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# Region of Variability of a Differential Operator Involving Bazilević Functions

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**Abstract**

***By making use of differential subordination, we, here, extend the region of variability of a differential operator involving Bazilević functions. Mathematica 7.0 is used to plot the extended regions of the complex plane.***

**Keywords:** *Analytic function, Bazilević function, Differential subordination.*
**2010 Mathematical Subject Classification:** 30C80, 30C45.

## 1 Introduction

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,\ \ (n \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}(n)$ is called a member of the class $\mathcal{B}(n, \alpha, \beta)$ if it satisfies

$$\Re\left(\frac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}}\right) > \beta,\ \ z \in \mathbb{E},$$

for some real numbers $\alpha$ and $\beta$ such that $\alpha > 0,\ 0 \leq \beta < 1$. A function $f \in \mathcal{A}(n)$ is said to belong to the class $\mathcal{S}^*(n, \alpha)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{E},$$

for some real number $\alpha,\ 0 \leq \alpha < 1$. Cho [1] proved the following results.

**Theorem 1.1** *If $f \in \mathcal{B}(n, 0, \beta),\ 1/2 \leq \beta < 1$, then*

$$\Re\left(\frac{f(z)}{z}\right) > \frac{n}{n + 2(1 - \beta)},\ z \in \mathbb{E}.$$

**Theorem 1.2** *If $f \in \mathcal{B}(n, 1, 0)$, then $\Re\left(\dfrac{f(z)}{z}\right) > \dfrac{n}{n+2}$ in $\mathbb{E}$.*

Owa [2] proved the following result.

**Theorem 1.3** *If $f \in \mathcal{B}(n, \alpha, \beta)$, then $\Re\left(\dfrac{f(z)}{z}\right)^{\alpha} > \dfrac{n + 2\alpha\beta}{n + 2\alpha}$ in $\mathbb{E}$.*

Denote $\mathcal{A} = \mathcal{A}(1)$ and therefore $\mathcal{A}$ is the class of analytic functions $f$ in $\mathbb{E}$, having the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

For two analytic functions $f$ and $g$ in the unit disk $\mathbb{E}$, we say that $f$ is subordinate to $g$ in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function $w$ analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function $g$ is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, $p$ be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and $h$ be univalent in $\mathbb{E}$. Then the function $p$ is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function $q$ is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all $p$ satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for each dominant $q$ of (1), is said to be the best dominant of (1).

The main objective of the paper is to extend the region of variability of the operator $\dfrac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}}$ involving Bazilević functions. For this, we establish a subordination theorem and discuss its particular cases to make comparisons with certain known results and consequently region of variability of the differential operators involving the above stated results of Cho [1] and Owa [2] has been extended. Extended regions have been shown by plotting using Mathematica 7.0.

## 2 Preliminaries

To prove our main result, we shall make use of the following lemma of Miller and Mocanu [3].

**Lemma 2.1** *([3], p.132, Theorem 3.4 h) Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\phi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
*(i) $h$ is convex, or*
*(ii) $Q$ is starlike.*
*In addition, assume that*
*(iii) $\Re \dfrac{zh'(z)}{Q(z)} > 0,\ z \in \mathbb{E}$.*
*If $p$ is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and*

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

*then $p \prec q$ and $q$ is the best dominant.*

## 3 Main Result

In what follows all the powers taken, are the principal ones.

**Theorem 3.1** *Let $\alpha, \beta$ be complex numbers such that $\beta \neq 0$ and let $q,\ q(z) \neq 0,\ z \in \mathbb{E}$, be a univalent function satisfying the condition*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\alpha}{\beta} - 1\right)\frac{zq'(z)}{q(z)}\right] > \max\{0, -\Re(\alpha)\}. \tag{2}$$

*If $f \in \mathcal{A}$, $\left(\dfrac{f(z)}{z}\right)^{\beta} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination*

$$\frac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}} \prec (q(z))^{\alpha/\beta}\left(1 + \frac{1}{\beta}\frac{zq'(z)}{q(z)}\right),\ z \in \mathbb{E}, \tag{3}$$

*then*

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec q(z),\ z \in \mathbb{E}.$$

**Proof.** Let us define the functions $\theta$ and $\phi$ as follows:

$$\theta(w) = w^{\frac{\alpha}{\beta}} \text{ and } \phi(w) = \frac{1}{\beta}w^{\frac{\alpha}{\beta}-1}.$$

Obviously, the functions $\theta$ and $\phi$ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Also define the functions $Q$ and $h$ as under:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{1}{\beta} zq'(z)(q(z))^{\frac{\alpha}{\beta}-1},$$

and

$$h(z) = \theta(q(z)) + Q(z) = (q(z))^{\alpha/\beta}\left(1 + \frac{1}{\beta}\frac{zq'(z)}{q(z)}\right).$$

A little calculation yields

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\alpha}{\beta} - 1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\alpha}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \alpha.$$

In view of (2), conditions (i) and (iii) of Lemma 2.1 are satisfied.
Write $\left(\frac{f(z)}{z}\right)^{\beta} = p(z)$ and therefore

$$\frac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}} = (p(z))^{\alpha/\beta}\left(1 + \frac{1}{\beta}\frac{zp'(z)}{p(z)}\right), \ z \in \mathbb{E}.$$

Also by (3), we obtain

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Hence, the result follows by the use of Lemma 2.1.

Setting $\beta = \alpha$ and the dominant $q(z) = \dfrac{1 + (1-2\gamma)z}{1-z}$, $z \in \mathbb{E}$ in the above theorem, we get:

**Theorem 3.2** *Let $\alpha$ be a non-zero complex number such that $\Re(\alpha) \geq 0$. If $f \in \mathcal{A}$, $\left(\frac{f(z)}{z}\right)^{\alpha} \neq 0, z \in \mathbb{E}$, satisfies*

$$\frac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}} \prec \frac{1 + (1-2\gamma)z}{1-z} + \frac{2(1-\gamma)z}{\alpha(1-z)^2}, \ 0 \leq \gamma < 1,$$

*then*

$$\left(\frac{f(z)}{z}\right)^{\alpha} \prec \frac{1 + (1-2\gamma)z}{1-z}, \ z \in \mathbb{E}.$$

**Remark 3.3** *We, here, show that the result in Theorem 3.2 extends the region of variability of the differential operator* $\dfrac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}}$ *over the result of Owa [2] stated in Theorem 1.3 to get the same conclusion. We justify our claim by considering the particular cases of Theorem 3.2 and Theorem 1.3. By taking* $\alpha = 1/2$ *and* $\gamma = 3/4$ *in Theorem 3.2, we obtain:*

*For all* $z$ *in* $\mathbb{E}$ *and* $f \in \mathcal{A}$, $\sqrt{\dfrac{f(z)}{z}} \neq 0, z \in \mathbb{E}$, *we have*

$$f'(z)\sqrt{\frac{z}{f(z)}} \prec \frac{2-z}{2(1-z)} + \frac{z}{(1-z)^2} = F(z) \quad \Rightarrow \quad \Re\left(\sqrt{\frac{f(z)}{z}}\right) > \frac{3}{4}. \tag{4}$$

*Writing* $n = 1$ *and* $\alpha = \beta = 1/2$ *in Theorem 1.3, then for* $f \in \mathcal{A}$, *it gives:*

$$\Re\left(f'(z)\sqrt{\frac{z}{f(z)}}\right) > \frac{1}{2} \quad \Rightarrow \quad \Re\left(\sqrt{\frac{f(z)}{z}}\right) > \frac{3}{4}, \; z \in \mathbb{E}. \tag{5}$$

*In Figure 3.1, the shaded region shows the image of the unit disk* $\mathbb{E}$ *under the*

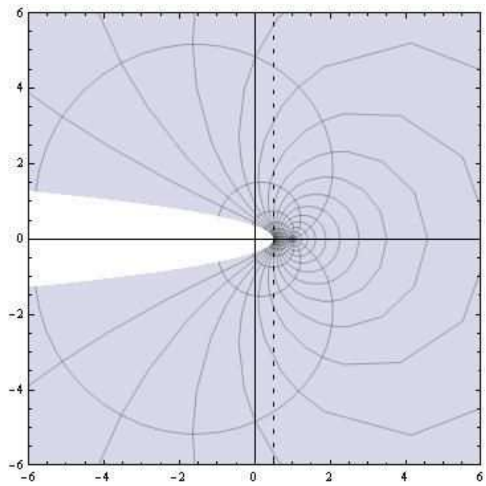


Figure 3.1

*function* $F(z)$ *(given in (4)) and we also plot the dashing line at* $\Re(w) = \dfrac{1}{2}$. *According to the result in (5), the operator* $f'(z)\sqrt{\dfrac{z}{f(z)}}$ *takes values right to the dashing line at* $\Re(w) = \dfrac{1}{2}$ *whereas by the result in (4), the same operator can take values in the total shaded region to conclude the same result. Thus, the shaded region left to the dashing line is the extension of the region of variability of the operator* $f'(z)\sqrt{\dfrac{z}{f(z)}}$ *for the required implication. This justifies our claim.*

Setting $\alpha = 0,\ \beta = 1$ and the dominant $q(z) = \dfrac{1+(1-2\gamma)z}{1-z},\ z \in \mathbb{E}$ in Theorem 3.1, we obtain:

**Theorem 3.4** *If $f \in \mathcal{A}$, $\dfrac{f(z)}{z} \neq 0, z \in \mathbb{E}$, satisfies*

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1-z} + \frac{(1-2\gamma)z}{1+(1-2\gamma)z},\ 0 \leq \gamma < 1,$$

*then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\gamma)z}{1-z},\ z \in \mathbb{E}.$$

**Remark 3.5** *In this remark, we show that Theorem 3.4 extends the region of variability of the differential operator $\dfrac{zf'(z)}{f(z)}$ over the result of Cho [1] stated in Theorem 1.1 to conclude the same result. To justify our claim, we consider the particular cases of Theorem 3.4 and Theorem 1.1 as follows. For $\gamma = 2/3$ in Theorem 3.4, we obtain:*

*For all $z$ in $\mathbb{E}$ and $f \in \mathcal{A}$, $\dfrac{f(z)}{z} \neq 0, z \in \mathbb{E}$, we have*

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1-z} - \frac{z}{3-z} = G(z) \quad \Rightarrow \quad \Re\left(\frac{f(z)}{z}\right) > \frac{2}{3}. \tag{6}$$

*Taking $n = 1$ and $\beta = 3/4$ in Theorem 1.1, then for $f \in \mathcal{A}$, it gives:*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{3}{4} \quad \Rightarrow \quad \Re\left(\frac{f(z)}{z}\right) > \frac{2}{3},\ z \in \mathbb{E}. \tag{7}$$

*In Figure 3.2, the image of the unit disk $\mathbb{E}$ under the function $G(z)$ (given in (6)) is shown by the shaded region and we also plot the dashing line at $\Re(w) = \dfrac{3}{4}$. According to the result in (7), the operator $\dfrac{zf'(z)}{f(z)}$ takes values right to the dashing line at $\Re(w) = \dfrac{3}{4}$ whereas by the result in (6), the same operator can take values in the total shaded region to conclude the same result. Thus, the shaded region left to the dashing line is the extension of the region of variability of the operator $\dfrac{zf'(z)}{f(z)}$ for the same conclusion. This justifies our claim.*

Setting $\alpha = \beta = 1$ and the dominant $q(z) = \dfrac{1+(1-2\gamma)z}{1-z},\ z \in \mathbb{E}$ in the Theorem 3.1, we obtain:

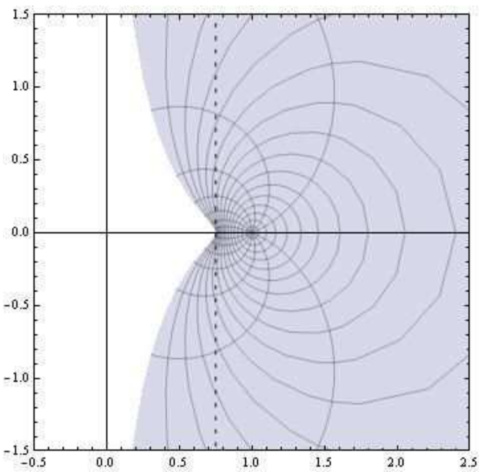

Figure 3.2

**Theorem 3.6** *If* $f \in \mathcal{A}$, $\dfrac{f(z)}{z} \neq 0, z \in \mathbb{E}$, *satisfies*

$$f'(z) \prec \frac{1+(1-2\gamma)z}{1-z} + \frac{2(1-\gamma)z}{(1-z)^2},\ 0 \leq \gamma < 1,$$

*then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\gamma)z}{1-z},\ z \in \mathbb{E}.$$

**Remark 3.7** *We, here, show that the result in Theorem 3.6 extends the region of variability of the differential operator* $f'(z)$ *over the result of Cho [1] stated in Theorem 1.2 to get the same conclusion. To make the comparison, we consider the particular cases of Theorem 3.6 and Theorem 1.2. Setting* $\gamma = 1/3$ *in Theorem 3.6, we obtain:*

*For all* $z$ *in* $\mathbb{E}$ *and* $f \in \mathcal{A}$, $\dfrac{f(z)}{z} \neq 0, z \in \mathbb{E}$, *we have*

$$f'(z) \prec \frac{3+z}{3(1-z)} + \frac{4z}{3(1-z)^2} = H(z) \quad \Rightarrow \quad \Re\left(\frac{f(z)}{z}\right) > \frac{1}{3}. \tag{8}$$

*Writing* $n = 1$ *in Theorem 1.2, then for* $f \in \mathcal{A}$, *it gives:*

$$\Re\left(f'(z)\right) > 0 \quad \Rightarrow \quad \Re\left(\frac{f(z)}{z}\right) > \frac{1}{3},\ z \in \mathbb{E}. \tag{9}$$

*The shaded region in Figure 3.3 shows the image of the unit disk* $\mathbb{E}$ *under the function* $H(z)$ *(given in (8)). According to the result in (9), the operator* $f'(z)$ *takes values in the right half plane whereas by the result in (8), the same*

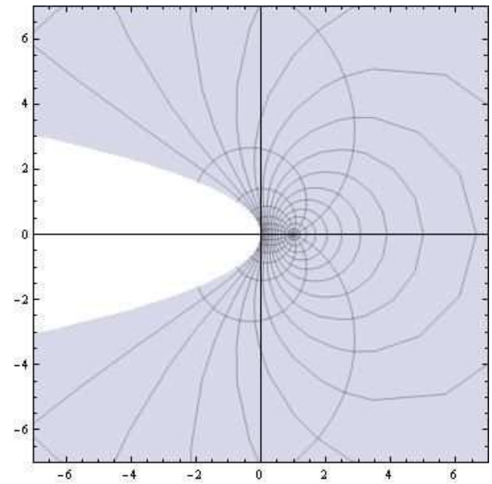


Figure 3.3

*operator can take values in the total shaded region to conclude the same result. Thus, the shaded region in the left half plane is the extension of the region of variability of the operator $f'(z)$ for having the same conclusion. This justifies our claim.*

# 4 Open Problem

The technique of differential subordination can similarly be applied to extend the regions of variability of differential operators involving with other classes of analytic functions.

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