*Int. J. Open Problems Complex Analysis, Vol. 5, No. 1, March 2013*
*ISSN 2074-2827; Copyright ©ICSRS Publication, 2013*
*www.i-csrs.org*

# Some Inclusion Relationships Associated with Generalized Srivastava-Attiya Operator

**R. M. El-Ashwah**
Department of Mathematics, Faculty of Science, University of Damietta
New Damietta 34517, Egypt. E-mail: r_elashwah@yahoo.com
**M. K. Aouf**
Department of Mathematics, Faculty of Science, University of Mansoura
Mansoura 33516, Egypt. E-mail: mkaouf127@yahoo.com
**A. Shamandy**
Department of Mathematics, Faculty of Science, University of Mansoura
Mansoura 33516, Egypt. E-mail: shamandy16@hotmail.com
**S. M. El-Deeb**
Department of Mathematics, Faculty of Science, University of Damietta
New Damietta 34517, Egypt. E-mail: shezaeldeeb@yahoo.com

**Abstract**

In this paper we introduce and study some new subclasses of $p$-valent starlike, convex, close-to-convex and quasi-convex functions defined by generalized Srivastava-Attiya operator. Inclusion relationships are established and integral operator of functions in these subclasses is discussed.

**Keywords:** $p$-Valent, starlike, convex, close-to-convex, quasi-convex, Srivastava-Attiya operator.
**2000 Mathematics Subject Classification. 30C45.**

## 1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and $p-$valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ . For simplicity, we write $\mathcal{A}(1) = \mathcal{A}$.

For functions $f(z) \in \mathcal{A}(p)$, given by (1), and $g(z)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z) \quad (z \in U;\ p \in \mathbb{N}). \tag{3}$$

A function $f \in \mathcal{A}(p)$ is said to be in the class $S_p^*(\alpha)$ of $p-$valently starlike of order $\alpha$ in $U$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leq \alpha < p;\ z \in U). \tag{4}$$

Also, a function $f \in \mathcal{A}(p)$ is said to be in the class $C_p(\alpha)$ of $p-$valently convex of order $\alpha$ in $U$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leq \alpha < p;\ z \in U). \tag{5}$$

It is easy to observe from (4) and (5) that

$$f(z) \in C_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha). \tag{6}$$

The class $S_p^*(\alpha)$ was introduced by Patil and Thakare [13] and The class $C_p(\alpha)$ was introduced by Owa [10].

Furthermore, a function $f \in \mathcal{A}(p)$, we say that $f \in K_p(\beta, \alpha)$ if there exists a function $g \in S_p^*(\alpha)$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \beta \quad (0 \leq \alpha, \beta < p;\ z \in U). \tag{7}$$

Functions in the class $K_p(\beta, \alpha)$ are called $p-$valently close-to-convex of order $\beta$ and type $\alpha$. The class $K_p(\beta, \alpha)$ was studied by Aouf [1]. We also say that a function $f \in \mathcal{A}(p)$ is in the class $K_p^*(\beta, \alpha)$ of $p-$valently quasi-convex of order $\beta$ and type $\alpha$ if there exists a function $g \in C_p(\alpha)$ such that

$$\Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta \quad (0 \leq \alpha, \beta < p;\ z \in U). \tag{8}$$

The class $K_p^*(\beta,\alpha)$ was studied by Aouf [2].

It follows from (7) and (8) that

$$f(z) \in K_p^*(\beta,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in K_p(\beta,\alpha). \tag{9}$$

Recently, Srivastava and Attiya [18] (see also [4, 5, 8, 14]) introduced and investigated the linear operator:

$$\mathcal{J}_{s,b}f(z){=}z{+}\sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^s a_k z^k \ \ \left(z{\in}U; b{\in}\mathbb{C}\backslash\mathbb{Z}^-{=}\{-1,-2,...\}\,; s{\in}\mathbb{C}; f{\in}\mathcal{A}\right).$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, Wang et al. [19] (see also [20]) introduced the linear operator:

$$\mathcal{J}_{s,b}^{\lambda,p}f(z) : \mathcal{A}\,(p) \to \mathcal{A}\,(p)\,,$$

which is defined as

$$\begin{aligned}\mathcal{J}_{s,b}^{\lambda,p}f(z) \quad &= \quad z^p + \sum_{k=1}^{\infty}\frac{(\lambda+p)_k}{k!}\left(\frac{p+b}{k+p+b}\right)^s a_{k+p}z^{k+p} \\ &\left(s\in\mathbb{C};\ b\in\mathbb{C}\backslash\mathbb{Z}^-;\ p\in\mathbb{N};\ \lambda>-p;\ z\in U\right),\end{aligned} \tag{10}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu=0;\ \theta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}), \\ \theta(\theta+1)....(\theta+\nu-1) & (\nu\in\mathbb{N};\ \theta\in\mathbb{C}). \end{cases} \tag{11}$$

It is readily verified from (10) that

$$z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)' = (p+\lambda)\,\mathcal{J}_{s,b}^{\lambda+1,p}f(z) - \lambda\mathcal{J}_{s,b}^{\lambda,p}f(z) \tag{12}$$

and

$$z\left(\mathcal{J}_{s+1,b}^{\lambda,p}f(z)\right)' = (p+b)\,\mathcal{J}_{s,b}^{\lambda,p}f(z) - b\mathcal{J}_{s+1,b}^{\lambda,p}f(z). \tag{13}$$

By specializing the parameters $\lambda, p, s$ and $b$, we obtain:

(i) $\mathcal{J}_{s,b}^{0,1}f(z) = \mathcal{J}_{s,b}f(z)$ $\left(s\in\mathbb{C},\ b\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$ (see Srivastava and Attiya [18]);

(ii) $\mathcal{J}_{1,b}^{0,1}f(z) = \mathcal{J}_b f(z)$ $(b>-1)$ (see Bernardi [3] and Libera [7]);

(iii) $\mathcal{J}_{s,1}^{0,1}f(z) = \mathcal{I}^s f(z)$ $(s>0)$ (see Jung [6]);

(iv) $\mathcal{J}_{\alpha,\beta}^{0,1} f(z) = \mathcal{P}_{\beta}^{\alpha} f(z)$ $(\alpha \geq 1,\ \beta > 1)$ (see Patel and Sahoo [11];
(v) $\mathcal{J}_{s,1}^{1-p,p} f(z) = D_p^s f(z)$ ($s$ is any integer) (see Patel and Sahoo [12]);
(vi) $\mathcal{J}_{s,1}^{1-p,p} f(z) = I_p^s f(z)$ $(s > 0)$ (see Shams et al. [17]).
Also, we note that:

$$\begin{aligned} \mathcal{J}_{s,b}^{1-p,p} f(z) &= \mathcal{J}_{s,b}^{p} f(z) \\ &= z^p + \sum_{k=1}^{\infty} \left( \frac{p+b}{k+p+b} \right)^s a_{k+p} z^{k+p} \end{aligned} \tag{14}$$

$$\left( s \in \mathbb{C}; b \in \mathbb{C} \backslash \mathbb{Z}^-; p \in \mathbb{N} \right).$$

We now define the following subclasses of $p$-valent function class $\mathcal{A}(p)$ by means of the linear operator $\mathcal{J}_{s,b}^{\lambda,p}$ given by (10).

**Definition 1.** In conjunction with (4) and (10),

$$S_{s,b}^{\lambda,p}(\alpha) = \left\{ f(z) \in \mathcal{A}(p) :\ \mathcal{J}_{s,b}^{\lambda,p} f(z) \in S_p^*(\alpha),\ 0 \leq \alpha < p \right\}.$$

**Definition 2.** In conjunction with (5) and (10),

$$C_{s,b}^{\lambda,p}(\alpha) = \left\{ f(z) \in \mathcal{A}(p) :\ \mathcal{J}_{s,b}^{\lambda,p} f(z) \in C_p(\alpha),\ 0 \leq \alpha < p \right\}.$$

**Definition 3.** In conjunction with (7) and (10),

$$K_{s,b}^{\lambda,p}(\beta,\alpha) = \left\{ f(z) \in \mathcal{A}(p) : \mathcal{J}_{s,b}^{\lambda,p} f(z) \in K_p(\beta,\alpha),\ 0 \leq \alpha, \beta < p \right\}.$$

**Definition 4.** In conjunction with (8) and (10),

$$K_{s,b}^{*,\lambda,p}(\beta,\alpha) = \left\{ f(z) \in \mathcal{A}(p) :\ \mathcal{J}_{s,b}^{\lambda,p} f(z) \in K_p^*(\beta,\alpha),\ 0 \leq \alpha, \beta < p \right\}.$$

Obviously, we know that

$$f(z) \in C_{s,b}^{\lambda,p}(\alpha) \Leftrightarrow \frac{z f'(z)}{p} \in S_{s,b}^{\lambda,p}(\alpha), \tag{15}$$

and

$$f(z) \in K_{s,b}^{*,\lambda,p}(\beta,\alpha) \Leftrightarrow \frac{z f'(z)}{p} \in K_{s,b}^{\lambda,p}(\beta,\alpha). \tag{16}$$

In order to prove our main results, we need the following lemma.

**Lemma 1 [9].** *Let $\Phi(u,v)$ be complex valued function, $\Phi : D \to \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that $\Phi(u,v)$ satisfies the following conditions:*

*(i)* $\Phi(u,v)$ *is continuous in* $D$;
*(ii)* $(1,0) \in D$ *and* $\Re\{\Phi(1,0)\} > 0$;

*(iii)* $\Re\{\Phi(iu_2, v_1)\} \leq 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \leq -\dfrac{(1+u_2^2)}{2}$.

*Let*

$$q(z) = 1 + q_1 z + q_2 z^2 + ... \tag{17}$$

*be regular in the unit disc* $U$ *such that* $(q(z), zq^{'}(z)) \in D$ *for all* $z \in U$. *If*

$$\Re\left\{\Phi(q(z), zq^{'}(z))\right\} > 0 \quad (z \in U),$$

*then*

$$\Re\{q(z)\} > 0 \quad (z \in U).$$

## 2. The Main Inclusion Relationships

Unless otherwise mentioned, we assume throughout this paper that :
$s \in \mathbb{C},\ b \in \mathbb{C}\backslash\mathbb{Z}^-,\ p \in \mathbb{N}$ and $\lambda > -p$.

In this section, we give several inclusion relationships for $p$-valent function classes, which are associated with the linear operator $\mathcal{J}_{s,b}^{\lambda,p}$.

**Theorem 1.** *Let* $0 \leq \alpha < p$ and $\operatorname{Re}\{b\} = b_1 > -\alpha$. *Then*

$$S_{s,b}^{\lambda+1,p}(\alpha) \subset S_{s,b}^{\lambda,p}(\alpha) \subset S_{s+1,b}^{\lambda,p}(\alpha). \tag{18}$$

**Proof.** We first prove that

$$S_{s,b}^{\lambda+1,p}(\alpha) \subset S_{s,b}^{\lambda,p}(\alpha). \tag{19}$$

Let $f(z) \in S_{s,b}^{\lambda+1,p}(\alpha)$ and set

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p} f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p} f(z)} = \alpha + (p-\alpha)q(z), \tag{20}$$

where $q(z)$ is given by (17). By using identity (12), we obtain

$$(p+\lambda)\frac{\mathcal{J}_{s,b}^{\lambda+1,p} f(z)}{\mathcal{J}_{s,b}^{\lambda,p} f(z)} = \lambda + \alpha + (p-\alpha)q(z). \tag{21}$$

Differentiating (21) logarithmically with respect to $z$, we have

$$\begin{aligned} \frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p} f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda+1,p} f(z)} &= \frac{z\left(\mathcal{J}_{s,b}^{\lambda,p} f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p} f(z)} + \frac{(p-\alpha)zq^{'}(z)}{\lambda+\alpha+(p-\alpha)q(z)} \\ &= \alpha + (p-\alpha)q(z) + \frac{(p-\alpha)zq^{'}(z)}{\lambda+\alpha+(p-\alpha)q(z)}. \end{aligned}$$

Let

$$\Phi(u,v)=(p-\alpha)u+\frac{(p-\alpha)v}{(p-\alpha)u+\lambda+\alpha}$$

with $u=q(z)=u_1+iu_2$ and $v=zq^{'}(z)=v_1+iv_2$, this show that $\Phi(u,v)$ satisfies the hypothese of Lemma 1. Consequently, we easily obtain the inclusion relationship (19). Now, we will prove the second part of relation (18), i.e.

$$S_{s,b}^{\lambda,p}(\alpha)\subset S_{s+1,b}^{\lambda,p}(\alpha). \tag{22}$$

Let $f(z)\in S_{s,b}^{\lambda,p}(\alpha)$ and set

$$\frac{z\left(\mathcal{J}_{s+1,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s+1,b}^{\lambda,p}f(z)}=\alpha+(p-\alpha)q(z), \tag{23}$$

where $q(z)$ is given by (17). By using identity (13), we obtain

$$(p+b)\frac{\mathcal{J}_{s,b}^{\lambda,p}f(z)}{\mathcal{J}_{s+1,b}^{\lambda,p}f(z)}=b+\alpha+(p-\alpha)q(z). \tag{24}$$

Differentiating (24) logarithmically with respect to $z$, we have

$$\begin{aligned}\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}f(z)} &= \frac{z\left(\mathcal{J}_{s+1,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s+1,b}^{\lambda,p}f(z)}+\frac{(p-\alpha)zq^{'}(z)}{b+\alpha+(p-\alpha)q(z)}\\ &= \alpha+(p-\alpha)q(z)+\frac{(p-\alpha)zq^{'}(z)}{b+\alpha+(p-\alpha)q(z)}.\end{aligned}$$

Let

$$\Phi(u,v)=(p-\alpha)u+\frac{(p-\alpha)v}{(p-\alpha)u+b+\alpha}$$

with $u=q(z)=u_1+iu_2$ and $v=zq^{'}(z)=v_1+iv_2$, this show that $\Phi(u,v)$. Then

(i) $\Phi(u,v)$ *is continuous in* $D=\left(\mathbb{C}\backslash\frac{-b-\alpha}{p-\alpha}\right)\times\mathbb{C}$;

*(ii)* $(1,0)\in D$ *and* $\Re\{\Phi(1,0)\}=p-\alpha>0$;

*(iii) for all* $(iu_2,v_1)\in D$ *and such that* $v_1\le-\frac{(1+u_2^2)}{2}$, we have

$$\begin{aligned}\Re\{\Phi(iu_2,v_1)\} &= \Re\left\{\frac{(p-\alpha)v_1}{(p-\alpha)iu_2+b_1+ib_2+\alpha}\right\}\\ &= \frac{(p-\alpha)(b_1+\alpha)v_1}{(b_1+\alpha)^2+(b_2+(p-\alpha)u_2)^2}\\ &\le -\frac{(p-\alpha)(b_1+\alpha)(1+u_2^2)}{2\left[(b_1+\alpha)^2+(b_2+(p-\alpha)u_2)^2\right]}\\ &< 0,\end{aligned}$$

which shows that $\Phi(u, v)$ satisfies the hypothese of Lemma 1. Consequently, we easily obtain the inclusion relationship (22). Combining the inclusion relationships (19) and (22), we complete the proof of Theorem 1.

**Theorem 2.** *Let* $0 \leq \alpha < p$ and $\operatorname{Re}\{b\} = b_1 > -\alpha$. *Then*

$$C_{s,b}^{\lambda+1,p}(\alpha) \subset C_{s,b}^{\lambda,p}(\alpha) \subset C_{s+1,b}^{\lambda,p}(\alpha). \tag{25}$$

**Proof.** Let $f(z) \in C_{s,b}^{\lambda+1,p}(\alpha)$. Then, by Definition 2, we have

$$\mathcal{J}_{s,b}^{\lambda+1,p} f(z) \in C_p(\alpha), \; 0 \leq \alpha < p.$$

Furthermore, in view of the relationship (6), we find that

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p} f(z)\right)'}{p} \in S_p^*(\alpha),$$

that is, that

$$\mathcal{J}_{s,b}^{\lambda+1,p}\left(\frac{zf'(z)}{p}\right) \in S_p^*(\alpha).$$

Thus, by using Definition 1 and Theorem 1, we have

$$\frac{zf'(z)}{p} \in S_{s,b}^{\lambda+1,p}(\alpha) \subset S_{s,b}^{\lambda,p}(\alpha),$$

which implies that

$$C_{s,b}^{\lambda+1,p}(\alpha) \subset C_{s,b}^{\lambda,p}(\alpha).$$

The right part of Theorem 2 can be proved by using similar arguments. The proof of Theorem 2 is thus completed.

**Theorem 3.** *Let* $0 \leq \alpha, \beta < p$. *Then*

$$K_{s,b}^{\lambda+1,p}(\beta, \alpha) \subset K_{s,b}^{\lambda,p}(\beta, \alpha) \subset K_{s+1,b}^{\lambda,p}(\beta, \alpha). \tag{26}$$

**Proof.** Let us begin by proving that

$$K_{s,b}^{\lambda+1,p}(\beta, \alpha) \subset K_{s,b}^{\lambda,p}(\beta, \alpha) \quad (0 \leq \alpha, \beta < p). \tag{27}$$

Let $f(z) \in K_{s,b}^{\lambda+1,p}(\beta, \alpha)$. Then there exists a function $\psi(z) \in S_p^*(\alpha)$ such that

$$\operatorname{Re}\left(\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p} f(z)\right)'}{\psi(z)}\right) > \beta \quad (z \in U).$$

We put $\mathcal{J}_{s,b}^{\lambda+1,p}g(z)=\psi(z)$, so that we have

$$g(z)\in S_{s,b}^{\lambda+1,p}(\alpha) \quad \text{and} \quad \mathrm{Re}\left(\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p}f(z)\right)'}{\mathcal{J}_{s,b}^{\lambda+1,p}g(z)}\right)>\beta \ \ (z\in U).$$

We next put

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}=\beta+(p-\beta)q(z), \tag{28}$$

where $q(z)$ is given by (17). Thus, by using identity (12), we obtain

$$\begin{aligned}
\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p}f(z)\right)'}{\mathcal{J}_{s,b}^{\lambda+1,p}g(z)} &= \frac{\mathcal{J}_{s,b}^{\lambda+1,p}\left(zf'(z)\right)}{\mathcal{J}_{s,b}^{\lambda+1,p}g(z)} \\
&= \frac{z\left[\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)\right]'+\lambda\left[\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)\right]}{z\left[\mathcal{J}_{s,b}^{\lambda,p}g(z)\right]'+\lambda\mathcal{J}_{s,b}^{\lambda,p}g(z)} \\
&= \frac{\dfrac{z\left[\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)\right]'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}+\lambda\dfrac{\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}}{\dfrac{z\left[\mathcal{J}_{s,b}^{\lambda,p}g(z)\right]'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}+\lambda}.
\end{aligned}$$

Since $g(z)\in S_{s,b}^{\lambda+1,p}(\alpha)$, by using Theorem 1, we can put

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}g(z)\right)'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}=\alpha+(p-\alpha)G(z),$$

where

$$G(z)=g_1(x,y)+ig_2(x,y) \quad \text{and} \quad \mathrm{Re}\left(G(z)\right)=g_1(x,y)>0 \qquad (z\in U).$$

Then

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p}f(z)\right)'}{\mathcal{J}_{s,b}^{\lambda+1,p}g(z)}=\frac{\dfrac{z\left[\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)\right]'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}+\lambda\left[\beta+(p-\beta)q(z)\right]}{\lambda+\alpha+(p-\alpha)G(z)}. \tag{29}$$

We thus find from (28) that

$$z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)' = \left[\beta + (p-\beta)q(z)\right]\mathcal{J}_{s,b}^{\lambda,p}g(z). \tag{30}$$

Differentiating both sides of (30) with respect to $z$, we obtain

$$z\left[\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)\right]' = z\left[\mathcal{J}_{s,b}^{\lambda,p}g(z)\right]'\left[\beta + (p-\beta)q(z)\right] + (p-\beta)zq'(z)\mathcal{J}_{s,b}^{\lambda,p}g(z)$$

$$\frac{z\left[\mathcal{J}_{s,b}^{\lambda,p}\left(zf'(z)\right)\right]'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)} = (p-\beta)zq'(z) + \left[\beta + (p-\beta)q(z)\right]\frac{z\left[\mathcal{J}_{s,b}^{\lambda,p}g(z)\right]'}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}$$

$$= (p-\beta)zq'(z) + \left[\beta + (p-\beta)q(z)\right]\left[\alpha + (p-\alpha)G(z)\right]. \tag{31}$$

By substituting (31) into (29), we have

$$\begin{aligned}\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p}f(z)\right)'}{\mathcal{J}_{s,b}^{\lambda+1,p}g(z)} &= \frac{(p-\beta)zq'(z) + \left[\beta + (p-\beta)q(z)\right]\left[\lambda + \alpha + (p-\alpha)G(z)\right]}{\lambda + \alpha + (p-\alpha)G(z)} \\ &= \frac{(p-\beta)zq'(z)}{\lambda + \alpha + (p-\alpha)G(z)} + \left[\beta + (p-\beta)q(z)\right]\end{aligned}$$

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda+1,p}f(z)\right)'}{\mathcal{J}_{s,b}^{\lambda+1,p}g(z)} - \beta = \left\{(p-\beta)q(z) + \frac{(p-\beta)zq'(z)}{\lambda + \alpha + (p-\alpha)G(z)}\right\}.$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, we define the function $\Phi(u, v)$ by

$$\Phi(u,v) = (p-\beta)u + \frac{(p-\beta)v}{\lambda + \alpha + (p-\alpha)G(z)}, \tag{32}$$

where $(u, v) \in D = (\mathbb{C}\backslash D^*) \times \mathbb{C}$ and

$$D^* = \left\{z : z \in \mathbb{C} \ \text{ and } \ \Re\left(G(z)\right) = g_1(x,y) > 1 - \frac{p+\lambda}{p-\alpha}\right\}.$$

Then it follows from (32) that

(i) $\Phi(u, v)$ *is continuous in* $D$;

*(ii)* $(1, 0) \in D$ *and* $\Re\left\{\Phi(1,0)\right\} = p - \beta > 0$;

*(iii) for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \leq -\dfrac{(1+u_2^2)}{2}$, we have

$$\begin{aligned}
\Re\{\Phi(iu_2, v_1)\} &= \Re\left\{\frac{(p-\beta)v_1}{(p-\alpha)G(z)+\lambda+\alpha}\right\} \\
&= \frac{(p-\beta)v_1\left[(p-\alpha)g_1(x,y)+\lambda+\alpha\right]}{\left[(p-\alpha)g_1(x,y)+\lambda+\alpha\right]^2+\left[(p-\alpha)g_2(x,y)\right]^2} \\
&\leq -\frac{(p-\beta)(1+u_2^2)\left[(p-\alpha)g_1(x,y)+\lambda+\alpha\right]}{2\left[(p-\alpha)g_1(x,y)+\lambda+\alpha\right]^2+2\left[(p-\alpha)g_2(x,y)\right]^2} \\
&< 0,
\end{aligned}$$

which shows that $\Phi(u, v)$ satisfies the hypothese of Lemma 1. Thus, in light of (28), we easily deduce the inclusion relationship (27).

The remainder of our proof of Theorem 3 would make use of the identity (13) in analogous manner and assume that

$$D^*=\left\{z{:}z\in\mathbb{C} \text{ and } \Re\left(G(z)\right)=g_1(x,y)>1-\frac{p+b_1}{p-\alpha}\right\}, \text{ where } b_1=\mathrm{Re}\{b\}. \quad (33)$$

We, therefore, choose to omit the details involved.

**Theorem 4.** *Let* $0 \leq \alpha, \beta < p$. *Then*

$$K_{s,b}^{*,\lambda+1,p}(\beta,\alpha) \subset K_{s,b}^{*,\lambda,p}(\beta,\alpha) \subset K_{s+1,b}^{*,\lambda,p}(\beta,\alpha). \quad (34)$$

**Proof.** Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (6), we can also prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (9).

# 3. A set of integral-preserving properties

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $J_{c,p}(f)(z)$ defined by Saitoh [15] (see also Saitoh et al. [16]):

$$\begin{aligned}
\mathcal{J}_{1,c}^{1-p,p}f(z) = J_{c,p}(f)(z) &= \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \\
&= \left(z^p+\sum_{k=1}^{\infty}\frac{c+p}{c+p+k}z^{k+p}\right)*f(z) \quad (c>-p;\ z\in U), \quad (35)
\end{aligned}$$

which satisfies the following relationship:

$$z\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)\right)^{'}=(c+p)\,\mathcal{J}_{s,b}^{\lambda,p}f(z)-c\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z). \tag{36}$$

**Theorem 5.** *Let* $c\geq 0$ and $0\leq\alpha<p$. *If* $f(z)\in S_{s,b}^{\lambda,p}(\alpha)$, *then* $J_{c,p}(f)(z)\in S_{s,b}^{\lambda,p}(\alpha)$.

**Proof.** Suppose that $f(z)\in S_{s,b}^{\lambda,p}(\alpha)$ and let

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)}=\alpha+(p-\alpha)\,h(z), \tag{37}$$

where $h(z)=1+c_1z+c_2z^2+...$, using the identity (36), we have

$$\frac{\mathcal{J}_{s,b}^{\lambda,p}f(z)}{\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)}=\frac{1}{c+p}\left\{c+\alpha+(p-\alpha)\,h(z)\right\}. \tag{38}$$

Differentiating (38) logarithmically with respect to $z$, we obtain

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}f(z)}=\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)}+\frac{(p-\alpha)\,zh^{'}(z)}{c+\alpha+(p-\alpha)\,h(z)}$$

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}f(z)}-\alpha=(p-\alpha)\,h(z)+\frac{(p-\alpha)\,zh^{'}(z)}{c+\alpha+(p-\alpha)\,h(z)}. \tag{39}$$

Now, we form the function $\Phi(u,v)$ by taking $u=h(z)$ and $v=zh^{'}(z)$ in (39) as:

$$\Phi(u,v)=(p-\alpha)\,u+\frac{(p-\alpha)\,v}{c+\alpha+(p-\alpha)\,u}.$$

It is easy to see that the function $\Phi(u,v)$ satisfies the conditions (i) and (ii) of Lemma 1 in $D=\left(\mathbb{C}\backslash\frac{-c-\alpha}{p-\alpha}\right)\times\mathbb{C}$. To verify the condition (iii), we proceed as follows:

$$\begin{aligned}\Re\left\{\Phi(iu_2,v_1)\right\} &= \Re\left\{-\frac{(p-\alpha)v_1}{(p-\alpha)iu_2+c+\alpha}\right\}\\ &= \frac{(p-\alpha)\,(c+\alpha)\,v_1}{(p-\alpha)^2u_2^2+(c+\alpha)^2}\\ &\leq -\frac{(p-\alpha)\,(c+\alpha)\,(1+u_2^2)}{2\left[(p-\alpha)^2u_2^2+(c+\alpha)^2\right]}\\ &< 0,\end{aligned}$$

where $v_1 = -\frac{1}{2}(1+u_2^2)$ and $(iu_2, v_1) \in D$. Therefore the function $\Phi(u,v)$ satisfies the conditions of Lemma 1. This shows that if $\Re\left\{\Phi(h(z), zh^{'}(z))\right\} > 0 \quad (z \in U)$, then $\Re\{h(z)\} > 0 \quad (z \in U)$, that is, if $f(z) \in S_{s,b}^{\lambda,p}(\alpha)$, then $J_{c,p}(f)(z) \in S_{s,b}^{\lambda,p}(\alpha)$. This completes the proof of Theorem 5.

**Theorem 6.** *Let* $c \geq 0$ and $0 \leq \alpha < p$. *If* $f(z) \in C_{s,b}^{\lambda,p}(\alpha)$, *then* $J_{c,p}(f)(z) \in C_{s,b}^{\lambda,p}(\alpha)$.

**Proof.** By applying Theorem 5, it follows that

$$\begin{aligned} f(z) &\in C_{s,b}^{\lambda,p}(\alpha) \Leftrightarrow \frac{zf^{'}(z)}{p} \in S_{s,b}^{\lambda,p}(\alpha) \\ &\Rightarrow J_{c,p}\left(\frac{zf^{'}(z)}{p}\right) \in S_{s,b}^{\lambda,p}(\alpha) \\ &\Leftrightarrow \frac{z}{p}\left(J_{c,p}f(z)\right)^{'} \in S_{s,b}^{\lambda,p}(\alpha) \\ &\Rightarrow J_{c,p}(f)(z) \in C_{s,b}^{\lambda,p}(\alpha), \end{aligned}$$

which proves Theorem 6.

**Theorem 7.** *Let* $c \geq 0$ and $0 \leq \alpha, \beta < p$. *If* $f(z) \in K_{s,b}^{\lambda,p}(\beta,\alpha)$, *then* $J_{c,p}(f)(z) \in K_{s,b}^{\lambda,p}(\beta,\alpha)$.

**Proof.** Suppose that $f(z) \in K_{s,b}^{\lambda,p}(\beta,\alpha)$. Then, by using Definition 3, there exists a function $g(z) \in S_{s,b}^{\lambda,p}(\alpha)$ such that

$$\Re\left(\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}\right) > \beta \quad (z \in U).$$

Thus, upon setting

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)} = \beta + (p-\beta)h(z), \tag{40}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + ...$ From (36) and (40), we have

$$(c+p)\,\mathcal{J}_{s,b}^{\lambda,p}f(z) = \mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)\left[\beta + (p-\beta)h(z)\right] + c\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z) \tag{41}$$

Differentiating both sides of (41) with respect to $z$, we obtain

$$\begin{aligned} (c+p)\,z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'} &= z\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)\right)^{'}\left[\beta + (p-\beta)h(z)\right] \\ &+(p-\beta)zh^{'}(z)\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)\right) + cz\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(f)(z)\right)^{'}, \end{aligned} \tag{42}$$

now apply (36) for the function g(z) and using (42), we obtain

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}=\beta+(p-\beta)h(z)+\frac{\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}\cdot\frac{(p-\beta)zh^{'}(z)}{c+p}. \tag{43}$$

Since $g(z)\in S_{s,b}^{\lambda,p}(\alpha)$, we know from Theorem 5 that $J_{c,p}g(z)\in S_{s,b}^{\lambda,p}(\alpha)$. So we can set

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}J_{c,p}(g)(z)}=\alpha+(p-\alpha)H(z), \tag{44}$$

where

$H(z)=h_1(x,y)+ih_2(x,y)$ and $\operatorname{Re}(H(z))=h_1(x,y)>0 \quad (z\in U)$.

Then we have

$$\frac{z\left(\mathcal{J}_{s,b}^{\lambda,p}f(z)\right)^{'}}{\mathcal{J}_{s,b}^{\lambda,p}g(z)}-\beta=(p-\beta)h(z)+\frac{(p-\beta)zh^{'}(z)}{c+\alpha+(p-\alpha)H(z)}. \tag{45}$$

Then, by setting $u=h(z)=u_1+iu_2$ and $v=zh^{'}(z)=v_1+iv_2$, we can define the function $\Phi(u,v)$ by

$$\Phi(u,v)=(p-\beta)u+\frac{(p-\beta)v}{c+\alpha+(p-\alpha)H(z)}. \tag{46}$$

It is easy to see that the function $\Phi(u,v)$ satisfies the conditions (i) and (ii) of Lemma 1 in $D=\mathbb{C}\times\mathbb{C}$. To verify the condition (iii), we proceed as follows:

$$\begin{aligned}\Re\{\Phi(iu_2,v_1)\} &= \Re\left\{\frac{(p-\beta)v_1}{c+\alpha+(p-\alpha)h_1(x,y)+i(p-\alpha)h_2(x,y)}\right\}\\ &= \frac{(p-\beta)v_1\left[c+\alpha+(p-\alpha)h_1(x,y)\right]}{\left[c+\alpha+(p-\alpha)h_1(x,y)\right]^2+\left[(p-\alpha)h_2(x,y)\right]^2}\\ &\le -\frac{(p-\beta)(1+u_2^2)\left[c+\alpha+(p-\alpha)h_1(x,y)\right]}{2\left[c+\alpha+(p-\alpha)h_1(x,y)\right]^2+2\left[(p-\alpha)h_2(x,y)\right]^2}\\ &< 0,\end{aligned}$$

where $v_1=-\frac{1}{2}(1+u_2^2)$ and $(iu_2,v_1)\in D$. Therefore the function $\Phi(u,v)$ satisfies the conditions of Lemma 1. This shows that if $\Re\left\{\Phi(h(z),zh^{'}(z))\right\}>0 \ (z\in U)$, then $\Re\{h(z)\}>0 \quad (z\in U)$, that is, if $f(z)\in K_{s,b}^{\lambda,p}(\beta,\alpha)$, then $J_{c,p}(f)(z)\in K_{s,b}^{\lambda,p}(\beta,\alpha)$. This completes the proof of Theorem 7.

**Theorem 8.** *Let* $c>0$ and $0\le\alpha,\beta<p$. *If* $f(z)\in K_{s,b}^{*,\lambda,p}(\beta,\alpha)$, *then*

$J_{c,p}(f)(z) \in K_{s,b}^{*,\lambda,p}(\beta,\alpha)$.

**Proof.** Just as we derived Theorem 6 from Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 from Theorem 7.

**Remark.** By specializing the parameters $s$, $\lambda$ and $b$, we obtain various results associated with operators $\mathcal{J}_{s,b}^{p}$ and $I_p^s f(z)$.

# 4. Open Problem

The inclusion results we established in this paper can be obtained by using Jack's Lemma (see I. S. Jack, Functions starlike and convex of order $\alpha$, J. London Math. Soc., 2(1971), no. 3, 469-474). Compare these results with the results given by using Jack's Lemma.

**Acknowledgements.** The authors thank the referees for their valuable suggestions which led to improvement of this paper.

# References

[1] M. K. Aouf, On a class of $p$-valent close-to-convex functions, Internat. J. Math. Math. Sci., 11(1988), no. 2, 259-266.

[2] M. K. Aouf, Some inclusion relationships associated with Dziok-Srivastava operator, Applied Math. Comput., 216(2010), 431-437.

[3] S. D. Bernardi, Convex and starlike univalent functions, Trans. Math. Soc., 135(1969), 429-446.

[4] N. E. Cho, I. H. Kim and H. M. Srivastava, Sandwich-type theorems for multivalent functions associated with Srivastava-Attiya operator, Appl. Math. Comput., 217(2010), 918-928.

[5] E. A. Elrifai, H. E. Darwish and A. R. Ahmed, Some applications of Srivastava-Attiya operator to $p$-valent starlike functions, J. Inequal. Appl. 2010(2010), Art. ID 790730, 1-11.

[6] J. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176(1993), no. 1, 138-147.

[7] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16(1965), 755-758.

[8] J.-L. Liu, Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator, Integral Transforms Spec. Funct., 19(2008), 893-901.

[9] S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(1978), 289-305.

[10] S. Owa, On certain classes of $p$-valent functions with negative coefficients, Simon Stevin, 59(1985), 385-402.

[11] J. Patel and P. Sahoo, Some applications of differential subordination to certain one-parameter families of integral operators, Indian J. Pure and Appl. Math., 35(2004), no. 10, 1167-1177.

[12] J. Patel and P. Sahoo, Certain subclasses of multivalent analytic functions, Indian J. Pure Appl. Math., 34(2003), no. 3, 487-500.

[13] D. A. Patil and N. K. Thakare, On convex hulls and extreme points of $p$-valent starlike and convex classes with applications, Bull. Math. Soc. Sci. Math. R. S. Roumaine (N. S.), 27(1983), no. 75, 145-160.

[14] D. Raducanu and H. M. Srivastava, A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function, Integral Transforms Spec. Funct., 18(2007), 933-943.

[15] H. Saitoh, On certain class of multivalent functions, Math. Japon., 37(1992), 871-875.

[16] H. Saitoh, S. Owa, T. Sekine, M. Nunokawa and R. Yamakawa, An application of certain integral operator, Appl. Math. Letters, 5(1992), 21-24.

[17] S. Shams, S. R. Kulkarni and J. M. Jahangiri, Subordination properties of $p$-valent functions defined by integral operators, Inernat. J. Math. Math. Sci. (2006), Art. ID 94572, 1-3.

[18] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, Integral Transforms Spec. Funct., 18(2007), 207-216.

[19] Z.-G. Wang, Q.-G. Li and Y.-P. Jiang, Certain subclasses of multivalent analytic functions involving the generalized Srivastava-Attiya operator, Integral Transforms Spec. Funct., 21(2010), 221-234.

[20] Z.-G. Wang, Z.-H. Liu and Y. Sun, Some properties of the generalized Srivastava-Attiya operator, Integral Transforms Spec. Funct., 23(2012), no. 3, 223-236.