right)}\right\}>\max\left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{7}$$

If $f\in\mathcal{A}$ satisfy the following subordination condition:

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zI_{\lambda}^{n-1}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}-\frac{2z\left[I_{\lambda}^{n}f(z)\right]^{2}}{\left[I_{\lambda}^{n+1}f(z)\right]^{3}}\right\}\prec q\left(z\right)+\gamma zq^{'}\left(z\right), \tag{8}$$

then

$$\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}\prec q\left(z\right)$$

and $q\left(z\right)$ is the best dominant.

Proof. Define a function $p\left(z\right)$ by

$$p\left(z\right)=\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}\qquad\left(z\in U\right). \tag{9}$$

Then the function $p(z)$ is analytic in U and $p(0)=1$. Therefore, differentiating (9) logarithmically with respect to z and using the identity (6) in the resulting equation, we have

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zI_{\lambda}^{n-1}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}-\frac{2z\left[I_{\lambda}^{n}f(z)\right]^{2}}{\left[I_{\lambda}^{n+1}f(z)\right]^{3}}\right\}=p(z)+\gamma zp'(z),$$

that is,

$$p(z)+\gamma zp'(z)\prec q(z)+\gamma zq'(z).$$

Therefore, Theorem 1 now follows by applying Lemma 1.

Putting $q(z)=\frac{1+Az}{1+Bz}$ $(A,B\in\mathbb{C},A\neq B,|B|<1)$ in Theorem 1, we obtain the following corollary.

Corollary 1. Let $A,B,\gamma\in\mathbb{C},A\neq B$, such that $|B|<1$ and $\Re\{\gamma\}>0$. If $f\in\mathcal{A}$ satisfy the following subordination condition:

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zI_{\lambda}^{n-1}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}-\frac{2z\left[I_{\lambda}^{n}f(z)\right]^{2}}{\left[I_{\lambda}^{n+1}f(z)\right]^{3}}\right\}\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^{2}},$$

then

$$\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}\prec\frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Now, by appealing to Lemma 2 it can be easily prove the following theorem.

Theorem 2. Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\gamma)>0$. If $f\in\mathcal{A}$ such that $\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}\in H[q(0),1]\cap Q$,

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zI_{\lambda}^{n-1}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}-\frac{2z\left[I_{\lambda}^{n}f(z)\right]^{2}}{\left[I_{\lambda}^{n+1}f(z)\right]^{3}}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq'(z)\prec\left(1+\frac{\gamma}{\lambda}\right)\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zI_{\lambda}^{n-1}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}-\frac{2z\left[I_{\lambda}^{n}f(z)\right]^{2}}{\left[I_{\lambda}^{n+1}f(z)\right]^{3}}\right\}$$

holds, then

$$q(z)\prec\frac{zI_{\lambda}^{n}f(z)}{\left[I_{\lambda}^{n+1}f(z)\right]^{2}}$$

and $q(z)$ is the best subordinant.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(A,B\in\mathbb{C},A\neq B,|B|<1)$ in Theorem 2, we have the following corollary.

Corollary 2. Let $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. If $f \in \mathcal{A}$ such that $\frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} \in H[q(0), 1] \cap Q$,

$$\left(1+\frac{\gamma}{\lambda}\right) \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zI_\lambda^{n-1} f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} - \frac{2z\left[I_\lambda^n f(z)\right]^2}{\left[I_\lambda^{n+1} f(z)\right]^3} \right\}$$

is univalent in U, and the following superordination condition

$$\frac{1+Az}{1+Bz} + \gamma \frac{(A-B)z}{(1+Bz)^2} \prec \left(1+\frac{\gamma}{\lambda}\right) \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zI_\lambda^{n-1} f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} - \frac{2z\left[I_\lambda^n f(z)\right]^2}{\left[I_\lambda^{n+1} f(z)\right]^3} \right\}$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2}$$

and $q(z)$ is the best subordinant.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem for I_λ^n.

Theorem 3. Let $q_1(z)$ be convex univalent in U with $q_1(0) = 1$, $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0, q_2(z)$ be univalent in U with $q_2(0) = 1$, and satisfies (7). If $f \in \mathcal{A}$ such that $\frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} \in H[q(0), 1] \cap Q$,

$$\left(1+\frac{\gamma}{\lambda}\right) \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zI_\lambda^{n-1} f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} - \frac{2z\left[I_\lambda^n f(z)\right]^2}{\left[I_\lambda^{n+1} f(z)\right]^3} \right\}$$

is univalent in U, and

$$\begin{aligned} q_1(z) + \gamma z q_1'(z) &\prec \left(1+\frac{\gamma}{\lambda}\right) \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zI_\lambda^{n-1} f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} - \frac{2z\left[I_\lambda^n f(z)\right]^2}{\left[I_\lambda^{n+1} f(z)\right]^3} \right\} \\ &\prec q_2(z) + \gamma z q_2'(z) \end{aligned}$$

holds, then

$$q_1(z) \prec \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Taking $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1, 2; -1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$ in Theorem 3, we have the following corollary.

Corollary 3. Let $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. If $f, g \in \mathcal{A}$ such that $\frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} \in H[q(0), 1] \cap Q$,

$$\left(1+\frac{\gamma}{\lambda}\right) \frac{zI_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zI_\lambda^{n-1} f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} - \frac{2z\left[I_\lambda^n f(z)\right]^2}{\left[I_\lambda^{n+1} f(z)\right]^3} \right\}$$

is univalent in U, and

$$\begin{aligned} & \frac{1+A_1 z}{1+B_1 z}+\frac{\gamma\left(A_1-B_1\right) z}{\left(1+B_1 z\right)^2} \\ \prec\ & \left(1+\frac{\gamma}{\lambda}\right) \frac{z I_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{z I_\lambda^{n-1} f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2}-\frac{2 z\left[I_\lambda^n f(z)\right]^2}{\left[I_\lambda^{n+1} f(z)\right]^3}\right\} \\ \prec\ & \frac{1+A_2 z}{1+B_2 z}+\frac{\gamma\left(A_2-B_2\right) z}{\left(1+B_2 z\right)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{z I_\lambda^n f(z)}{\left[I_\lambda^{n+1} f(z)\right]^2} \prec \frac{1+A_2 z}{1+B_2 z}$$

and $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are, respectively, the best subordinant and the best dominant.

Remark: Taking $\lambda=1$ in Theorems 1, 2, 3 and Corollary 1, respectively, we obtain the results of Cotîrlă [5, Theorems 3.5, 3.6, 3.7 and Example 3.2, respectively].

3 Open Problem

Find the sufficient conditions for normalized analytic functions $f(z)$ and α to satisfy:

$$q_1(z) \prec \frac{z}{I_\lambda^{n+1} f(z)}\left(\frac{I_\lambda^n f(z)}{I_\lambda^{n+1} f(z)}\right)^\alpha \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=q_2(0)=1$.

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References

[1] R. M. Ali, V. Ravichandran and K. G. Subramanian, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci. 15 (2004), no. 1, 87-94.

[2] T. Bulboacă, Classes of first order differential superordinations, Demonstratio Math. 35 (2002), no. 2, 287-292.

[3] T. Bulboacă, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[4] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., 15 (1984), 737-745.

[5] L.-I. Cotîrlă, Differential subordinations and superordinations for certain analytic functions defined by an integral operator, Carpathian J. Math., 25(2009), no. 1, 49-54.

[6] S. S. Miller and P. T. Mocanu, Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[7] S. S. Miller and P. T. Mocanu, Subordinates of differential superordinations, Complex Variables, 48 (2003), no. 10, 815-826.

[8] J. Patel, Inclusion relations and convolution properties of certain subclasses of analytic functions de ned by a generalized Salagean operator, Bull. Belg. Math. Soc. Simon Stevin 15(2008), 33-47.

[9] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Sco., 49 (1975), 109-115.

[10] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag) 1013 , (1983), 362 - 372 .

[11] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differantial sandwich theorems for some subclasses of analytic functions, J. Austr.Math. Anal. Appl., 3 (2006), no. 1, Art. 8, 1-11.

[12] N. Tuneski, On certain sufficient conditions for starlikeness, Internat. J. Math. Math. Sci., 23 (2000), no. 8, 521-527.A. Alb Lupaş, *A note on a certain subclass of analytic functions defined by multiplier transformation*, Journal of Computational Analysis and Applications, Vol. 12, No. 1-B, 2010, 369-373.