*Int. J. Open Problems Complex Analysis, Vol. 5, No. 2, July 2013*
*ISSN 2074-2827;* 
*www.i-csrs.org*

# On Certain Subclass of p-Valent Functions with Negative Coefficients Defined by Convolution

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**Abstract**

In this paper we introduce and study new class of analytic $p$-valent functions with negative coefficients defined by convolution. We obtain coefficients inequalities, distortion theorems, extreme points and radii of close to convexity, starlikeness and convexity for the class $\mathcal{T}_p^*(f, g; A, B; m, q)$. Also we investigate several application involving an integral operator. Finally, we obtain integral means for functions belonging to this class.

**Keywords and Phrases:** Analytic and $p$-valent functions, negative coefficients.

**2000 Mathematical Subject Classification:** 30C45, 30A20, 34A40.

## 1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

that are analytic and $p-$valent in the open unit disc $U = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Let $g(z) \in \mathcal{A}(p)$, be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k. \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be $p$-valent starlike of order $\alpha$, denoted by $\mathcal{S}_p^*(\alpha)$, if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (0 \leq \alpha < p;\ z \in U). \tag{1.4}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be $p$-valent convex of order $\alpha$, denoted by $\mathcal{K}_p(\alpha)$, if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ (0 \leq \alpha < p;\ z \in U). \tag{1.5}$$

From (1.4) and (1.5) it follows that

$$f(z) \in \mathcal{K}_p(\alpha) \text{ if and only if } \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha). \tag{1.6}$$

The classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were introduced and studied by Owa [14].

For two functions $f$ and $g$, analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$, and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that $f(z) \prec g(z) \ \Rightarrow \ f(0) = g(0)$ and $f(U) \subset g(U)$. In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [13]).

For each $f(z) \in \mathcal{A}(p)$, we have

$$f^{(q)}(z) = \delta(p,q) z^{p-q} + \sum_{k=p+1}^{\infty} \delta(k,q) a_k z^{k-q}, \tag{1.7}$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)...........(p-q+1) & (q \neq 0), \\ 1 & (q = 0). \end{cases} \tag{1.8}$$

For $-1 \leq B < A \leq 1$, $-1 \leq B < 0$, $m, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, $p > q + m$ and $g(z)$ is given by (1.2) with $b_k \geq 0$ $(k \geq p+1)$, we let $\mathcal{A}_p(f, g; A, B; m, q)$ be the subclass of $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1), the functions $g(z)$ of the form (1.2) and satisfying the analytic criterion:

$$\frac{z^m (f * g)^{(q+m)}(z)}{(f * g)^{(q)}(z)} \prec \delta(p-q, m) \frac{1 + Az}{1 + Bz} \ (z \in U). \tag{1.9}$$

In other words, $f(z) \in \mathcal{A}_p(f, g; A, B; m, q)$ if and only if there exists function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that

$$\left| \frac{\dfrac{z^m (f * g)^{(q+m)}(z)}{(f * g)^{(q)}(z)} - \delta(p-q, m)}{B\left[\dfrac{z^m (f * g)^{(q+m)}(z)}{(f * g)^{(q)}(z)}\right] - A\,\delta(p-q, m)} \right| < 1. \tag{1.10}$$

Also denote by $\mathcal{T}(p)$, the subclass of $\mathcal{A}(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \ (a_k \geq 0;\ p \in \mathbb{N}). \tag{1.11}$$

Further, we define the class

$$\mathcal{T}_p^*(f, g; A, B; m, q) = \mathcal{A}_p(f, g; A, B; m, q) \cap \mathcal{T}(p). \tag{1.12}$$

We note that for suitable choices of $g(z)$, $m$, $q$, $A$, $B$ and $p$, we obtain the following subclasses:

(1) $\mathcal{T}_p^*(f, \frac{z^p}{1-z}; A, B; 1, 0) = T_p^*(A, B, 0)$ $(-1 \leq A < B \leq 1$ and $0 < B \leq 1)$ (see Aouf [1, with $\alpha = 0$]);

(2) $\mathcal{T}_p^*\left(f, \frac{z^p}{1-z}; \beta\left[1 - \frac{2\alpha}{(p-q)}\right], -\beta; 1, q\right) = S_1(p, q, \alpha, \beta)$ $(0 \leq \alpha < (p-q)$, $0 < \beta \leq 1$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $p > q)$ (see Aouf [2, with $n = 1$]);

(3) $\mathcal{T}_p^*\left(f, \frac{z^p}{1-z}; A, B; 1, j-1\right) = T^*(A, B, 0, p, j)$ $(-1 \leq A < B \leq 1$, $0 < B \leq 1$, $1 \leq j \leq p$ and $p \in \mathbb{N})$ (see Aouf [3, with $\alpha = 0$]);

(4) $\mathcal{T}_p^*\left(f, g; \left[1 - \frac{2\alpha}{(p-q)}\right], -1; 1, q\right) = TS_g^*(p, q, 1, \alpha)$ $(0 \leq \alpha < (p-q)$, $q \in \mathbb{N}_0$, $p > q$ and $p \in \mathbb{N})$ (see Aouf and Mostafa [4, with $n = 1$]);

(5) $\mathcal{T}_p^*\left(f, \dfrac{z^p}{1-z}; \gamma\left[1-\dfrac{2\alpha}{\delta(p-q,m)}\right], -\gamma; m, q\right) = \mathcal{A}_{1,p}^*(m,q,\alpha,\gamma)$ $(0 \le \alpha < \delta(p-q,m),\ 0 < \gamma \le 1,\ p \in \mathbb{N},\ m,q \in \mathbb{N}_0$ and $m+q<p)$ (see Liu and Liu [12, with $n=1$]);
(6) $\mathcal{T}_p^*(f, \dfrac{z^p}{1-z}; \left[1-\dfrac{2\alpha}{(p-q)}\right], -1; 1, q) = \mathcal{S}_1(p,q,\alpha)$ $(0 \le \alpha < p-q,\ q \in \mathbb{N}_0,\ p \in \mathbb{N}$ and $p > q)$ (see Chen et al. [6, with $n=1$]);
(7) $\mathcal{T}_p^*(f, \dfrac{z^p}{1-z}; \left[1-\dfrac{2\alpha}{(p-q)}\right], -1; 1, q) = SC_p^1(q,0,\alpha)$ $(0 \le \alpha < p-q,\ q \in \mathbb{N}_0,\ p \in \mathbb{N}$ and $p > q)$ (see Irmak et al. [10, with $n=1$ and $\lambda=0$]);
(8) $\mathcal{T}_1^*(f, \dfrac{z}{1-z}; (1-2\alpha)\beta, -\beta; 1, 0) = S^*(\alpha,\beta)$ $(0 \le \alpha < 1$ and $0 \le \beta < 1)$ (see Gupta and Jain [9]).

Also, we note that:
(1) $\mathcal{T}_p^*(f, z^p + \sum\limits_{k=p+1}^{\infty} \Gamma_{k,p}(\alpha_1) z^k; A, B; m, q) = \mathcal{T}_p^*(\alpha_1, \beta_1; A, B; m, q)$

$$= \left\{ f \in \mathcal{T}(p) : \frac{z^m (H_{p,\ell,s}(\alpha_1) f(z))^{(q+m)}}{(H_{p,\ell,s}(\alpha_1) f(z))^{(q)}} \prec \delta(p-q,m)\frac{1+Az}{1+Bz}\ (\ell \le s+1;\ \ell, s \in \mathbb{N}_0;\ -1 \le B < A \le 1;\ -1 \le B < 0;\ z \in U) \right\},$$

where the operator

$$H_{p,\ell,s}(\alpha_1)(z) = z^p + \sum_{k=p+1}^{\infty} \Gamma_{k,p}(\alpha_1) z^k,$$

$$\Gamma_{k,p}(\alpha_1) = \frac{(\alpha_1)_{k-p} \ldots\ldots\ldots (\alpha_\ell)_{k-p}}{(\beta_1)_{k-p} \ldots\ldots\ldots (\beta_s)_{k-p}} \frac{1}{(k-p)!}\ , \tag{1.13}$$

$\alpha_1, \ldots.., \alpha_\ell$ and $\beta_1, \ldots.., \beta_s$ are real parameters, $\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $j = 1, \ldots.s$, was introduced and studied by Dziok and Srivastava [7];
(2) $\mathcal{T}_p^*(f, z^p + \sum\limits_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n z^k; A, B; m, q) = \mathcal{T}_p^*(\lambda, \ell; A, B; m, q)$

$$= \left\{ f \in \mathcal{T}(p) : \frac{z^m (I_p^n(\lambda,\ell) f(z))^{(q+m)}}{(I_p^n(\lambda,\ell) f(z))^{(q)}} \prec \delta(p-q,m)\frac{1+Az}{1+Bz}\ (\lambda \ge 0;\ \ell \ge 0;\ n \in \mathbb{Z} = \{0, \pm1, \pm2, \ldots\};\ -1 \le B < A \le 1;\ -1 \le B < 0;\ p \in \mathbb{N};\ z \in U) \right\},$$

where the operator

$$I_p^n(\lambda,\ell)(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n z^k, \tag{1.14}$$

was introduced and studied by Prajapat [15], (see also, El-Ashwah and Aouf [8] and Catas [5]).

## 2. Coefficients estimates

Unless otherwise mentioned, we assume throughout this paper that $-1 \le B < A \le 1,\ -1 \le B < 0,\ m, q \in \mathbb{N}_0,\ p \in \mathbb{N},\ p > q+m,\ k \ge p+1$ and the function $g(z)$ is given by (1.2) with $b_k > 0$.

**Theorem 1.** A function $f(z)$ of the form (1.1) is in the class $\mathcal{A}_p(f, g; A, B; m, q)$ if

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)\,|a_k|\,b_k \le (A-B)\delta(p-q,m)\delta(p,q). \tag{2.1}$$

**Proof**. Assume that the inequality (2.1) holds true, then for $z \in U$, we have

$$\left| \frac{\dfrac{z^m (f*g)^{(q+m)}(z)}{(f*g)^{(q)}(z)} - \delta(p-q,m)}{B\left[\dfrac{z^m (f*g)^{(q+m)}(z)}{(f*g)^{(q)}(z)}\right] - A\ \delta(p-q,m)} \right|$$

$$= \left| \frac{\sum_{k=p+1}^{\infty} [\delta(k-q,m)-\delta(p-q,m)]\delta(k,q)a_k b_k z^{k-q}}{\sum_{k=p+1}^{\infty} [B\ \delta(k-q,m)-A\ \delta(p-q,m)]\delta(k,q)a_k b_k z^{k-q} - (A-B)\delta(p-q,m)\delta(p,q)z^{p-q}} \right|.$$

This last expression is bounded above by 1 if

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)\,|a_k|\,b_k \le (A-B)\delta(p-q,m)\delta(p,q).$$

Hence $f(z) \in \mathcal{A}_p(f, g; A, B; m, q)$. This completes the proof of Theorem 1.

**Theorem 2.** A function $f(z)$ of the form (1.11) is in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$ if and only if

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)a_k b_k \le (A-B)\delta(p-q,m)\delta(p,q). \tag{2.2}$$

**Proof**. In view of Theorem 1, we need only to prove the necessity. If $f(z) \in T_p^*(f, g; A, B; m, q)$, then

$$\left| \frac{\dfrac{z^m (f*g)^{(q+m)}(z)}{(f*g)^{(q)}(z)} - \delta(p-q,m)}{B\left[\dfrac{z^m (f*g)^{(q+m)}(z)}{(f*g)^{(q)}(z)}\right] - A\ \delta(p-q,m)} \right|$$

$$= \left| \frac{\sum\limits_{k=p+1}^{\infty} [\delta(k-q,m)-\delta(p-q,m)]\delta(k,q)a_k b_k z^{k-q}}{\sum\limits_{k=p+1}^{\infty} [B\ \ \delta(k-q,m)-A\ \ \delta(p-q,m)]\delta(k,q)a_k b_k z^{k-q}+(A-B)\delta(p-q,m)\delta(p,q)z^{p-q}} \right| < 1.$$

Since $Re\{z\} \le |z|$ $(z \in U)$, we thus find that

$$\mathrm{Re}\left\{ \frac{\sum\limits_{k=p+1}^{\infty} [\delta(k-q,m)-\delta(p-q,m)]\delta(k,q)a_k b_k z^{k-q}}{\sum\limits_{k=p+1}^{\infty} [B\ \ \delta(k-q,m)-A\ \ \delta(p-q,m)]\delta(k,q)a_k b_k z^{k-q}+(A-B)\delta(p-q,m)\delta(p,q)z^{p-q}} \right\} < 1.$$

Letting $z \longrightarrow 1^-$ along the real axis, we have the desired inequality

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)a_k b_k \le (A-B)\delta(p-q,m)\delta(p,q).$$

This completes the proof of Theorem 2.

**Corollary 1.** Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$. Then

$$a_k \le \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}. \tag{2.3}$$

The result is sharp for the function

$$f(z) = z^p - \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k} z^k. \tag{2.4}$$

# 3. Distortion theorems

**Theorem 3.** Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$. Then for $|z| = r < 1$, we have

$$|f(z)| \ge r^p - \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} r^{p+1}, \tag{3.1}$$

and

$$|f(z)| \le r^p + \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} r^{p+1}, \tag{3.2}$$

provided that $b_k \geq b_{p+1}$ $(k \geq p+1)$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} z^{p+1}. \tag{3.3}$$

**Proof.** Since

$$\begin{aligned} &[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1} \\ &\qquad \leq [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k. \end{aligned}$$

Then using Theorem 2, we have

$$[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1} \sum_{k=p+1}^{\infty} a_k$$

$$\begin{aligned} &\leq \sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)a_k b_k \\ &\leq (A-B)\delta(p-q,m)\delta(p,q), \end{aligned}$$

that is,

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}}. \tag{3.4}$$

From (1.11) and (3.4), we have

$$\begin{aligned} |f(z)| &\geq r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\geq r^p - \tfrac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} r^{p+1}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |f(z)| &\leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\leq r^p + \tfrac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} r^{p+1}. \end{aligned} \tag{3.6}$$

Since each of equalities in (3.1) and (3.2) is satisfied by the function $f(z)$ given by (3.3), our proof of Theorem 3 is thus completed.

**Theorem 4**. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$. Then for $|z|=r<1$, we have

$$|f'(z)| \geq pr^{p-1} - \frac{(p+1)(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} r^p, \tag{3.7}$$

and

$$|f'(z)| \leq pr^{p-1} + \frac{(p+1)(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} r^p, \tag{3.8}$$

provided that $b_k \geq b_{p+1} (k \geq p+1)$. The result is sharp for the function $f(z)$ given by (3.3).

**Proof**. From Theorem 2 and (3.4), we have

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(p+1)(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}}.$$

Since the remaining part of the proof is similar to the proof of Theorem 3, we omit the details.

# 4. Convex linear combinations

**Theorem 5.** Let $\mu_\upsilon \geq 0$ for $\upsilon = 1,2,........,\ell$ and $\sum_{\upsilon=1}^{\ell} \mu_\upsilon \leq 1$. If the functions $f_\upsilon(z)$ defined by

$$f_\upsilon(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\upsilon} z^k \ (a_{k,\upsilon} \geq 0;\ \upsilon = 1,2,...,\ell), \tag{4.1}$$

are in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$ for every $\upsilon = 1,2,...,\ell$, then the function $f(z)$ defined by

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \left( \sum_{\upsilon=1}^{\ell} \mu_\upsilon a_{k,\upsilon} \right) z^k, \tag{4.2}$$

is also in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$.

**Proof.** Since $f_\upsilon(z)$ are in the class $T_p^*(f,g;A,B;m,q)$, it follows from Theorem 2 that

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q) a_{k,\upsilon} b_k \leq (A-B)\delta(p-q,m)\delta(p,q),$$

for every $\upsilon = 1, 2, ........, \ell$. Hence

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m) - (1-A)\delta(p-q,m)]\delta(k,q)\left(\sum_{\upsilon=1}^{\ell} \mu_\upsilon a_{k,\upsilon}\right) b_k$$

$$\begin{aligned} &= \sum_{\upsilon=1}^{\ell} \mu_\upsilon \left(\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m) - (1-A)\delta(p-q,m)]\delta(k,q) a_{k,\upsilon} b_k\right) \\ &\leq (A-B)\delta(p-q,m)\delta(p,q)\sum_{\upsilon=1}^{\ell}\mu_\upsilon \leq (A-B)\delta(p-q,m)\delta(p,q). \end{aligned}$$

From Theorem 2, it follows that $f(z) \in \mathcal{T}_p^*(f, g; A, B; m, q)$. This completes the proof of Theorem 5.

**Corollary 2**. The class $\mathcal{T}_p^*(f, g; A, B; m, q)$ is closed under convex linear combinations.

**Proof.** Let the functions $f_\upsilon(z)$ $(\upsilon = 1, 2)$ defined by (4.1) be in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$. Then it is sufficient to show that the function

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \ (0 \leq \lambda \leq 1),$$

is in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$. Since for $0 \leq \lambda \leq 1$,

$$h(z) = z^p - \sum_{k=p+1}^{\infty} [\lambda a_{k,1} + (1-\lambda) a_{k,2}] z^k, \tag{4.3}$$

with the aid of Theorem 2, we have

$$\sum_{k=p+1}^{\infty} [(1-B)\delta(k-q,m) - (1-A)\delta(p-q,m)]\delta(k,q)[\lambda a_{k,1} + (1-\lambda) a_{k,2}] b_k$$

$$\begin{aligned} &\leq \lambda(A-B)\delta(p-q,m)\delta(p,q) + (1-\lambda)(A-B)\delta(p-q,m)\delta(p,q) \\ &= (A-B)\delta(p-q,m)\delta(p,q), \end{aligned}$$

which implies that $h(z) \in \mathcal{T}_p^*(f, g; A, B; m, q)$. This completes the proof of Corollary 2.

**Theorem 6.** Let $f_p(z) = z^p$ and

$$f_k(z) = z^p - \frac{(A-B)\ \delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m) - (1-A)\delta(p-q,m)]\delta(k,q) b_k} z^k. \tag{4.4}$$

Then $f(z)$ is in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=p}^{\infty}\mu_k f_k(z), \tag{4.5}$$

where $\mu_k \geq 0$ and $\sum\limits_{k=p}^{\infty}\mu_k=1$.

**Proof**. Assume that

$$\begin{aligned} f(z) &= \sum_{k=p}^{\infty}\mu_k f_k(z) \\ &= z^p-\sum_{k=p+1}^{\infty}\frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}\mu_k z^k. \end{aligned} \tag{4.6}$$

Then it follows that

$$\begin{aligned} &\sum_{k=p+1}^{\infty}\frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)}\cdot \\ &\qquad\cdot\frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}\mu_k \\ &\qquad\leq \sum_{k=p+1}^{\infty}\mu_k=(1-\mu_p)\leq 1. \end{aligned}$$

Hence from Theorem 2 we have $f(z)\in\mathcal{T}_p^*(f,g;A,B;m,q)$.

Conversely, assume that the function $f(z)$ defined by (1.11) belongs to the class $\mathcal{T}_p^*(f,g;A,B;m,q)$. Then

$$a_k\leq\frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}.$$

Setting

$$\mu_k=\frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)a_k b_k}{(A-B)\delta(p-q,m)\delta(p,q)},$$

where

$$\mu_p=1-\sum_{k=p+1}^{\infty}\mu_k\ .$$

We can see that $f(z)$ can be expressed in the form (4.5). This completes the proof of Theorem 6.

**Corollary 3**. The extreme points of the class $\mathcal{T}_p^*(f,g;A,B;m,q)$ are the functions $f_p(z)=z^p$ and

$$f_k(z)=z^p-\frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}z^k. \tag{4.7}$$

# 5. Radii of close-to-convexity, starlikeness and convexity

**Theorem 7.** Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$. Then $f(z)$ is $p$-valent close-to-convex of order $\delta$ $(0\le\delta<p)$ *in* $|z|\le r_1$, where

$$r_1=\inf_{k\ge p+1}\left\{\frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)}\left(\frac{p-\delta}{k}\right)\right\}^{\frac{1}{k-p}}. \tag{5.1}$$

The result is sharp and the extremal function is given by (2.4).
**Proof.** We must show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\delta \;\; for \;\; |z|\le r_1 \;, \tag{5.2}$$

where $r_1$ is given by (5.1). Indeed we find from (1.11) that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le\sum_{k=p+1}^{\infty}ka_k\,|z|^{k-p}\,.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\delta,$$

if

$$\sum_{k=p+1}^{\infty}\left(\frac{k}{p-\delta}\right)a_k\,|z|^{k-p}\le 1. \tag{5.3}$$

But by using Theorem 2, (5.3) will be true if

$$\left(\frac{k}{p-\delta}\right)|z|^{k-p}\le\frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)}.$$

Then

$$|z|\le\left\{\frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)}\left(\frac{p-\delta}{k}\right)\right\}^{\frac{1}{k-p}}.$$

This completes the proof of Theorem 7.

**Theorem 8.** Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$. Then $f(z)$ is $p$-valent starlike of order $\delta$ $(0 \leq \delta < p)$ in $|z| \leq r_2$, where

$$r_2 = \inf_{k \geq p+1} \left\{ \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \left(\frac{p-\delta}{k-\delta}\right) \right\}^{\frac{1}{k-p}}. \tag{5.4}$$

The result is sharp and the extremal function is given by (2.4).

**Proof.** We must show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \ for \ \ |z| \leq r_2, \tag{5.5}$$

where $r_2$ is given by (5.4). Indeed we find from (1.11) that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq \frac{\sum\limits_{k=p+1}^{\infty} (k-p)a_k \left|z\right|^{k-p}}{1 - \sum\limits_{k=p+1}^{\infty} a_k \left|z\right|^{k-p}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta,$$

if

$$\sum_{k=p+1}^{\infty} \left(\frac{k-\delta}{p-\delta}\right) a_k \left|z\right|^{k-p} \leq 1. \tag{5.6}$$

But by using Theorem 2, (5.6) will be true if

$$\left(\frac{k-\delta}{p-\delta}\right) |z|^{k-p} \leq \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)}.$$

Then

$$|z| \leq \left\{ \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \left(\frac{p-\delta}{k-\delta}\right) \right\}^{\frac{1}{k-p}}.$$

This completes the proof of Theorem 8.

**Corollary 4.** Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f,g;A,B;m,q)$. Then $f(z)$ is $p$-valent convex of order $\delta$ $(0 \leq \delta < p)$ in $|z| \leq r_3$, where

$$r_3 = \inf_{k \geq p+1} \left\{ \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \left(\frac{p(p-\delta)}{k(k-\delta)}\right) \right\}^{\frac{1}{k-p}}. \tag{5.7}$$

The result is sharp and the extremal function is given by (2.4).

# 6. Integral operators

In view of Theorem 2, we see that $z^p - \sum_{k=p+1}^{\infty} d_k z^k$ is in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$ as long as $0 \leq d_k \leq a_k$ for all $k$ $(k \geq p+1)$. In particular, we have

**Theorem 9.** Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$ and $c$ be a real number such that $c > -p$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \ (c > -p), \tag{6.1}$$

also belongs to the class $\mathcal{T}_p^*(f, g; A, B; m, q)$.

**Proof.** From the representation (6.1) of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=p+1}^{\infty} d_k z^k,$$

where

$$d_k = \left(\frac{c+p}{k+c}\right) a_k \leq a_k.$$

This leads to $F(z) \in \mathcal{T}_p^*(f, g; A, B; m, q)$. This completes the proof of Theorem 9.

Putting $c = 1 - p$ in Theorem 9, we get the following corollary.

**Corollary 5**. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$ and let $F(z)$ be defined by

$$F(z) = \frac{1}{z^{1-p}} \int_0^z \frac{f(t)}{t^p} dt. \tag{6.2}$$

Then $F(z) \in \mathcal{T}_p^*(f, g; A, B; m, q)$.

The converse of Theorem 9 is not true. This leads to a radius of $p-$valence result.

**Theorem 10.** Let the function

$$F(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \ (a_k \geq 0), \tag{6.3}$$

be in the class $\mathcal{T}_p^*(f, g; A, B; m, q)$ and let $c$ be a real number such that $c > -p$. Then the function $f(z)$ given by

$$f(z) = \frac{z^{1-c}[z^c\ F(z)]'}{p+c} = z^p - \sum_{k=p+1}^{\infty} \left(\frac{k+c}{p+c}\right) a_k z^k \ (c > -p), \tag{6.4}$$

is $p$-valent in $|z| < R_p^*$, where

$$R_p^* = \inf_{k \geq p+1} \left\{ \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \left(\frac{p(p+c)}{k(k+c)}\right) \right\}^{\frac{1}{k-p}}. \tag{6.5}$$

The result is sharp.

**Proof.** In order to obtain the required result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \ for \ \ |z| < R_p^* \ ,$$

where $R_p^*$ is given by (6.5). Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} \frac{k(k+c)}{(p+c)} a_k \left|z\right|^{k-p}.$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p,$$

if

$$\sum_{k=p+1}^{\infty} \frac{k(k+c)}{p(p+c)} a_k \left|z\right|^{k-p} \leq 1. \tag{6.6}$$

But Theorem 2 confirms that

$$\sum_{k=p+1}^{\infty} \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)a_k b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \leq 1.$$

Hence (6.6) will be satisfied if

$$\frac{k(k+c)}{p(p+c)} \left|z\right|^{k-p} \leq \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)}.$$

Then

$$|z| \leq \left\{ \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \left(\frac{p(p+c)}{k(k+c)}\right) \right\}^{\frac{1}{k-p}}. \tag{6.7}$$

The required result follows now from (6.7). The result is sharp for the function

$$f(z) = z^p - \frac{(k+c)(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)(p+c)b_k} z^k. \tag{6.8}$$

# 7. Integral means

In this section integral means for functions belonging to the class $\mathcal{T}_p^*(f,g;A,B;m,q)$ are obtained.

In 1925, Littlewood [11] proved the following lemma.

**Lemma 1**. If the functions $f$ and $g$ are analytic in $U$ with $g \prec f$, then for $\eta > 0$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta. \tag{7.1}$$

Applying Lemma 1, Theorem 2, and Corollary 3, we prove the following theorem.

**Theorem 11.** Suppose $f(z) \in \mathcal{T}_p^*(f,g;A,B;m,q)$, $\eta > 0$, $0 < r < 1$ and $f_{p+1}(z)$ is defined by

$$f_{p+1}(z) = z^p - \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} z^{p+1}. \tag{7.2}$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^{\eta} d\theta \le \int_0^{2\pi} |f_{p+1}(z)|^{\eta} d\theta \tag{7.3}$$

**Proof.** For $f(z)$ given by (1.11), (7.3) is equivalent to prove that

$$\int_0^{2\pi} \left| z^p - \sum_{k=p+1}^{\infty} a_k z^k \right|^{\eta} d\theta \le \int_0^{2\pi} \left| z^p - \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} z^{p+1} \right|^{\eta} d\theta.$$

By using Lemma 1, it suffices to show that

$$z^p - \sum_{k=p+1}^{\infty} a_k z^k \prec z^p - \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} z^{p+1}.$$

Setting

$$\sum_{k=p+1}^{\infty} a_k z^k = \frac{(A-B)\delta(p-q,m)\delta(p,q)}{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}} w^{p+1}(z).$$

By using Theorem 2, we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=p+1}^{\infty} \frac{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}}{(A-B)\delta(p-q,m)\delta(p,q)} a_k z^k \right|^{\frac{1}{p+1}} \\ &\leq |z| \sum_{k=p+1}^{\infty} \frac{[(1-B)\delta(p-q+1,m)-(1-A)\delta(p-q,m)]\delta(p+1,q)b_{p+1}}{(A-B)\delta(p-q,m)\delta(p,q)} a_k \\ &\leq |z| \sum_{k=p+1}^{\infty} \frac{[(1-B)\delta(k-q,m)-(1-A)\delta(p-q,m)]\delta(k,q)a_k b_k}{(A-B)\delta(p-q,m)\delta(p,q)} \\ &\leq |z|. \end{aligned}$$

This completes the proof of Theorem 11.

**Remarks.**
(1) Putting $g(z) = \dfrac{z^p}{1-z}$, $m = 1$ and $q = 0$ in our results, we obtain some analogous results obtained by Aouf [1, with $\alpha = 0$];
(2) Putting $g(z) = \dfrac{z^p}{1-z}$, $A = \beta \left[1 - \dfrac{2\alpha}{(p-q)}\right]$, $B = -\beta$ and $m = 1$ ($0 \leq \alpha < (p-q)$ and $0 < \beta \leq 1$) in our results, we obtain some analogous results obtained by Aouf [2, with $n = 1$];
(3) Putting $g(z) = \dfrac{z^p}{1-z}$, $m = 1$ and $q = j-1$ ($1 \leq j \leq p$) in our results, we obtain some analogous results obtained by Aouf [3, with $\alpha = 0$];
(4) Putting $A = \left[1 - \dfrac{2\alpha}{(p-q)}\right]$, $B = -1$ and $m = 1$ ($0 \leq \alpha < p-q$) in our results, we obtain some analogous results obtained by Aouf and Mostafa [4, with $n = 1$];
(5) Putting $g(z) = \dfrac{z^p}{1-z}$, $A = \gamma \left[1 - \dfrac{2\alpha}{\delta(p-q,m)}\right]$ and $B = -\gamma$ ($0 \leq \alpha < \delta(p-q,m)$ and $0 < \gamma \leq 1$) in our results, we obtain some analogous results obtained by Liu and Liu [12, with $n = 1$];
(6) Putting $g(z) = \dfrac{z^p}{1-z}$, $A = \left[1 - \dfrac{2\alpha}{(p-q)}\right]$, $B = -1$ and $m = 1$ ($0 \leq \alpha < p-q$) in our results , we obtain some analogous results obtained by Chen et al. [6, with $n = 1$];
(7) Putting $g(z) = \dfrac{z^p}{1-z}$, $A = \left[1 - \dfrac{2\alpha}{(p-q)}\right]$, $B = -1$ and $m = 1$ ($0 \leq \alpha < p-q$) in our results , we obtain some analogous results obtained by Irmak et al. [10, with $n = 1$ and $\lambda = 0$];
(8) Putting $g(z) = \dfrac{z}{1-z}$, $A = (1-2\alpha)\beta$, $B = -\beta$, $m = 1$, $q = 0$ and $p = 1$ ($0 \leq \alpha < 1$ and $0 \leq \beta < 1$) in our results, we obtain some analogous

results obtained by Gupta and Jain [9].

# Applications.

We can derive corresponding results by taking $g(z)$ in (1.9) as follows:
(1) $g(z) = z^p + \sum_{k=p+1}^{\infty} \Gamma_{k,p}(\alpha_1) z^k$ (see [7]), where $\Gamma_{k,p}(\alpha_1)$ is given by (1.13);
(2) $g(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n z^k$ (see [5], [8], [15]), where $\lambda \geq 0,\ \ell \geq 0,\ n \in \mathbb{Z}$.

## 8. Open Problems

The authors suggest to study :
(1) Partial sums for functions in the class $\mathcal{A}_p(f, g; A, B; m, q)$;
(2) The class $\mathcal{A}_p(f, g; A, B; m, q)$ can be redefined by using varying arguments to get new class. So, new results similar to or parallel to what obtained in this paper can be derived for the new class;
(3) For $f(z) \in \sum_p$ the class of $p$-valent meromorphic functions, construct the analogous class $\sum_p(f, g; A, B; m, q)$.

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