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A note on special strong differential superordinations using a multiplier transformation and Ruscheweyh derivative

Alina Alb Lupaş

Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
e-mail: dalb@uoradea.ro

Abstract

In the present paper we establish several strong differential superordinations regarding the extended new operator $RI^{\alpha}_{m,\lambda,l}$ ***defined by using the extended Ruscheweyh derivative and the extended multiplier transformation,*** $RI^{\alpha}_{m,\lambda,l} : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$, $RI^{\alpha}_{m,\lambda,l} f(z,\zeta) = (1-\alpha)R^m f(z,\zeta) + \alpha I(m,\lambda,l) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, ***where*** $R^m f(z,\zeta)$ ***denote the extended Ruscheweyh derivative,*** $I(m,\lambda,l)$ ***is the extended multiplier transformation and*** $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$ ***is the class of normalized analytic functions.***

Keywords: *strong differential superordination, convex function, best subordinant, extended differential operator.*

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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,$$

$z \in U, \zeta \in \overline{U}\}$,
for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Denote by

$$K_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}) : Re \frac{z f''_z(z,\zeta)}{f'_z(z,\zeta)} + 1 > 0\}$$

the class of convex function in $U \times \overline{U}$.

We extend the Ruscheweyh derivative [15] and the multiplier transformation studied in [1], [2] to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ introduced in [14].

Definition 1.1 *[5] For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\ (m+1) R^{m+1} f(z,\zeta) &= z\left(R^m f(z,\zeta)\right)'_z + m R^m f(z,\zeta), z \in U, \zeta \in \overline{U}. \end{aligned}$$

Remark 1.2 *[5] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.3 *[8]For $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, the operator $I(m,\lambda,l) f(z,\zeta)$ is defined by the following infinite series*

$$I(m,\lambda,l) f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m a_j(\zeta) z^j, z \in U, \zeta \in \overline{U}.$$

Remark 1.4 *[8] It follows from the above definition that*

$$(l+1) I(m+1,\lambda,l) f(z,\zeta) =$$
$$[l+1-\lambda] I(m,\lambda,l) f(z,\zeta) + \lambda z \left(I(m,\lambda,l) f(z,\zeta)\right)'_z, z \in U, \zeta \in \overline{U}.$$

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [13].

Definition 1.5 *[13] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$, such that $H(z,\zeta)=f(w(z),\zeta)$, for all $\zeta\in\overline{U}$. In such a case we write $H(z,\zeta)\prec\prec f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

Remark 1.6 *[13] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.5 is equivalent to $H(0,\zeta)=f(0,\zeta)$, for all $\zeta\in\overline{U}$, and $H\left(U\times\overline{U}\right)\subset f\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.7 *[7] We denote by Q^* the set of functions that are analytic and injective on $\overline{U}\times\overline{U}\backslash E(f,\zeta)$, where $E(f,\zeta)=\{y\in\partial U:\lim_{z\to y}f(z,\zeta)=\infty\}$, and are such that $f'_z(y,\zeta)\neq 0$ for $y\in\partial U\times\overline{U}\backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta)=a$ is denoted by $Q^*(a)$.*

We have need the following lemmas to study the strong differential superordinations.

Lemma 1.8 *[7] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $Re\,\gamma\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)$ is univalent in $U\times\overline{U}$ and*

$$h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta),\quad z\in U,\zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and is the best subordinant.

Lemma 1.9 *[7] Let $q(z,\zeta)$ be a convex function in $U\times\overline{U}$ and let $h(z,\zeta)=q(z,\zeta)+\frac{1}{\gamma}zq'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $Re\,\gamma\geq 0$.*

If $p\in\mathcal{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)$ is univalent in $U\times\overline{U}$ and

$$q(z,\zeta)+\frac{1}{\gamma}zq'_z(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta),z\in U,\zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is the best subordinant.

We extend the differential operator studied in [3], [4] to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$

Definition 1.10 *[10] Let $\alpha, \lambda, l \geq 0$, $n, m \in \mathbb{N}$. Denote by $RI^{\alpha}_{m,\lambda,l}$ the operator given by $RI^{\alpha}_{m,\lambda,l} : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$RI^{\alpha}_{m,\lambda,l} f(z,\zeta) = (1-\alpha) R^m f(z,\zeta) + \alpha I\left(m,\lambda,l\right) f(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

Remark 1.11 *[10] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j\left(\zeta\right) z^j$, then $RI^{\alpha}_{m,\lambda,l} f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left\{\alpha \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m + (1-\alpha) C^m_{m+j-1}\right\} a_j\left(\zeta\right) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.12 *For $\alpha = 0$, $RI^{0}_{m,\lambda,l} f(z,\zeta) = R^m f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$, and for $\alpha = 1$, $RI^{1}_{m,\lambda,l} f\left(z,\zeta\right) = I\left(m,\lambda,l\right) f\left(z,\zeta\right)$, where $z \in U$, $\zeta \in \overline{U}$, which was studied in [8], [9], [11]. For $l = 0$, we obtain $RI^{\alpha}_{m,\lambda,0} f\left(z,\zeta\right) = RD^m_{1,\alpha} f\left(z,\zeta\right)$ which was studied in [6], [12] and for $l = 0$ and $\lambda = 1$, we obtain $RI^{\alpha}_{m,1,0} f\left(z,\zeta\right) = L^m_{\alpha} f\left(z,\zeta\right)$ which was studied in [5], [7].*

For $m = 0$, $RI^{\alpha}_{0,\lambda,l} f\left(z,\zeta\right) = (1-\alpha) R^0 f\left(z,\zeta\right) + \alpha I\left(0,\lambda,l\right) f\left(z,\zeta\right) = f\left(z,\zeta\right) = R^0 f\left(z,\zeta\right) = I\left(0,\lambda,l\right) f\left(z,\zeta\right)$, where $z \in U$, $\zeta \in \overline{U}$.

2 Main results

Theorem 2.1 *Let $h\left(z,\zeta\right)$ be a convex function in $U \times \overline{U}$ with $h\left(0,\zeta\right) = 1$. Let $m \in \mathbb{N}$, $\lambda, \alpha, l \geq 0$, $f\left(z,\zeta\right) \in \mathcal{A}^*_{n\zeta}$, $F\left(z,\zeta\right) = I_c\left(f\right)\left(z,\zeta\right) = \frac{c+2}{z^{c+1}} \int_0^z t^c f\left(t,\zeta\right) dt$, $z \in U$, $\zeta \in \overline{U}$, $Rec > -2$, and suppose that $\left(RI^{\alpha}_{m,\lambda,l} f\left(z,\zeta\right)\right)'_z$ is univalent in $U \times \overline{U}$, $\left(RI^{\alpha}_{m,\lambda,l} F\left(z,\zeta\right)\right)'_z \in \mathcal{H}^*\left[1,n,\zeta\right] \cap Q^*$ and*

$$h\left(z,\zeta\right) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} f\left(z,\zeta\right)\right)'_z, z \in U, \zeta \in \overline{U}, \tag{1}$$

then

$$q\left(z,\zeta\right) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} F\left(z,\zeta\right)\right)'_z, z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof We have

$$z^{c+1} F\left(z,\zeta\right) = (c+2) \int_0^z t^c f\left(t,\zeta\right) dt$$

and differentiating it, with respect to z, we obtain $(c+1) F\left(z,\zeta\right) + zF'_z\left(z,\zeta\right) = (c+2) f\left(z,\zeta\right)$ and

$$(c+1) RI^{\alpha}_{m,\lambda,l} F\left(z,\zeta\right) + z \left(RI^{\alpha}_{m,\lambda,l} F\left(z,\zeta\right)\right)'_z = (c+2) RI^{\alpha}_{m,\lambda,l} f\left(z,\zeta\right),$$

$z \in U, \zeta \in \overline{U}$.

Differentiating the last relation with respect to z we have

$$\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}+\frac{1}{c+2}z\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)''_{z^{2}}=\left(RI^{\alpha}_{m,\lambda,l}f\left(z,\zeta\right)\right)'_{z}, z\in U,\zeta\in\overline{U}. \quad (2)$$

Using (2), the strong differential superordination (1) becomes

$$h\left(z,\zeta\right)\prec\prec\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}+\frac{1}{c+2}z\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)''_{z^{2}}. \quad (3)$$

Denote

$$p\left(z,\zeta\right)=\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}, z\in U,\zeta\in\overline{U}. \quad (4)$$

Replacing (4) in (3) we obtain

$$h\left(z,\zeta\right)\prec\prec p\left(z,\zeta\right)+\frac{1}{c+2}zp'_{z}\left(z,\zeta\right), z\in U,\zeta\in\overline{U}.$$

Using Lemma 1.8 for $\gamma=c+2$, we have

$q\left(z,\zeta\right)\prec\prec p\left(z,\zeta\right), z\in U,\zeta\in\overline{U}, i.e. q\left(z,\zeta\right)\prec\prec\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}, z\in U,\zeta\in\overline{U},$

where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is convex and it is the best subordinant.

Corollary 2.2 *Let* $h\left(z,\zeta\right)=\frac{1+(2\beta-\zeta)z}{1+z}$, *where* $\beta\in[0,1)$. *Let* $m\in\mathbb{N}$, $\lambda,\alpha,l\geq 0$, $f\left(z,\zeta\right)\in\mathcal{A}^{*}_{n\zeta}$, $F\left(z,\zeta\right)=I_{c}\left(f\right)\left(z,\zeta\right)=\frac{c+2}{z^{c+1}}\int_0^z t^{c}f\left(t,\zeta\right)dt$, $z\in U$, $\zeta\in\overline{U}$, $Re c>-2$, *and suppose that* $\left(RI^{\alpha}_{m,\lambda,l}f\left(z,\zeta\right)\right)'_{z}$ *is univalent in* $U\times\overline{U}$, $\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}\in\mathcal{H}^{*}\left[1,n,\zeta\right]\cap Q^{*}$ *and*

$$h\left(z,\zeta\right)\prec\prec\left(RI^{\alpha}_{m,\lambda,l}f\left(z,\zeta\right)\right)'_{z}, z\in U,\zeta\in\overline{U}, \quad (5)$$

then

$$q\left(z,\zeta\right)\prec\prec\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}, z\in U,\zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{(c+2)(1+\zeta-2\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function q is convex and it is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.1 and considering $p(z,\zeta)=\left(RI^{\alpha}_{m,\lambda,l}F\left(z,\zeta\right)\right)'_{z}$, the strong differential superordination (5) becomes

$$h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p\left(z,\zeta\right)+\frac{1}{c+2}zp'_{z}\left(z,\zeta\right),\quad z\in U,\zeta\in\overline{U}.$$

By using Lemma 1.8 for $\gamma = c+2$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z \frac{1+(2\beta-\zeta)t}{1+t} t^{\frac{c+2}{n}-1} dt$$

$$= 2\beta - \zeta + \frac{(c+2)(1+\zeta-2\beta)}{nz^{\frac{c+2}{n}}} \int_0^z \frac{t^{\frac{c+2}{n}-1}}{t+1} dt \prec\prec \left(RI^{\alpha}_{m,\lambda,l} F(z,\zeta)\right)'_z,$$

$z \in U, \zeta \in \overline{U}$.

The function q is convex and it is the best subordinant.

Theorem 2.3 *Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q'_z(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$, $Re c > -2$. Let $m \in \mathbb{N}$, $\lambda, \alpha, l \geq 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta) dt$, $z \in U$, $\zeta \in \overline{U}$, and suppose that $\left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z$ is univalent in $U \times \overline{U}$, $\left(RI^{\alpha}_{m,\lambda,l} F(z,\zeta)\right)'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z, z \in U, \zeta \in \overline{U}, \tag{6}$$

then

$$q(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} F(z,\zeta)\right)'_z, z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is the best subordinant.

Proof Following the same steps as in the proof of Theorem 2.1 and considering $p(z,\zeta) = \left(RI^{\alpha}_{m,\lambda,l} F(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$, the strong differential superordination (6) becomes

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), z \in U, \zeta \in \overline{U}.$$

Using Lemma 1.9 for $\gamma = c+2$, we have
$q(z,\zeta) \prec\prec p(z,\zeta), z \in U, \zeta \in \overline{U}$, i.e.

$$q(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} F(z,\zeta)\right)'_z, z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is the best subordinant.

Theorem 2.4 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $m \in \mathbb{N}$, $\lambda, \alpha, l \geq 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z$ is univalent and $\frac{RI^{\alpha}_{m,\lambda,l} f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z, \quad z \in U, \zeta \in \overline{U}, \tag{7}$$

then

$$q(z,\zeta) \prec\prec \frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}, \quad z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. *The function* q *is convex and it is the best subordinant.*

Proof By using the properties of operator $RI^{\alpha}_{m,\lambda,l}$, we have $RI^{\alpha}_{m,\lambda,l}f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C^m_{m+j-1} \right\} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C^m_{m+j-1} \right\} a_j(\zeta) z^j}{z}$ $= 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots$, $z \in U$, $\zeta \in \overline{U}$.

We deduce that $p \in \mathcal{H}^*[1,n,\zeta]$.

Let $RI^{\alpha}_{m,\lambda,l}f(z,\zeta) = zp(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Differentiating with respect to z we obtain $\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z = p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Then (7) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), z \in U, \zeta \in \overline{U}, \quad i.e. \quad q(z,\zeta) \prec\prec \frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}, z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant.

Corollary 2.5 *Let* $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U \times \overline{U}$, *where* $0 \le \beta < 1$. *Let* $m \in \mathbb{N}$, $\lambda, \alpha, l \ge 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ *and suppose that* $\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z$ *is univalent and* $\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. *If*

$$h(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z, \quad z \in U, \zeta \in \overline{U}, \tag{8}$$

then

$$q(z,\zeta) \prec\prec \frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}, \quad z \in U, \zeta \in \overline{U},$$

where q *is given by* $q(z,\zeta) = 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{t+1} dt$, $z \in U$, $\zeta \in \overline{U}$. *The function* q *is convex and it is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta)=\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}$, the strong differential superordination (8) becomes

$$h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\quad z\in U,\zeta\in\overline{U}.$$

By using Lemma 1.8 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z\frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt$$

$$=2\beta-\zeta+\frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt\prec\prec\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z},\quad z\in U,\zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem 2.6 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$. If $m\in\mathbb{N}$, $\lambda,\alpha,l\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, suppose that $\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'$ is univalent and $\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z,\quad z\in U,\zeta\in\overline{U},\qquad(9)$$

then

$$q(z,\zeta)\prec\prec\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z},\quad z\in U,\zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is the best subordinant.

Proof Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta)=\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}$, the strong differential superordination (9) becomes

$$q(z,\zeta)+zq'_z(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\quad z\in U,\zeta\in\overline{U}.$$

Using Lemma 1.9 for $\gamma=1$, we have
$q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z},z\in U,\zeta\in\overline{U},$$

and q is the best subordinant.

Theorem 2.7 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $m\in\mathbb{N}$, $\lambda,\alpha,l\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z$ is univalent and $\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z,\quad z\in U,\zeta\in\overline{U},\tag{10}$$

then

$$q(z,\zeta)\prec\prec\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)},\quad z\in U,\zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof For $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$ we have $RI^{\alpha}_{m,\lambda,l}f(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Consider

$$p(z,\zeta)=\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}=$$

$$\frac{z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}+(1-\alpha)C^{m+1}_{m+j}\right\}a_j(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}+(1-\alpha)C^{m}_{m+j-1}\right\}a_j(\zeta)z^j}.$$

We have $p'_z(z,\zeta)=\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)\right)'_z}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}-p(z,\zeta)\cdot\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}$ and we obtain $p(z,\zeta)+z\cdot p'_z(z,\zeta)=\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z$.

Relation (10) becomes

$$h(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\quad z\in U,\zeta\in\overline{U}.$$

By using Lemma 1.8 for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),z\in U,\zeta\in\overline{U},i.e.q(z,\zeta)\prec\prec\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)},z\in U,\zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Corollary 2.8 *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \leq \beta < 1$. Let $\lambda, \alpha, l \geq 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z$ is univalent and $\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z, \quad z \in U, \zeta \in \overline{U}, \tag{11}$$

then

$$q(z,\zeta) \prec\prec \frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}, \quad z \in U, \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{t+1}dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof Following the same steps as in the proof of Theorem 2.7 and considering $p(z,\zeta) = \frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}$, the strong differential superordination (11) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt$$

$$= 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{t+1}dt \prec\prec \frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}, \quad z \in U, \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem 2.9 *Let $q(z,\zeta)$ be a convex function and h be defined by $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta)$. Let $\lambda, \alpha, l \geq 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z$ is univalent and $\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z, \quad z \in U, \zeta \in \overline{U}, \tag{12}$$

then

$$q(z,\zeta) \prec\prec \frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}, \quad z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is the best subordinant.

Proof Following the same steps as in the proof of Theorem 2.7 and considering $p(z,\zeta) = \frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}$, the strong differential superordination (12) becomes

$$h\left(z,\zeta\right) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

By using Lemma 1.9 for $\gamma = 1$, we have
$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}$, i.e.

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt \prec\prec \frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f\left(z,\zeta\right)}, \quad z \in U, \zeta \in \overline{U},$$

and q is the best subordinant.

Theorem 2.10 *Let $h\left(z,\zeta\right)$ be a convex function in $U \times \overline{U}$ with $h(0,\zeta) = 1$ and let $\lambda, \alpha, l \geq 0$, $m \in \mathbb{N}$, $f\left(z,\zeta\right) \in \mathcal{A}^*_{n\zeta}$,*
$\frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f\left(z,\zeta\right) - \frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f\left(z,\zeta\right) + \frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f\left(z,\zeta\right) +$
$\frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2)\right] I\left(m+2,\lambda,l\right) f\left(z,\zeta\right) -$
$\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)\left(\bar{l}+1\right)}{\lambda^2} - (m+1)(2m+1)\right] \cdot I\left(m+1,\lambda,l\right) f\left(z,\zeta\right) +$
$\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2\right] I\left(m,\lambda,l\right) f\left(z,\zeta\right)$ *is univalent and*
$[RD^m_{\lambda,\alpha}f(z,\zeta)]'_z \in \mathcal{H}^*\left[1,n,\zeta\right] \cap Q^*$. *If*

$$h(z,\zeta) \prec\prec \frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f\left(z,\zeta\right) - \frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f\left(z,\zeta\right) \tag{13}$$

$$+\frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f\left(z,\zeta\right) + \frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2)\right] I\left(m+2,\lambda,l\right) f\left(z,\zeta\right) -$$

$$\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(l+1)}{\lambda^2} - (m+1)(2m+1)\right] I\left(m+1,\lambda,l\right) f\left(z,\zeta\right) +$$

$$\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2\right] I\left(m,\lambda,l\right) f\left(z,\zeta\right), z \in U, \zeta \in \overline{U},$$

holds, then

$$q(z,\zeta) \prec\prec [RI^{\alpha}_{m,\lambda,l}f(z,\zeta)]'_z, \quad z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have $RI^{\alpha}_{m,\lambda,l} f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left\{\alpha \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m + (1-\alpha) C^m_{m+j-1}\right\} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Let

$$p(z,\zeta) = \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z \tag{14}$$

$$= 1 + \sum_{j=n+1}^{\infty} \left\{\alpha \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m + (1-\alpha) C^m_{m+j-1}\right\} j a_j(\zeta) z^{j-1}$$

$$= 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \dots.$$

By using the properties of operators $RI^{\alpha}_{m,\lambda,l}$, R^m and $I(m,\lambda,l)$, after a short calculation, we obtain

$p(z,\zeta) + zp'_z(z,\zeta) = \frac{(m+1)(m+2)}{z} RI^{\alpha}_{m+2,\lambda,l} f(z,\zeta) - \frac{(m+1)(2m+1)}{z} RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta) + \frac{m^2}{z} RI^{\alpha}_{m,\lambda,l} f(z,\zeta) + \frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2)\right] I(m+2,\lambda,l) f(z,\zeta) - \frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(l+1)}{\lambda^2} - (m+1)(2m+1)\right] I(m+1,\lambda,l) f(z,\zeta) + \frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2\right] I(m,\lambda,l) f(z,\zeta)$.

Using the notation in (14), the strong differential superordination becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta).$$

By using Lemma 1.8 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), z \in U, \zeta \in \overline{U}, i.e. q(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z, z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant.

Corollary 2.11 *Let* $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U \times \overline{U}$, *where* $0 \le \beta < 1$. *Let* $\lambda, \alpha, l \ge 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, *suppose that* $\frac{(m+1)(m+2)}{z} RI^{\alpha}_{m+2,\lambda,l} f(z,\zeta) - \frac{(m+1)(2m+1)}{z} RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta) + \frac{m^2}{z} RI^{\alpha}_{m,\lambda,l} f(z,\zeta) + \frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2)\right] I(m+2,\lambda,l) f(z,\zeta) - \frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(l+1)}{\lambda^2} - (m+1)(2m+1)\right] I(m+1,\lambda,l) f(z,\zeta) + \frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2\right] I(m,\lambda,l) f(z,\zeta)$ *is univalent in* $U \times \overline{U}$ *and* $[RI^{\alpha}_{m,\lambda,l} f(z,\zeta)]'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. *If*

$$h(z,\zeta) \prec\prec \frac{(m+1)(m+2)}{z} RI^{\alpha}_{m+2,\lambda,l} f(z,\zeta) - \frac{(m+1)(2m+1)}{z} RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta) \tag{15}$$

$$+\frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f(z,\zeta)+\frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2}-(m+1)(m+2)\right]I(m+2,\lambda,l)f(z,\zeta)-$$
$$\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(l+1)}{\lambda^2}-(m+1)(2m+1)\right]I(m+1,\lambda,l)f(z,\zeta)+$$
$$\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2}-m^2\right]I(m,\lambda,l)f(z,\zeta),z\in U,\zeta\in\overline{U},$$

then
$$q(z,\zeta)\prec\prec\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z,\quad z\in U,\zeta\in\overline{U},$$
where q *is given by* $q(z,\zeta)=2\beta-\zeta+\frac{1+\zeta-\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function* q *is convex and it is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta)=\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z$, the strong differential superordination (15) becomes
$$h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\quad z\in U,\zeta\in\overline{U}.$$
By using Lemma 1.8 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.
$$q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z\frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt$$
$$=2\beta-\zeta+\frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt\prec\prec\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z,\quad z\in U,\zeta\in\overline{U}.$$
The function q is convex and it is the best subordinant.

Theorem 2.12 *Let* $q(z,\zeta)$ *be a convex function in* $U\times\overline{U}$ *and* $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$. *Let* $\lambda,\alpha,l\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, *suppose that* $\frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)+$ $\frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f(z,\zeta)+\frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2}-(m+1)(m+2)\right]I(m+2,\lambda,l)f(z,\zeta)-$ $\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(l+1)}{\lambda^2}-(m+1)(2m+1)\right]I(m+1,\lambda,l)f(z,\zeta)+$ $\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2}-m^2\right]I(m,\lambda,l)f(z,\zeta)$ *is univalent in* $U\times\overline{U}$ *and* $[RI^{\alpha}_{m,\lambda,l}f(z,\zeta)]'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. *If*
$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec \tag{16}$$
$$\frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)+$$

$$\frac{m^2}{z} RI^{\alpha}_{m,\lambda,l} f(z,\zeta) + \frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2)\right] I(m+2,\lambda,l) f(z,\zeta) -$$

$$\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(l+1)}{\lambda^2} - (m+1)(2m+1)\right] I(m+1,\lambda,l) f(z,\zeta) +$$

$$\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2\right] I(m,\lambda,l) f(z,\zeta), z \in U, \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z, \quad z \in U, \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. *The function q is the best subordinant.*

Proof Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta) = \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z$, the strong differential superordination (16) becomes

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

By using Lemma 1.9 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z, \quad z \in U, \zeta \in \overline{U}.$$

The function q is the best subordinant.

3 Open Problem

The definitions, theorems and corollaries we established in this paper can be extended by using the notion of strong superordination. One can use the operator from definition 1.10 and the same technique to prove earlier results and to obtain a new set of results.

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