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Some Properties for Certain Class of Analytic Functions Defined by Convolution with Varying Arguments

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Abstract

In this paper, we introduce a new class $V(g, \lambda, A, B)$ ***of analytic functions with varying arguments in the open unit disc*** $U = \{z \in \mathbb{C} : |z| < 1\}$ ***defined by convolution. The object of the present paper is to determine coefficient estimates, extreme points, distortion theorems for functions belonging to this class.***

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For functions f given by (1) and $g(z) \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z).$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [4] and [14]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $\lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1$ and for all $z \in U$, let $S(g, \lambda, A, B)$ denotes the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1) and $g(z)$ of the form (2) satisfying the analytic criterion:

$$(1-\lambda)\frac{(f*g)(z)}{z} + \lambda (f*g)'(z) \prec \frac{1+Az}{1+Bz}. \tag{3}$$

It is noticed that for suitable choice of g, λ, A and B we obtain the following subclasses studied by various authors.

(1) Putting $A = 2\alpha - 1$ and $B = 1$, the class $S(g, \lambda, 2\alpha-1, 1)$ reduces to the class $S(f, g; \lambda, \alpha)$ $(0 \leq \alpha < 1, \lambda \geq 0)$ (see Aouf et al. [3], with $b = 1$);
(2) Putting $g(z) = z + \sum\limits_{n=2}^{\infty} n^k a_n z^n$ or $(b_n = n^k, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\})$, the class $S(z + \sum\limits_{n=2}^{\infty} n^k a_n z^n, \lambda, A, B)$ reduces to the class $G_k(\lambda, A, B)$ (see Sivasubramanian et al. [20], with $b = 1$);
(3) Putting $g(z) = \frac{z}{1-z}, A = 2\alpha - 1$ and $B = 1$, the class $S(\frac{z}{1-z}, \lambda, 2\alpha - 1, 1)$ reduces to the class $B(\lambda, \alpha)$ $(0 \leq \alpha < 1, 0 \leq \lambda \leq 1)$ (see Chunyi and Owa [9]);
(4) Putting $g(z) = \frac{z}{1-z}, A = 2\alpha - 1$, $B = 1$ and $\lambda = 0$, the class $S(\frac{z}{1-z}, 0, 2\alpha - 1, 1)$ reduces to the class $B(\alpha)$ $(0 \leq \alpha < 1)$ (see Chen [7, 8] and Goel [12]);
(5) Putting $g(z) = \frac{z}{(1-z)^2}, A = 2\alpha - 1$, $B = 1$ and $\lambda = 1$, the class $S(\frac{z}{(1-z)^2}, 1, 2\alpha - 1, 1)$ reduces to the class $C(\alpha)$ $(0 \leq \alpha < 1)$ (see Srivastava and Owa [21]).

Also we note that:
(1) Putting $\lambda = 0$ in (1.3), the class $S(g, 0, A, B)$ reduces to the class $S(g, A, B) = \left\{ \frac{(f*g)(z)}{z} \prec \frac{1+Az}{1+Bz}, -1 \leq A < B \leq 1, 0 < B \leq 1, z \in U \right\}$;
(2) Putting $\lambda = 1$ in (1.3), the class $S(g, 1, A, B)$ reduces to the class $C(g, A, B) = \left\{ (f*g)'(z) \prec \frac{1+Az}{1+Bz}, -1 \leq A < B \leq 1, 0 < B \leq 1, z \in U \right\}$;
(3) Putting $g(z) = \frac{z}{1-z}$ or $(b_n = 1)$ in (1.3), the class $S(\frac{z}{1-z}, \lambda, A, B)$ reduces to the class $S(\lambda, A, B) = \left\{ (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \frac{1+Az}{1+Bz}, \lambda \geq 0, -1 \leq A < B \right.$

$\leq 1, 0 < B \leq 1, z \in U\}$;

(4) Putting $g(z) = z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n$ $(or\ b_n = \Psi_n(\alpha_1))$, where

$$\Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1} \cdots\cdots\cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots\cdots (\beta_s)_{n-1}(n-1)!} \tag{4}$$

$$(\alpha_i > 0,\ i = 1, \ldots, q; \beta_j > 0,\ j = 1, \ldots, s;\ q \leq s+1; q, s \in \mathbb{N}_0),$$

the class $S(z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n; \lambda, A, B)$ reduces to the class $S_{q,s}([\alpha_1]; \lambda, A, B)$

$$= \left\{ f \in A : (1-\lambda) \frac{H_{q,s}(\alpha_1) f(z)}{z} + \lambda \left(H_{q,s}(\alpha_1) f(z)\right)' \prec \frac{1+Az}{1+Bz}, z \in U \right\},$$

where $H_{q,s}(\alpha_1)$ is the Dziok-Srivastava operator (see [10] and [11]) which contains well known operators such as Carlson-Shaffer linear operator (see [5]), the Bernardi-Libera-Livingston operator (see [13]), Srivastava - Owa fractional derivative operator (see [16]), the Ruscheweyh derivative operator (see [17]) and the Noor integral operator (see [15]);

(5) Putting $g(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+l+\gamma(n-1)}{1+l}\right)^m z^n$ $\left(or\ b_n = \left(\frac{1+l+\gamma(n-1)}{1+l}\right)^m, \gamma \geq 0,\ l \geq 0,\ m \in \mathbb{N}_0\right)$, the class $S(z + \sum_{n=2}^{\infty} \left(\frac{1+l+\gamma(n-1)}{1+l}\right)^m z^n; \lambda, A, B)$ reduces to the class $S(\gamma, l, m; \lambda, A, B)$

$$= \left\{ f \in A : (1-\lambda) \frac{I^m(\gamma, l) f(z)}{z} + \lambda \left(I^m(\gamma, l) f(z)\right)' \prec \frac{1+Az}{1+Bz}, z \in U \right\},$$

where $I^m(\gamma, l) f(z)$ is the extended multiplier transformation (see [6]), for $l = 0, \gamma \geq 0$, the operator $I_m(\gamma, 0) = D_\gamma^m$ was introduced and studied by Al-Oboudi (see [1]) and for $l = \gamma = 0$, the operator $I_m(0,0) = D^m$, where D^m is Salagean differential operator (see [18]).

Definition 1 [19]. A function $f(z)$ defined by (1) is said to be in the class $V(\theta_n)$ if $f(z) \in A$ and $\arg(a_n) = \theta_n$ for all $n \geq 2$. If furthermore, there exists a real number β such that

$$\theta_n + (n-1)\beta \equiv \pi \pmod{2\pi},$$

then $f(z)$ is said to be in the class $V(\theta_n; \beta)$. The union of $V(\theta_n; \beta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers β is denoted by V.

Let $V(g,\lambda,A,B)$ denote the subclass of V consisting of functions $f(z)$ in $S(g,\lambda,A,B)$.

We note that:
(1) $V(\frac{z}{1-z},\lambda,2\alpha-1,1)=V_{\lambda}(\alpha)$ $(0\le\alpha<1,\lambda\ge0)$ (see Aouf et al. [2]);
(2) $V(\frac{z}{1-z},0,2\alpha-1,1)=B_{\alpha}$ $(0\le\alpha<1)$ (see Srivastava and Owa [21]);
(3) $V(\frac{z}{(1-z)^2},1,2\alpha-1,1)=C_{\alpha}$ $(0\le\alpha<1)$ (see Srivastava and Owa [21]).

Also we note that:
(1) $V(g,\lambda,2\alpha-1,1)=V(f,g,\lambda,\alpha)$ $(0\le\alpha<1,\lambda\ge0)$ denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $S(f,g;\lambda,\alpha)$;
(2) $V(z+\sum\limits_{n=2}^{\infty}n^k a_n z^n,1,\lambda,A,B)=V_k(\lambda,1,A,B)=V_k(\lambda,A,B)$ $(k\in\mathbb{N}_0,\lambda\ge0,-1\le A<B\le1,0<B\le1)$ denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $G_k(\lambda,A,B)$;
(3) $V(g,0,A,B)=V(g,A,B)$ $(-1\le A<B\le1,0<B\le1)$ denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $S(g,A,B)$;
(4) $V(g,1,A,B)=VC(g,A,B)$ $(-1\le A<B\le1,0<B\le1)$ denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $C(g,A,B)$;
(5) $V(\frac{z}{1-z},\lambda,A,B)=V(\lambda,A,B)$ $(\lambda\ge0,-1\le A<B\le1,0<B\le1)$ denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $S(\lambda,A,B)$;
(6) $V(z+\sum\limits_{n=2}^{\infty}\Psi_n(\alpha_1)z^n;\lambda,A,B)=V_{q,s}([\alpha_1];\lambda,A,B)$ (where $\Psi_n(\alpha_1)$ is given by (4), $\lambda\ge0,-1\le A<B\le1,0<B\le1$) denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $S_{q,s}([\alpha_1];\lambda,A,B)$;
(7) $V(z+\sum\limits_{n=2}^{\infty}\left(\frac{1+\gamma(n-1)+l}{1+l}\right)^m z^n;\lambda,A,B)=V(\gamma,l,m;\lambda,A,B)$ $(\lambda\ge0,-1\le A<B\le1,0<B\le1)$ denotes the subclass of V, consisting of functions $f(z)$ belonging to the class $S(\gamma,l,m;\lambda,A,B)$.

2 Coefficient Estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\lambda\ge0,-1\le A<B\le1,0<B\le1,b_n>0$ and $g(z)$ is defined by (2).
Theorem 1. Let the function $f(z)$ be of the form (1), if

$$\sum_{n=2}^{\infty}\left[1+\lambda(n-1)\right](1+B)\,b_n\left|a_n\right|\le(B-A)\,,\tag{5}$$

then $f(z)\in S(g,\lambda,A,B)$.

Proof. A function $f(z)$ of the form (1) belongs to the class $S(g,\lambda,A,B)$ if and only if there exists a function $w, |w(z)| \leq |z|$, such that

$$(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)=\frac{1+Aw(z)}{1+Bw(z)},$$

or, equivalently,

$$\left|\frac{(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)-1}{B\left[(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)\right]-A}\right|<1. \tag{6}$$

Thus, it is sufficient to prove that

$$\left|(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)-1\right|-\left|B\left[(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)\right]-A\right|<0.$$

Indeed, letting $|z|=r(0\leq r<1)$ we have

$$\begin{aligned}
&\left|(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)-1\right|-\left|B\left[(1-\lambda)\frac{(f*g)(z)}{z}+\lambda(f*g)'(z)\right]-A\right|\\
=\ &\left|\sum_{n=2}^{\infty}[1+\lambda(n-1)]\,b_na_nz^n\right|-\left|(B-A)z+\sum_{n=2}^{\infty}B[1+\lambda(n-1)]\,b_na_nz^n\right|\\
\leq\ &r\left(\sum_{n=2}^{\infty}[1+\lambda(n-1)]\,b_n|a_n|r^{n-1}-(B-A)+\sum_{n=2}^{\infty}B[1+\lambda(n-1)]\,b_n|a_n|r^{n-1}\right)\\
<\ &\sum_{n=2}^{\infty}[1+\lambda(n-1)](1+B)\,b_n|a_n|-(B-A).
\end{aligned}$$

In view of (5) the last inequality is less the zero, hence $f(z)\in S(g,\lambda,A,B)$. This completes the proof of Theorem 1.

Theorem 2. *Let the function $f(z)$ be of the form (1), then $f(z)$ is in the class $V(g,\lambda,A,B)$ if and only if*

$$\sum_{n=2}^{\infty}[1+\lambda(n-1)](1+B)\,b_n|a_n|\leq(B-A).$$

Proof. In view of Theorem 1 we need only to show that each function $f(z)$ from the class $V(g,\lambda,A,B)$ satisfies the coefficient inequality (5). Let $f(z)\in V(g,\lambda,A,B)$. Then, by (6) and (1), we have

$$\left|\frac{\sum_{n=2}^{\infty}[1+\lambda(n-1)]\,b_na_nz^{n-1}}{(B-A)+\sum_{n=2}^{\infty}B[1+\lambda(n-1)]\,b_na_nz^{n-1}}\right|<1.$$

Since $f(z) \in V$, $f(z)$ lies in the class $V(\theta_n, \beta)$ for some sequence $\{\theta_n\}$ and a real number β such that $\theta_n + (n-1)\beta \equiv \pi \,(\mathrm{mod}\, 2\pi) \quad (n \geq 2)$. Set $z = re^{i\beta}$ in the above inequality, we get

$$\left| \frac{-\sum\limits_{n=2}^{\infty} [1+\lambda(n-1)]\, b_n \,|a_n|\, r^{n-1}}{(B-A) - \sum\limits_{n=2}^{\infty} B\,[1+\lambda(n-1)]\, b_n \,|a_n|\, r^{n-1}} \right| < 1.$$

Since $Re\,\{w(z)\} < |w(z)| < 1$, then

$$\mathrm{Re}\left\{ \frac{\sum\limits_{n=2}^{\infty} [1+\lambda(n-1)]\, b_n \,|a_n|\, r^{n-1}}{(B-A) - \sum\limits_{n=2}^{\infty} B\,[1+\lambda(n-1)]\, b_n \,|a_n|\, r^{n-1}} \right\} < 1.$$

It is clear that the denominator of the left hand said cannot vanish for $r \in [0,1)$. Moreover, it is positive for $r = 0$ and in consequence for $r \in [0,1)$. Thus, we have

$$\sum_{n=2}^{\infty} [1+\lambda(n-1)]\,(1+B)\, b_n \,|a_n|\, r^{n-1} \leq (B-A)\,,$$

which, upon letting $r \to 1^-$, readily yields the assertion (5). This completes the proof of Theorem 2.

Corollary 1. *Let the function $f(z)$ defined by (1) be in the class $V(g,\lambda,A,B)$, then*

$$|a_n| \leq \frac{(B-A)}{[1+\lambda(n-1)]\,(1+B)\, b_n} \;\; (n \geq 2). \tag{7}$$

The result is sharp for the function

$$f(z) = z + \frac{(B-A)}{[1+\lambda(n-1)]\,(1+B)\, b_n} e^{i\theta_n} z^n \;\; (n \geq 2). \tag{8}$$

3 Distortion theorems

Theorem 3. *Let the function $f(z)$ defined by (1) be in the class $V(g,\lambda,$ $A,B)$. Then*

$$|z| - \frac{(B-A)}{(1+B)(1+\lambda)\, b_2} |z|^2 \leq |f(z)| \leq |z| + \frac{(B-A)}{(1+B)(1+\lambda)\, b_2} |z|^2\,, \tag{9}$$

provided $b_n \geq b_2$ $(n \geq 2)$. *The result is sharp.*
Proof. Since

$$\Phi(n) = [1 + \lambda(n-1)](1+B)\, b_n, \tag{10}$$

is an increasing function of $n\,(n \geq 2)$, from Theorem 1, we have

$$(1+B)(1+\lambda)\, b_2 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} [1+\lambda(n-1)](1+B)\, b_n |a_n| \leq (B-A),$$

that is

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(B-A)}{(1+B)(1+\lambda)\, b_2}.$$

Thus

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{(B-A)}{(1+B)(1+\lambda)\, b_2} |z|^2. \end{aligned}$$

Similarly, we get

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\geq |z| - \frac{(B-A)}{(1+B)(1+\lambda)\, b_2} |z|^2. \end{aligned}$$

This completes the proof of Theorem 3. Finally the result is sharp for the function

$$f(z) = z + \frac{(B-A)}{(1+B)(1+\lambda)\, b_2} e^{i\theta_2} z^2 \tag{11}$$

at $z = \pm |z|\, e^{-i\theta_2}$.

Corollary 2. Under the hypotheses of Theorem 3, $f(z)$ is included in a disc with center at the origin and radius r_1 given by

$$r_1 = 1 + \frac{(B-A)}{(1+B)(1+\lambda)\, b_2}.$$

Theorem 4. *Let the function* $f(z)$ *defined by (1) belong to the class* $V(g, \lambda, A, B)$. *Then*

$$1 - \frac{2(B-A)}{(1+B)(1+\lambda)\, b_2} |z| \leq \left| f'(z) \right| \leq 1 + \frac{2(B-A)}{(1+B)(1+\lambda)\, b_2} |z|. \tag{12}$$

provided $b_n \geq b_2$ $(n \geq 2)$. *The result is sharp for the function f(z) given by (11) at* $z = \pm |z| e^{-i\theta_2}$

Proof. Since $\{n\Phi(n)\}$, where $\Phi(n)$ given by (10) is increasing function of $n\,(n \geq 2)$, in view of Theorem 1, we have

$$\frac{(1+B)(1+\lambda)b_2}{2}\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty}[1+\lambda(n-1)](1+B)b_n|a_n| \leq (B-A),$$

that is

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{2(B-A)}{(1+B)(1+\lambda)b_2}.$$

Thus

$$\begin{aligned}\left|f'(z)\right| &= \left|1+\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq 1+|z|\sum_{n=2}^{\infty} n|a_n| \\ &\leq 1+\frac{2(B-A)}{(1+B)(1+\lambda)b_2}|z|.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\left|f'(z)\right| &\geq 1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1} \geq 1-|z|\sum_{n=2}^{\infty} n|a_n| \\ &\geq 1-\frac{2(B-A)}{(1+B)(1+\lambda)b_2}|z|.\end{aligned}$$

Finally the result is sharp for the function $f(z)$ given by (11). This completes the proof of Theorem 4.

Corollary 3. Let the function $f(z)$ defined by (1) be in the class $V(g,\lambda,A,B)$. Then $f'(z)$ is included in a disc with center at the origin and radius r_2 given by

$$r_2 = 1+\frac{2(B-A)}{(1+B)(1+\lambda)b_2}.$$

4 Extreme points

Theorem 5. Let the function $f(z)$ defined by (1) be in the class $V(g,\lambda,A,B)$, *with* $\arg(a_n) = \theta_n$ where $[\theta_n + (n-1)\beta] \equiv \pi \,(\mathrm{mod}\, 2\pi)$. *Define* $f_1(z) = z$ and

$$f_n(z) = z + \frac{(B-A)}{[1+\lambda(n-1)](1+B)b_n}e^{i\theta_n}z^n.$$

Then $f(z)$ is in the class $V(g,\lambda,A,B)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z), \tag{13}$$

where $\mu_n \geq 0$ $(n\geq 1)$ and $\sum\limits_{n=1}^{\infty}\mu_n=1$.

Proof. If $f(z)=\sum\limits_{n=1}^{\infty}\mu_n f_n(z)$ with $\sum\limits_{n=1}^{\infty}\mu_n=1$ and $\mu_n\geq 0$, then

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[1+\lambda\left(n-1\right)\right]\left(1+B\right)b_n\frac{(B-A)}{\left[1+\lambda\left(n-1\right)\right]\left(1+B\right)b_n}\mu_n\\ =\ &\sum_{n=2}^{\infty}\left(B-A\right)\mu_n=\left(B-A\right)\left(1-\mu_1\right)\leq\left(B-A\right).\end{aligned}$$

So, by Theorem 1, we have $f(z)\in V(g,\lambda,A,B)$.
Conversely, let the function $f(z)$ defined by (1.1) belongs to the class $V(g,\lambda,A,B)$. Then a_n are given by (7). Setting

$$\mu_n=\frac{\left[1+\lambda\left(n-1\right)\right]\left(1+B\right)b_n}{(B-A)}\left|a_n\right|, \tag{14}$$

and

$$\mu_1=1-\sum_{n=2}^{\infty}\mu_n.$$

From Theorem 1, $\sum\limits_{n=1}^{\infty}\mu_n\leq 1$ and so $\mu_n\geq 0$. Since $\mu_n f_n(z)=\mu_n z+a_n z^n$, then

$$\sum_{n=1}^{\infty}\mu_n f_n(z)=z+\sum_{n=2}^{\infty}a_n z^n=f(z).$$

This completes the proof of Theorem 5.

Remark. Specializing $g(z),\lambda,A$ and B, in the above results, we obtain the corresponding results for the corresponding classes $V(f,g,\lambda,\alpha)$, $V_k(\lambda,A,B)$ $V(g,A,B)$, $VC(g,A,B)$, $V(\lambda,A,B)$, $V_{q,s}([\alpha_1];\lambda,A,B)$ and $V(\gamma,l,m;\lambda,A,B)$ defined in the introduction.

5 Open problems

(i) We can generalize this study when the functions $f\left(z\right)$and $g\left(z\right)$ are p-valent functions.

(ii) For meromorphic functions $f_1, f_2 \in \sum$, where $f_1(z) = z^{-1} + \sum\limits_{n=1}^{\infty} a_n z^n$ and $f_2(z) = z^{-1} + \sum\limits_{n=1}^{\infty} b_n z^n$,the Hadamard product (or convolution) is given by

$$(f_1 * f_2)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n = (f_2 * f_1)(z).$$

We suggest to study the class

$$\begin{aligned} \sum V(f_2, \lambda, A, B) &= (1+\lambda) z (f_1 * f_2)(z) + \lambda z^2 (f_1 * f_2)'(z) \prec \frac{1+Az}{1+Bz}, \\ &\quad (\lambda \geq 0; -1 \leq A < B \leq 1, 0 < B \leq 1). \end{aligned}$$

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