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The convexity order of an integral operator using some special classes of functions

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Abstract

The aim of this paper is to prove the convexity order for an integral operator, using functions from some special classes of functions such as $\mathcal{PH}(|\beta|)$ ***and*** $\mathcal{PV}(|\alpha|, |\beta|)$***.***

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1 Introduction and preliminaries

We consider the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

denoted by $\mathcal{A}$, which are analityc in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. By $\mathcal{S}$ we denote the class of all analytical functions that are univalent in $\mathcal{U}$. V. Pescar has defined some new classes of analytic functions in [1].

Definition 1.1. [1] The class of univalent functions $\mathcal{PH}(|\beta|)$, β be a complex number, $\beta \neq 0$, is the class defined by

$$\mathcal{PH}(|\beta|) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 2|\beta|(\sqrt{2}-1) \right| < Re\left[\sqrt{2}\frac{zf'(z)}{f(z)} + 2|\beta|(\sqrt{2}-1) \right] \right\} \tag{2}$$

for all $z \in \mathcal{U}$.

Definition 1.2. [1] The class of univalent functions $\mathcal{PV}(|\alpha|, |\beta|)$, α, β be complex numbers, $\alpha \neq 0$, $|\beta| \leq 1$, is the class defined by

$$\mathcal{PV}(|\alpha|, |\beta|) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - (|\alpha| + |\beta|) \right| \leq Re\frac{zf'(z)}{f(z)} + |\alpha| - |\beta| \right\} \quad (3)$$

for all $z \in \mathcal{U}$.

In this paper we consider an integral operator derived from the integral operator introduced by Ularu and Breaz in [2]. For the analytical functionsf_i and g_i and the complex numbers γ_i we define the integral operator

$$I(f,g)(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)} \right)^{|\gamma_i|} dt \quad (4)$$

for $i = \overline{1,n}$.
We will use functions from the classes $\mathcal{PH}(|\beta|)$ and $\mathcal{PH}(|\alpha|, |\beta|)$ defined by Pescar in [1] and we will find the convexity order for the operator defined by (4).

2 The convexity order

Theorem 2.1. *Let γ_i be complex numbers for $i = \overline{1,n}$. If the functions $f_i \in \mathcal{PH}(|\beta|)$, $g_i \in \mathcal{PH}(|\alpha|)$ and if*

$$0 < 1 - 2(\sqrt{2} - 1) \sum_{i=1}^n |\gamma_i|(|\beta| + |\alpha|) < 1,$$

then the integral operator $I(f,g)(z)$ defined by (4) is convex and has the convexity order $1 - 2(\sqrt{2}-1)\sum_{i=1}^n |\gamma_i|(|\beta| + |\alpha|)$, for $i = \overline{1,n}$.

Proof. Using the form of (4) and after some calculus we obtain that

$$\frac{zI''(f,g)(z)}{I'(f,g)(z)} = \sum_{i=1}^n |\gamma_i| \left(\frac{zf_i'(z)}{f_i(z)} - \frac{zg_i'(z)}{g_i(z)} \right), \quad (5)$$

for $i = \overline{1,n}$ and for all $z \in \mathcal{U}$.
From (5) and using the definition of the class $\mathcal{PH}(\beta)$ results that

$$\sqrt{2}\mathrm{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)} + 1 \right) = \sqrt{2} \sum_{i=1}^n |\gamma_i| \left[\mathrm{Re}\frac{zf_i'(z)}{f_i(z)} - \mathrm{Re}\frac{zg_i'(z)}{g_i(z)} \right] + \sqrt{2}. \quad (6)$$

Because $f_i \in \mathcal{PH}(|\beta|)$ and $g_i \in \mathcal{PH}(|\alpha|)$ we have

$$\begin{aligned}&\sqrt{2}\mathrm{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}+1\right) > \\ &> \sqrt{2}\sum_{i=1}^{n}\gamma_i\left[\left|\frac{zf_i'(z)}{f_i(z)}-2|\beta|(\sqrt{2}-1)\right|-2|\beta|(\sqrt{2}-1)\right]- \\ &-\sqrt{2}\sum_{i=1}^{n}\gamma_i\left[\left|\frac{zg_i'(z)}{g_i(z)}+2|\alpha|(\sqrt{2}-1)\right|-2|\alpha|(\sqrt{2}-1)\right]+\sqrt{2}.\end{aligned}$$

Since $\left|\frac{zf_i'(z)}{f_i(z)}-2|\beta|(\sqrt{2}-1)\right|>0$ and $\left|\frac{zg_i'(z)}{g_i(z)}-2|\alpha|(\sqrt{2}-1)\right|>0$ results that

$$\sqrt{2}\mathrm{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}+1\right) > \sqrt{2}\sum_{i=1}^{n}|\gamma_i|[-2|\beta|(\sqrt{2}-1)-2|\alpha|(\sqrt{2}-1)]+\sqrt{2}.$$

From the above relation we obtain that

$$\mathrm{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right) > 1-2(\sqrt{2}-1)\sum_{i=1}^{n}|\gamma_i|(|\beta|+|\alpha|).$$

So we obtain that the operator $I(f,g)(z)$ is convex by order $1-2(\sqrt{2}-1)\sum\limits_{i=1}^{n}|\gamma_i|(|\beta|+|\alpha|)$. □

For $n=1$ in Theorem 2.1 we obtain:

Corollary 2.1. *We consider the complex number γ. If the functions $f \in \mathcal{PH}(|\beta|)$ and $g \in \mathcal{PH}(|\alpha|)$ and if*

$$0 < 1-2(\sqrt{2}-1)|\gamma|(|\beta|+|\alpha|) < 1,$$

then the integral operator $I(f,g)(z) = \int\limits_0^z \left(\frac{f(t)}{g(t)}\right)^{|\gamma|} dt$ is convex by order $1-2(\sqrt{2}-1)|\gamma|(|\beta|+|\alpha|)$.

If we consider $\alpha=\beta$ in Theorem 2.1 we obtain:

Corollary 2.2. *For $i=\overline{1,n}$ we consider the complex numbers γ_i. If the functions $f_i \in \mathcal{PH}(|\beta|)$ and $g_i \in \mathcal{PH}(|\beta|)$ and if*

$$0 < 1-4|\beta|(\sqrt{2}-1)\sum_{i=1}^{n}|\gamma_i| < 1,$$

then the integral operator $I(f,g)(z)$ defined by (4) is convex and has the convexity order $1-4|\beta|(\sqrt{2}-1)\sum\limits_{i=1}^{n}|\gamma_i|$, for $i=\overline{1,n}$.

Theorem 2.2. *Let γ_i be complex numbers for $i = \overline{1,n}$. If the functions $f_i \in \mathcal{PV}(|\alpha|, |\beta|)$, $g_i \in \mathcal{PV}(|\alpha_1|, |\beta_1|)$ and if*

$$0 < 1 - \sum_{i=1}^{n} |\gamma_i|(|\alpha| - |\beta| + |\alpha_1| - |\beta_1|) < 1,$$

then the integral operator $I(f,g)(z)$ defined by (4) is convex of order $1 - \sum_{i=1}^{n} |\gamma_i|(|\alpha| - |\beta| + |\alpha_1| - |\beta_1|)$, for $i = \overline{1,n}$.

Proof. From (4) and because the functions $f_i(z) \in \mathcal{PV}(|\alpha|, |\beta|)$ and $g_i(z) \in \mathcal{PV}(|\alpha_1|, |\beta_1|)$, for $i = \overline{1,n}$ and for all $z \in \mathcal{U}$ we obtain:

$$\operatorname{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)} + 1\right) >$$

$$> \sum_{i=1}^{n} |\gamma_i| \left[\left|\frac{zf_i'(z)}{f_i(z)} - (|\alpha| + |\beta|)\right| - |\alpha| + |\beta| - \left|\frac{zg_i'(z)}{g_i(z)} - (|\alpha_1| + |\beta_1|)\right| - |\alpha_1| + |\beta_1|\right] + 1.$$

Because $\left|\frac{zf_i'(z)}{f_i(z)} - (|\alpha| + |\beta|)\right| > 0$ and $\left|\frac{zg_i'(z)}{g_i(z)} - (|\alpha_1| + |\beta_1|)\right| > 0$ we obtain that

$$\operatorname{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)} + 1\right) > 1 - \sum_{i=1}^{n} |\gamma_i|[|\alpha| - |\beta| + |\alpha_1| - |\beta_1|].$$

From the above relation and from hypothesis results that the operator is convex by order $1 - \sum_{i=1}^{n} |\gamma_i|[|\alpha| - |\beta| + |\alpha_1| - |\beta_1|]$.m4 □

For $n = 1$ in Theorem 2.2 we obtain:

Corollary 2.3. *We consider the complex number γ. If the functions $f \in \mathcal{PV}(|\alpha|, |\beta|)$ and $g \in \mathcal{PV}(|\alpha_1|, |\beta_1|)$ and if*

$$0 < 1 - |\gamma|(|\alpha| - |\beta| + |\alpha_1| - |\beta_1|) < 1,$$

then the integral operator $I(f,g)(z) = \int_0^z \left(\frac{f(t)}{g(t)}\right)^{\gamma} dt$ is convex of order $1 - |\gamma|(|\alpha| - |\beta| + |\alpha_1| - |\beta_1|)$.

For $\alpha = \alpha_1$ and $\beta = \beta_1$ in Theorem 2.2 we obtain:

Corollary 2.4. *For $i = \overline{1,n}$ we consider the complex numbers γ_i. If the functions $f_i \in \mathcal{PV}(|\alpha|, |\beta|)$ and $g_i \in \mathcal{PV}(|\alpha|, |\beta|)$ and if*

$$0 < 1 - \sum_{i=1}^{n} |\gamma_i|(2|\alpha| - 2|\beta|) < 1,$$

then the integral operator $I(f,g)(z)$ defined by (4) is convex of order $1 - \sum_{i=1}^{n} |\gamma_i|(2|\alpha| - 2|\beta|)$, for $i = \overline{1,n}$.

3 Open Problem

First open problem is to prove the convexity order for the operator $I(f, g)(z)$ defined before by (4), for functions that are in other special classes of functions. The second open problem is to prove the starlikeness for this operator and that the operator belongs to some other special classes of functions defined in this area.

References

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