Some Properties for Certain Class of Analytic Functions Defined by Convolution with Varying Arguments

A. O. Mostafa, M. K. Aouf, A. Shamandy and E. A. Adwan

Department of Mathematics, Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: adelaeg254@yahoo.com, mkaouf127@yahoo.com
aashamandy@hotmail.com, eman.a2009@yahoo.com

Abstract

In this paper, we introduce a new class $V(g, \lambda, A, B)$ of analytic functions with varying arguments in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ defined by convolution. The object of the present paper is to determine coefficient estimates, extreme points, distortion theorems for functions belonging to this class.

Keywords: Analytic functions, varying arguments, extreme points.
2000 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  (1)

which are analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$. For functions $f$ given by (1) and $g(z) \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$  (2)
Putting to the class

Also we note that:

If \( f \) and \( g \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \) if there exists a Schwarz function \( w \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(w(z)), \ z \in U \). Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence (cf., e.g., [4] and [14]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

For \( \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1 \) and for all \( z \in U \), let \( S(\lambda; A, B) \) denotes the subclass of \( A \) consisting of functions \( f(z) \) of the form (1) and \( g(z) \) of the form (2) satisfying the analytic criterion:

\[
(1 - \lambda) \left( \frac{f * g}(z) \right) + \lambda (f * g)'(z) < \frac{1 + Az}{1 + Bz}.
\]

It is noticed that for suitable choice of \( \lambda, A \) and \( B \) we obtain the following subclasses studied by various authors.

1. Putting \( A = 2\alpha - 1 \) and \( B = 1 \), the class \( S(\lambda; 2\alpha - 1, 1) \) reduces to the class \( S(f, g; \lambda, \alpha) (0 \leq \alpha < 1, \lambda \geq 0) \) (see Aouf et al. [3], with \( b = 1 \));

2. Putting \( g(z) = z + \sum_{n=2}^{\infty} nk^2a_nz^n \) or \((b_n = nk, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\})\), the class \( S(z + \sum_{n=2}^{\infty} nk^2a_nz^n, \lambda, A, B) \) reduces to the class \( G_k(\lambda, A, B) \) (see Sivasubramanian et al. [20], with \( b = 1 \));

3. Putting \( g(z) = \frac{z}{1 - z}, A = 2\alpha - 1 \) and \( B = 1 \), the class \( S(\frac{z}{1 - z}, \lambda, 2\alpha - 1, 1) \) reduces to the class \( B(\lambda, \alpha) (0 \leq \alpha < 1, 0 \leq \lambda \leq 1) \) (see Chunyi and Owa [9]);

4. Putting \( g(z) = \frac{z}{1 - z}, A = 2\alpha - 1, B = 1 \) and \( \lambda = 0 \), the class \( S(\frac{z}{1 - z}, 0, 2\alpha - 1, 1) \) reduces to the class \( B(\alpha) (0 \leq \alpha < 1) \) (see Chen [7, 8] and Goel [12]);

5. Putting \( g(z) = \frac{z}{(1 - z)^2}, A = 2\alpha - 1, B = 1 \) and \( \lambda = 1 \), the class \( S(\frac{z}{(1 - z)^2}, 1, 2\alpha - 1, 1) \) reduces to the class \( C(\alpha) (0 \leq \alpha < 1) \) (see Srivastava and Owa [21]).

Also we note that:

1. Putting \( \lambda = 0 \) in (1.3), the class \( S(0, A, B) \) reduces to the class \( S(g, 0, A, B) = \left\{ \frac{(f \circ g)(z)}{z} < \frac{1 + Az}{1 + Bz}, -1 \leq A < B \leq 1, 0 < B \leq 1, z \in U \right\} \);

2. Putting \( \lambda = 1 \) in (1.3), the class \( S(1, A, B) \) reduces to the class \( C(g, A, B) = \left\{ (f \circ g)'(z) < \frac{1 + Az}{1 + Bz}, -1 \leq A < B \leq 1, 0 < B \leq 1, z \in U \right\} \);

3. Putting \( g(z) = \frac{z}{1 - z} \) or \((b_n = 1)\) in (1.3), the class \( S(\frac{z}{1 - z}, \lambda, A, B) \) reduces to the class \( S(\lambda, A, B) = \left\{ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) < \frac{1 + Az}{1 + Bz}, \lambda \geq 0, -1 \leq A < B \right\} \).
\[ \leq 1, 0 < B \leq 1, z \in U \} ; \\
\text{(4) Putting } g(z) = z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n \quad \text{(or } b_n = \Psi_n(\alpha_1)) , \text{ where} \\
\Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{\beta_{n-1} \cdots (\beta_s)_{n-1} (n-1)!} \quad \text{(4)} \\
\] 

\((\alpha_i > 0, \ i = 1, \ldots , q; \beta_j > 0, \ j = 1, \ldots , s; \ q \leq s + 1; q, s \in \mathbb{N}_0), \)

the class \( S(z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n ; \lambda, A, B) \) reduces to the class \( S_{q,s}([\alpha_1]; \lambda, A, B) \)

\[ = \left\{ f \in A : (1 - \lambda) \frac{H_{q,s}(\alpha_1) f(z)}{z} + \lambda (H_{q,s}(\alpha_1) f(z))' < \frac{1 + Az}{1 + Bz} , z \in U \right\} , \]

where \( H_{q,s}(\alpha_1) \) is the Dziok-Srivastava operator (see [10] and [11]) which contains well known operators such as Carlson-Shafer linear operator (see [5]), the Bernardi-Libera-Livingston operator (see [13]), Srivastava-Owa fractional derivative operator (see [16]), the Ruscheweyh derivative operator (see [17]) and the Noor integral operator (see [15]);

\[ \text{(5) Putting } g(z) = z + \sum_{n=2}^{\infty} \left( \frac{1 + l + \gamma(n-1)}{1 + l} \right)^m z^n \quad \text{(or } b_n = \left( \frac{1 + l + \gamma(n-1)}{1 + l} \right)^m , \gamma \geq 0, \ l \geq 0, \ m \in \mathbb{N}_0), \text{ the class } S(z + \sum_{n=2}^{\infty} \left( \frac{1 + l + \gamma(n-1)}{1 + l} \right)^m z^n ; \lambda, A, B) \text{ reduces to the class } S(\gamma, l, m; \lambda, A, B) \]

\[ = \left\{ f \in A : (1 - \lambda) \frac{I^m(\gamma,l) f(z)}{z} + \lambda (I^m(\gamma,l) f(z))' < \frac{1 + Az}{1 + Bz} , z \in U \right\} , \]

where \( I^m(\gamma,l) f(z) \) is the extended multiplier transformation (see [6]), for \( l = 0, \gamma \geq 0, \) the operator \( I_m(\gamma,0) = D^m_\gamma \) was introduced and studied by Al-Oboudi (see [1]) and for \( l = \gamma = 0, \) the operator \( I_m(0,0) = D^m \), where \( D^m \) is Salagean differential operator (see [18]).

**Definition 1** [19]. A function \( f(z) \) defined by (1) is said to be in the class \( V(\theta_n) \) if \( f(z) \in A \) and \( \arg (a_n) = \theta_n \) for all \( n \geq 2. \) If furthermore, there exists a real number \( \beta \) such that

\[ \theta_n + (n - 1)\beta \equiv \pi \pmod{2\pi} , \]

then \( f(z) \) is said to be in the class \( V(\theta_n; \beta). \) The union of \( V(\theta_n; \beta) \) taken over all possible sequences \( \{\theta_n\} \) and all possible real numbers \( \beta \) is denoted by \( V. \)
Let \( V(g, \lambda, A, B) \) denote the subclass of \( V \) consisting of functions \( f(z) \) in \( S(g, \lambda, A, B) \).

We note that:

1. \( V(\frac{z}{1+z}, \lambda, 2\alpha - 1, 1) = V_{\lambda}(\alpha) \) \( (0 \leq \alpha < 1, \lambda \geq 0) \) (see Aouf et al. [2]);
2. \( V(\frac{z}{1-z}, 0, 2\alpha - 1, 1) = B_{\alpha} \) \( (0 \leq \alpha < 1) \) (see Srivastava and Owa [21]);
3. \( V(\frac{1}{1-z^2}, 1, 2\alpha - 1, 1) = C_{\alpha} \) \( (0 \leq \alpha < 1) \) (see Srivastava and Owa [21]).

Also we note that:

1. \( V(g, \lambda, 2\alpha - 1, 1) = V(f, g, \lambda, \alpha) \) \( (0 \leq \alpha < 1, \lambda \geq 0) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( S(f, g; \lambda, \alpha) \);
2. \( V(z + \sum_{n=2}^{\infty} n^k a_n z^n, 1, \lambda, A, B) = V_k(\lambda, 1, A, B) = V_k(\lambda, A, B) \) \( (k \in \mathbb{N}_0, \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( C_k(\lambda, A, B) \);
3. \( V(g, 0, A, B) = V(g, A, B) \) \( (-1 \leq A < B \leq 1, 0 < B \leq 1) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( S(g, A, B) \);
4. \( V(g, 1, A, B) = VC(g, A, B) \) \( (-1 \leq A < B \leq 1, 0 < B \leq 1) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( C(g, A, B) \);
5. \( V(\frac{z}{1-z^2}, \lambda, A, B) = V(\lambda, A, B) \) \( (\lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( S(\lambda, A, B) \);
6. \( V(z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1) z^n; \lambda, A, B) = V_{q,s}(\alpha_1; \lambda, A, B) \) (where \( \Psi_n(\alpha_1) \) is given by (4), \( \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( S_{q,s}(\alpha_1; \lambda, A, B) \);
7. \( V(z + \sum_{n=2}^{\infty} \left(\frac{1+\gamma(n-1)+l}{1+l}\right)^m z^n; \lambda, A, B) = V(\gamma, l, m; \lambda, A, B) \) \( (\lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1) \) denotes the subclass of \( V \), consisting of functions \( f(z) \) belonging to the class \( S(\gamma, l, m; \lambda, A, B) \).

## 2 Coefficient Estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, \( \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1, b_n > 0 \) and \( g(z) \) is defined by (2).

**Theorem 1.** Let the function \( f(z) \) be of the form (1), if

\[
\sum_{n=2}^{\infty} [1 + \lambda(n-1)](1 + B) b_n |a_n| \leq (B - A),
\]

then \( f(z) \) belongs to \( S(\gamma, l, m; \lambda, A, B) \).
Proof. A function $f(z)$ of the form (1) belongs to the class $S(g, \lambda, A, B)$ if and only if there exists a function $w, |w(z)| \leq |z|$, such that

$$(1 - \lambda) \left( \frac{f*g}{z} \right)(z) + \lambda (f*g)'(z) = \frac{1 + Aw(z)}{1+Bw(z)},$$

or, equivalently,

$$(1 - \lambda) \left( \frac{f*g}{z} \right)(z) + \lambda (f*g)'(z) - 1 < 0.$$  

Thus, it is sufficient to prove that

$$\left| (1 - \lambda) \left( \frac{f*g}{z} \right)(z) + \lambda (f*g)'(z) - 1 \right| < B \left[ (1 - \lambda) \left( \frac{f*g}{z} \right)(z) + \lambda (f*g)'(z) \right] - A.$$  

Indeed, letting $|z| = r(0 \leq r < 1)$ we have

$$\left| (1 - \lambda) \left( \frac{f*g}{z} \right)(z) + \lambda (f*g)'(z) - 1 \right| < B \left[ (1 - \lambda) \left( \frac{f*g}{z} \right)(z) + \lambda (f*g)'(z) \right] - A.$$  

This completes the proof of Theorem 1.

Theorem 2. Let the function $f(z)$ be of the form (1), then $f(z)$ is in the class $V(g, \lambda, A, B)$ if and only if

$$\sum_{n=2}^{\infty} [1 + \lambda (n - 1)] b_n a_n z^n \leq (B - A).$$  

Proof. In view of Theorem 1 we need only to show that each function $f(z)$ from the class $V(g, \lambda, A, B)$ satisfies the coefficient inequality (5). Let $f(z) \in V(g, \lambda, A, B)$. Then, by (6) and (1), we have

$$\left| \sum_{n=2}^{\infty} [1 + \lambda (n - 1)] b_n a_n z^n \right| < 1.$$  

In view of (5) the last inequality is less than zero, hence $f(z) \in S(g, \lambda, A, B)$. This completes the proof of Theorem 1.
Since \( f(z) \in V \), \( f(z) \) lies in the class \( V(\theta_n, \beta) \) for some sequence \( \{\theta_n\} \) and a real number \( \beta \) such that \( \theta_n + (n-1)\beta \equiv \pi \mod {2\pi} \) \( (n \geq 2) \). Set \( z = re^{i\beta} \) in the above inequality, we get

\[
\left| \frac{-\sum_{n=2}^{\infty} [1 + \lambda (n - 1)] b_n |a_n| r^{n-1}}{(B - A) - \sum_{n=2}^{\infty} B [1 + \lambda (n - 1)] b_n |a_n| r^{n-1}} \right| < 1.
\]

Since \( \text{Re} \{ w(z) \} < |w(z)| < 1 \), then

\[
\text{Re} \left\{ \frac{\sum_{n=2}^{\infty} [1 + \lambda (n - 1)] b_n |a_n| r^{n-1}}{(B - A) - \sum_{n=2}^{\infty} B [1 + \lambda (n - 1)] b_n |a_n| r^{n-1}} \right\} < 1.
\]

It is clear that the denominator of the left hand said cannot vanish for \( r \in [0, 1) \). Moreover, it is positive for \( r = 0 \) and in consequence for \( r \in [0, 1) \). Thus, we have

\[
\sum_{n=2}^{\infty} [1 + \lambda (n - 1)] (1 + B) b_n |a_n| r^{n-1} \leq (B - A),
\]

which, upon letting \( r \to 1^- \), readily yields the assertion (5). This completes the proof of Theorem 2.

**Corollary 1.** Let the function \( f(z) \) defined by (1) be in the class \( V(g, \lambda, A, B) \), then

\[
|a_n| \leq \frac{(B - A)}{[1 + \lambda (n - 1)] (1 + B) b_n} \quad (n \geq 2).
\]

The result is sharp for the function

\[
f(z) = z + \frac{(B - A)}{[1 + \lambda (n - 1)] (1 + B) b_n} e^{i\theta_n} z^n \quad (n \geq 2).
\]

### 3 Distortion theorems

**Theorem 3.** Let the function \( f(z) \) defined by (1) be in the class \( V(g, \lambda, A, B) \). Then

\[
|z| - \frac{(B - A)}{(1 + B) (1 + \lambda) b_2} |z|^2 \leq |f(z)| \leq |z| + \frac{(B - A)}{(1 + B) (1 + \lambda) b_2} |z|^2,
\]
provided $b_n \geq b_2 \ (n \geq 2)$. The result is sharp.

**Proof.** Since

$$\Phi(n) = [1 + \lambda(n - 1)] (1 + B) b_n, \quad (10)$$

is an increasing function of $n \ (n \geq 2)$, from Theorem 1, we have

$$(1 + B) (1 + \lambda) b_2 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} [1 + \lambda(n - 1)] (1 + B) b_n |a_n| \leq (B - A),$$

that is

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(B - A)}{(1 + B) (1 + \lambda) b_2}.$$ 

Thus

$$|f(z)| = |z + \sum_{n=2}^{\infty} a_n z^n| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |z| + \frac{(B - A)}{(1 + B) (1 + \lambda) b_2} |z|^2.$$ 

Similarly, we get

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\geq |z| - \frac{(B - A)}{(1 + B) (1 + \lambda) b_2} |z|^2.$$ 

This completes the proof of Theorem 3. Finally the result is sharp for the function

$$f(z) = z + \frac{(B - A)}{(1 + B) (1 + \lambda) b_2} e^{i\theta_2} z^2 \quad (11)$$

at $z = \pm |z| e^{-i\theta_2}$.

**Corollary 2.** Under the hypotheses of Theorem 3, $f(z)$ is included in a disc with center at the origin and radius $r_1$ given by

$$r_1 = 1 + \frac{(B - A)}{(1 + B) (1 + \lambda) b_2}.$$

**Theorem 4.** Let the function $f(z)$ defined by (1) belong to the class $V(g, \lambda, A, B)$. Then

$$1 - \frac{2(B - A)}{(1 + B) (1 + \lambda) b_2} |z| \leq |f'(z)| \leq 1 + \frac{2(B - A)}{(1 + B) (1 + \lambda) b_2} |z|. \quad (12)$$
provided $b_n \geq b_2$ ($n \geq 2$). The result is sharp for the function $f(z)$ given by (11) at $z = \pm |z| e^{-i\theta_2}$

**Proof.** Since \( \{n\Phi(n)\} \), where \( \Phi(n) \) given by (10) is increasing function of \( n \) ($n \geq 2$), in view of Theorem 1, we have

\[
\frac{(1 + B)(1 + \lambda)b_2}{2} \sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} [1 + \lambda(n - 1)] (1 + B) b_n |a_n| \leq (B - A),
\]

that is

\[
\sum_{n=2}^{\infty} n |a_n| \leq \frac{2(B - A)}{(1 + B)(1 + \lambda)b_2}.
\]

Thus

\[
|f'(z)| = |1 + \sum_{n=2}^{\infty} n a_n z^{n-1}| \leq 1 + |z| \sum_{n=2}^{\infty} n |a_n|
\]

\[
\leq 1 + \frac{2(B - A)}{(1 + B)(1 + \lambda)b_2} |z|.
\]

Similarly, we get

\[
|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \geq 1 - |z| \sum_{n=2}^{\infty} n |a_n|
\]

\[
\geq 1 - \frac{2(B - A)}{(1 + B)(1 + \lambda)b_2} |z|.
\]

Finally the result is sharp for the function $f(z)$ given by (11). This completes the proof of Theorem 4.

**Corollary 3.** Let the function $f(z)$ defined by (1) be in the class $V(g, \lambda, A, B)$. Then $f'(z)$ is included in a disc with center at the origin and radius $r_2$ given by

\[
r_2 = 1 + \frac{2(B - A)}{(1 + B)(1 + \lambda)b_2}.
\]

### 4 Extreme points

**Theorem 5.** Let the function $f(z)$ defined by (1) be in the class $V(g, \lambda, A, B)$, with $\arg(a_n) = \theta_n$ where $[\theta_n + (n - 1)\beta] \equiv \pi \mod 2\pi$. Define $f_1(z) = z$ and $f_n(z) = z + \frac{(B - A)}{[1 + \lambda(n - 1)](1 + B)b_n} e^{i\theta_n} z^n$. 


Then \( f(z) \) is in the class \( V(g, \lambda, A, B) \) if and only if it can be expressed in the form
\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),
\]
where \( \mu_n \geq 0 \) \( (n \geq 1) \) and \( \sum_{n=1}^{\infty} \mu_n = 1 \).

**Proof.** If \( f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \) with \( \sum_{n=1}^{\infty} \mu_n = 1 \) and \( \mu_n \geq 0 \), then
\[
\sum_{n=1}^{\infty} [1 + \lambda(n - 1)] (1 + B) b_n \frac{(B - A)}{[1 + \lambda(n - 1)] (1 + B) b_n} \mu_n
= \sum_{n=2}^{\infty} (B - A) \mu_n = (B - A) (1 - \mu_1) \leq (B - A).
\]
So, by Theorem 1, we have \( f(z) \in V(g, \lambda, A, B) \).
Conversely, let the function \( f(z) \) defined by (1.1) belongs to the class \( V(g, \lambda, A, B) \). Then \( a_n \) are given by (7). Setting
\[
\mu_n = \frac{[1 + \lambda(n - 1)] (1 + B) b_n}{(B - A)} |a_n|,
\]
and
\[
\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.
\]
From Theorem 1, \( \sum_{n=1}^{\infty} \mu_n \leq 1 \) and so \( \mu_n \geq 0 \). Since \( \mu_n f_n(z) = \mu_n z + a_n z^n \), then
\[
\sum_{n=1}^{\infty} \mu_n f_n(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z).
\]
This completes the proof of Theorem 5.

**Remark.** Specializing \( g(z), \lambda, A \) and \( B \), in the above results, we obtain the corresponding results for the corresponding classes \( V(f, g, \lambda, \alpha), V_k(\lambda, A, B) \), \( V(g, A, B) \), \( VC(g, A, B) \), \( V(\lambda, A, B), V_{q,s}(\alpha_1; \lambda, A, B) \) and \( V(\gamma, l, m; \lambda, A, B) \) defined in the introduction.

### 5 Open problems

(i) We can generalize this study when the functions \( f(z) \) and \( g(z) \) are \( p \)-valent functions.
(ii) For meromorphic functions $f_1, f_2 \in \mathbb{C}$, where $f_1(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n$ and $f_2(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n$, the Hadamard product (or convolution) is given by
\[(f_1 \ast f_2)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n = (f_2 \ast f_1)(z).\]

We suggest to study the class
\[\sum V(f_2, \lambda, A, B) = (1 + \lambda) z (f_1 \ast f_2)(z) + \lambda z^2 (f_1 \ast f_2)'(z) < \frac{1 + A^2}{1 + B^2},\]
\[(\lambda \geq 0; -1 \leq A < B \leq 1, 0 < B \leq 1).\]

References


