Int. J. Open Problems Complex Analysis, Vol. 5, No. 1, March 2013
ISSN 2074-2827;
www.i-csrs.org

Multivalent Functions with Varying Arguments

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Abstract

Silverman [4] was defined the class of univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for which $arg(a_k)$ prescribed in such way that $f(z)$ is univalent if and only if $f(z)$ is starlike. In this paper we introduce the subclass of $p-$valent functions with varying arguments, especially $p-$valent starlike functions and $p-$valent convex functions, moreover we give some interesting properties of functions in these classes, including coefficients estimates, distortion theorems and extreme functions.

Keywords: *analytic, varying arguments, extreme points.*
2000 Mathematical Subject Classification: 30C45.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ \ (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in A(p)$ is said to be $p-$valent starlike of order α if it satisfies the inequality:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ \left(0 \leq \alpha < p; p \in \mathbb{N}; z \in U\right), \tag{2}$$

or, equivalently,

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha}\right| < 1, \tag{3}$$

we denote by $S_p(\alpha)$ the class of all p-valent starlike functions of order α. Also a function $f(z) \in A(p)$ is said to be p-valent convex of order α if it satisfies the inequality:

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ \left(0 \leq \alpha < p; p \in \mathbb{N}; z \in U\right), \tag{4}$$

or, equivalently,

$$\left|\frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha}\right| < 1, \tag{5}$$

we denote by $K_p(\alpha)$ the class of all p-valently convex functions of order α, the classes $S_p(\alpha)$ and $K_p(\alpha)$ were introduced by Patil and Thakare [3] and Owa [2] studied these classes with negative coefficients, further from (2) and (3), we can see that

$$f(z) \in K_p(\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in S_p(\alpha) \ (0 \leq \alpha < p; p \in \mathbb{N}). \tag{6}$$

Silverman [4] defined the class of univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for which $arg(a_k)$ prescribed in such way that $f(z)$ is univalent if and only if $f(z)$ is starlike. In this paper we introduce the subclass of $p-$valent functions with varying arguments as follows:

Definition 1. A function $f(z)$ of the form (1) is said to be in the class $V_p(\theta_k)$ if $f \in A(p)$ and $arg(a_k) = \theta_k$ for all $k \geq p+1$. If furthermore there exist a real number δ such that $\theta_k + (k-p)\delta \equiv \pi \ (mod\ 2\pi)$ for all $k \geq p+1$, then $f(z)$ is said to be in the class $V_p(\theta_k, \delta)$. The union of $V_p(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V_p.

Note that:

(i) $V_p(\theta_k + 2k\pi) = V_p(\theta_k)$, k is an integer;

(ii) $V_p(\pi, 0) = T_p = \left\{ f \in A(p) : f(z) = z^p - \sum\limits_{k=p+1}^{\infty} |a_k| z^k \ (p \in \mathbb{N}) \right\}$, the set of $p-$valent functions with negative coefficients.
Let $V_p(\alpha)$ denote the subclass of V_p consisting of functions $f(z) \in S_p(\alpha)$, also let $\overline{V_p}(\alpha)$ denote the subclass of V_p consisting of functions $f(z) \in K_p(\alpha)$, which are the classes of p-valent starlike functions with varying arguments and p-valent convex functions with varying arguments, respectively.
We note that $V_1(\alpha) = V^*(\alpha)$, which was introduced and studied by Silverman [4].
In this paper we obtain coefficient bounds for functions in the classes $V_p(\alpha)$ and $\overline{V_p}(\alpha)$, further we obtain distortion bounds and the extreme points for functions in these classes.

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \alpha < p,\ p \in \mathbb{N}$ and $z \in U$.
We shall need the following lemmas given by Owa [2] and Aouf [1, with $A = -1$ and $B = 1$]:
Lemma 1. The sufficient condition for $f(z)$ given by (1) to be in the class $S_p(\alpha)$ is that

$$\sum_{k=p+1}^{\infty} (k - \alpha) |a_k| \leq p - \alpha. \tag{7}$$

Lemma 2. The sufficient condition for $f(z)$ given by (1) to be in the class $K_p(\alpha)$ is that

$$\sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right) (k - \alpha) |a_k| \leq (p - \alpha). \tag{8}$$

Theorem 1. Let $f(z)$ of the form (1), then $f(z) \in V_p(\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} (k - \alpha) |a_k| \leq p - \alpha. \tag{9}$$

Proof. In view of Lemma 1, we need only to show that function $f(z) \in V_p(\alpha)$ satisfies the coefficient inequality (9). If $f(z) \in V_p(\alpha)$, then from (3), we have

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{\dfrac{zf'(z)}{f(z)} + p - 2\alpha} \right| < 1,$$

thus we have

$$\left| \frac{\sum\limits_{k=p+1}^{\infty} (k-p)\, a_k z^{k-p}}{2(p-\alpha) + \sum\limits_{k=p+1}^{\infty} (k+p-2\alpha)\, a_k z^{k-p}} \right| < 1. \tag{10}$$

Since $f(z) \in V_p$, $f(z)$ lies in the class $V_p(\theta_k, \delta)$ for some sequence $\{\theta_k\}$ and a real number δ such that

$$\theta_k + (k-p)\delta \equiv \pi(mod 2\pi) \;\; (k \geq p+1).$$

Set $z = re^{i\delta}$ in (10), we get

$$\frac{\sum\limits_{k=p+1}^{\infty} (k-p)\, |a_k|\, r^{k-p}}{2(p-\alpha) - \sum\limits_{k=p+1}^{\infty} (k+p-2\alpha)\, |a_k|\, r^{k-p}} < 1.$$

Letting $r \longrightarrow 1$, then we have

$$\sum_{k=p+1}^{\infty} (k-\alpha)\, |a_k| \leq p-\alpha.$$

Hence the proof of Theorem 1 is completed.

Corollary 1. If $f(z) \in V_p(\alpha)$, then

$$|a_k| \leq \frac{p-\alpha}{k-\alpha} \;\; (k \geq p+1). \tag{11}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{p-\alpha}{k-\alpha} e^{i\theta_k} z^k \;\; (k \geq p+1). \tag{12}$$

Using the same technique as used in Theorem 1, we have the following theorem:

Theorem 2. Let $f(z)$ of the form (1), then $f(z) \in \overline{V_p}(\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right) (k-\alpha)\, |a_k| \leq (p-\alpha). \tag{13}$$

Corollary 2. If $f(z) \in \overline{V_p}(\alpha)$, then

$$|a_k| \leq \frac{p(p-\alpha)}{k(k-\alpha)} \;\; (k \geq p+1). \tag{14}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{p(p-\alpha)}{k(k-\alpha)} e^{i\theta_k} z^k \;\; (k \geq p+1). \tag{15}$$

3. Distortion theorems

Theorem 3. Let the function $f(z)$ defined by (1) be in the class $V_p(\alpha)$. Then

$$|z|^p - \frac{p-\alpha}{p+1-\alpha}|z|^{p+1} \le |f(z)| \le |z|^p + \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}. \tag{16}$$

The result is sharp.
Proof. We employ the same technique as used by Silverman [4]. In view of Theorem 1, since

$$\Phi(k) = k - \alpha, \tag{17}$$

is an increasing function of $k(k \ge p+1)$, in view of Lemma 1, we have

$$(p+1-\alpha)\sum_{k=p+1}^{\infty}|a_k| \le \sum_{k=p+1}^{\infty}(k-\alpha)|a_k| \le p-\alpha,$$

that is

$$\sum_{k=p+1}^{\infty}|a_k| \le \frac{p-\alpha}{p+1-\alpha}. \tag{18}$$

Thus we have

$$|f(z)| \le |z|^p + |z|^{p+1}\sum_{k=p+1}^{\infty}|a_k|,$$

$$|f(z)| \le |z|^p + \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}.$$

Similarly, we get

$$|f(z)| \ge |z|^p - |z|^{p+1}\sum_{k=p+1}^{\infty}|a_k|,$$

$$|f(z)| \ge |z|^p - \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}.$$

Finally the result is sharp for the function

$$f(z) = z^p + \frac{p-\alpha}{p+1-\alpha}e^{i\theta_{p+1}}z^{p+1}, \tag{19}$$

at $z = \pm|z|e^{-i\theta_{p+1}}$.
This completes the proof of Theorem 3.
Corollary 3. Under the hypotheses of Theorem 3, $f(z)$ is included in a disc with center at the origin and radius r_1 given by

$$r_1 = 1 + \frac{p-\alpha}{p+1-\alpha}. \tag{20}$$

Theorem 4. Let the function $f(z)$ defined by (1) be in the class $V_p(\alpha)$. Then

$$\left\{p-\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}\le\left|f^{'}(z)\right|\le\left\{p+\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}. \tag{21}$$

The result is sharp.

Proof. Similarly for $\Phi(k)$ defined by (17) it is clear that $\frac{\Phi(k)}{k}$ is an increasing function of $k(k\ge p+1;p\in\mathbb{N})$, in view of Theorem 1, we have

$$\frac{\Phi(p+1)}{p+1}\sum_{k=p+1}^{\infty}k|a_k|\le\sum_{k=p+1}^{\infty}\frac{\Phi(k)}{k}k|a_k|=\sum_{k=p+1}^{\infty}\Phi(k)|a_k|\le p-\alpha,$$

that is

$$\sum_{k=p+1}^{\infty}k|a_k|\le\frac{(p+1)(p-\alpha)}{p+1-\alpha}.$$

Thus we have

$$\begin{aligned}\left|f^{'}(z)\right| &\le p|z|^{p-1}+|z|^p\sum_{k=p+1}^{\infty}k|a_k|\le\left\{p+|z|\sum_{k=p+1}^{\infty}k|a_k|\right\}|z|^{p-1}\\ &\le\left\{p+\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}.\end{aligned}$$

Similarly

$$\begin{aligned}\left|f^{'}(z)\right| &\ge p|z|^{p-1}-|z|^p\sum_{k=p+1}^{\infty}k|a_k|\ge\left\{p-|z|\sum_{k=p+1}^{\infty}k|a_k|\right\}|z|^{p-1}\\ &\le\left\{p-\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}.\end{aligned}$$

Finally, we can see that the assertions of Theorem 4 are sharp for the function $f(z)$ defined by (19). This completes the proof of Theorem 4.

Corollary 4. Under the hypotheses of Theorem 4, $f^{'}(z)$ is included in a disc with center at the origin and radius r_2 given by

$$r_2=p+\frac{(p+1)(p-\alpha)}{p+1-\alpha}. \tag{22}$$

Using the same technique as used in Theorem 3 and Theorem 4 , in view of Lemma 2, we have the following theorems for functions in the class $\overline{V_p}(\alpha)$:

Theorem 5. Let the function $f(z)$ defined by (1) be in the class $\overline{V_p}(\alpha)$. Then

$$|z|^p-\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1}\le|f(z)|\le|z|^p+\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1}. \tag{23}$$

The result is sharp for the function

$$f(z) = z^p + \frac{p\,(p-\alpha)}{(p+1)\,(p+1-\alpha)} e^{i\theta_{p+1}} z^{p+1}, \tag{24}$$

at $z = \pm |z|\, e^{-i\theta_{p+1}}$.

Corollary 5. Under the hypotheses of Theorem 5, $f(z)$ is included in a disc with center at the origin and radius r_3 given by

$$r_3 = 1 + \frac{p\,(p-\alpha)}{(p+1)\,(p+1-\alpha)}. \tag{25}$$

Theorem 6. Let the function $f(z)$ defined by (1) be in the class $\overline{V_p}(\alpha)$. Then

$$\left\{p - \frac{p\,(p-\alpha)}{p+1-\alpha}\,|z|\right\} |z|^{p-1} \leq \left|f^{'}(z)\right| \leq \left\{p + \frac{p\,(p-\alpha)}{p+1-\alpha}\,|z|\right\} |z|^{p-1}. \tag{26}$$

The result is sharp for the function given by (24).

Corollary 6. Under the hypotheses of Theorem 6, $f^{'}(z)$ is included in a disc with center at the origin and radius r_4 given by

$$r_4 = p + \frac{p\,(p-\alpha)}{p+1-\alpha}. \tag{27}$$

4. Extreme points

Theorem 7. Let the function $f(z)$ defined by (1) be in the class $V_p(\alpha)$, with $arg(a_k) = \theta_k$, where $\theta_k + (k-p)\delta \equiv \pi \ (mod\ 2\pi)\,(k \geq p+1)$. Define

$$f_p(z) = z^p$$

and

$$f_k(z) = z^p + \frac{p-\alpha}{k-\alpha} e^{i\theta_k} z^k \quad (k \geq p+1). \tag{28}$$

Then $f(z) \in V_p(\alpha)$ if and only if $f(z)$ can expressed in the form $f(z) = \sum_{k=p}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum_{k=p}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum_{k=p}^{\infty} \mu_k f_k(z)$ with $\mu_k \geq 0$ and $\sum_{k=p}^{\infty} \mu_k = 1$, then

$$\begin{aligned} &\sum_{k=p+1}^{\infty} (k-\alpha)\left(\frac{p-\alpha}{k-\alpha}\right)\mu_k \\ = &\sum_{k=p+1}^{\infty} (p-\alpha)\,\mu_k = (p-\alpha)\left(1-\mu_p\right) \leq (p-\alpha). \end{aligned}$$

Hence $f(z) \in V_p(\alpha)$.
Conversely, let the function $f(z)$ defined by (1) be in the class $V_p(\alpha)$, define

$$\mu_k = \frac{k-\alpha}{p-\alpha}\,|a_k| \quad (k \geq p+1) \tag{29}$$

and

$$\mu_p = 1 - \sum_{k=p+1}^{\infty} \mu_k.$$

From Theorem 1, $\sum\limits_{k=p+1}^{\infty} \mu_k \leq 1$ and so $\mu_p \geq 0$. Since $\mu_k f_k(z) = \mu_k z^p + a_k z^k$, then

$$\sum_{k=p}^{\infty} \mu_k f_k(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k = f(z).$$

This completes the proof of Theorem 7.
Similarly we can get the following theorem.
Theorem 8. Let the function $f(z)$ defined by (1) be in the class $\overline{V_p}(\alpha)$, with $\arg\, a_k = \theta_k$, where $\theta_k + (k-p)\delta \equiv \pi \ (mod\ 2\pi)\, (k \geq p+1)$. Define

$$f_p(z) = z^p$$

and

$$f_k(z) = z^p + \frac{p\,(p-\alpha)}{k\,(k-\alpha)} e^{i\theta_k} z^k \quad (k \geq p+1)\,. \tag{30}$$

Then $f(z) \in \overline{V_p}(\alpha)$ if and only if $f(z)$ can expressed in the form $f(z) = \sum\limits_{k=p}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum\limits_{k=p}^{\infty} \mu_k = 1$.

Remark.
Putting $p = 1$ in all the above results, we obtain the corresponding results obtained by Silverman [4].

5. Open problem

Find the distortion theorems, radii of starlikeness, convexity and close-to-convexity and modified-Hadamard products of functions belonging to the classes $S_p(\alpha)$ and $K_p(\alpha)$, also several applications involving an integral operator and certain fractional calculus operators can also be considered.

Acknowledgements. The authors thank the referees for their valuable suggestions which led to improvement of this paper.

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