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Convolution Properties of Convex Harmonic Functions

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Abstract

In this paper, we examine the convolutions of convex harmonic functions with some other classes of univalent harmonic functions defined by certain coefficient conditions and prove that such convolutions belong to some well known classes of univalent harmonic functions.

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1 Introduction

A continuous function $f(x+iy) = u(x,y)+iv(x,y)$ defined in a domain $D \subset \mathbb{C}$ (Complex plane) is harmonic in D if u and v are real harmonic in D. Clunie and Shiel-Small [1] showed that such function can be written in the form $f = h + \bar{g}$, where h and g are analytic.We call g the co-analytic part and h, the analytic part of f. In the unit disc $E = \{z : |z| < 1\}$, g and h can be expanded in Taylor series as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

The mapping f is sense-preserving and locally one-to-one in E iff the Jacobian of the mapping, given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2,$$

is positive. So,the condition for f to be sense-preserving and locally one-to-one is that $|h'(z)| > |g'(z)|$ in E, or equivalently, if the dilatation function $w(z)=\frac{g'(z)}{h'(z)}$ satisfies $|w(z)| < 1$ in E. In such case, we say that f is locally univalent.

We denote by S_H, the class of harmonic, sense preserving and univalent functions f in E, normalized by the conditions $f(0) = 0$ and $f_z(0) = 1$.Denoted by K_H, S_H^* and C_H the subclasses of S_H consisting of harmonic functions which map E onto convex, starlike and close-to-convex domains, respectively. The classical family S of normalized univalent functions in E is a subclass of S_H. We let K, S^* and C denote the subclasses of S consisting of functions which are convex, starlike (w.r.t. origin) and close-to-convex in E, respectively. Finally, let S_H^0 be the subclass of S_H whose members f satisfy additional condition, $f_{\bar{z}}(0) = \bar{b_1} = 0$ and K_H^0, S_H^{0*} and C_H^0 be the subclasses of S_H^0 of convex, starlike and close-to-convex mappings, respectively.

For analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, their convolution (or Hadamard product) is defined as $(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n$. In case of harmonic functions

$$F(z) = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n \tag{1}$$

and

$$f(z) = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n, \tag{2}$$

we define their convolution as,

$$\begin{aligned} (F * f)(z) &= (H * h)(z) + \overline{(G * g)(z)} \\ &= z + \textstyle\sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n \end{aligned}$$

In 1973, Ruscheweyh and Shiel-Small [6] proved the following:
(i) If ϕ and $\psi \in K$, then $\phi * \psi$ also belongs to K.
(ii) If $\phi \in K$ and $\psi \in C$, then $\phi * \psi \in C$.

The above results do not extend naturally to harmonic case i.e the convolution of a function $F \in K_H^0$ with another harmonic function f may not preserve the properties of f. For example let;

$$F = H + \overline{G}, \quad \text{where } H + G = \frac{z}{1-z} \text{ with dilatation } W(z) = -z$$

and

$$f = h + \overline{g}, \quad \text{where } h + g = \frac{z}{1-z} \text{ and dialatation } w(z) = -z^n, \quad n \in N$$

Then, both F and$f$ belong to K_H^0, but their convolution $f * F$ is not even univalent in E for $n \geq 3$ (see M.Dorff [3])

Although, some results on convolution of harmonic functions are available in literature, but still very little is known in this direction. Clunie and Shiel-Small [1] proved that if $\phi \in K$ and $F \in K_H$ then,

$$(\alpha\overline{\phi} + \phi) * F \in C_H \quad (|\alpha| \leq 1)$$

They posed a question: if $F \in K_H$, then what is the collection of harmonic functions f, such that $F * f \in K_H$? Ruscheweyh and Salinas [5] presented a partial reply to their question. They proved that if ϕ is analytic in the unit disk E then $F * \phi = Re(F) * \phi + \overline{Im(F) * \phi} \in K_H$ for all $F \in K_H$ iff for each real number γ, the function $(\phi + i\gamma z\phi')$ is convex in the direction of imaginary axis.(A domain Ω is said to be convex in direction $\phi, 0 \leq \phi < \pi$, if every line parallel to the line through 0 and $e^{i\phi}$ has a connected intersection with Ω).

In this paper, we investigate the properties of the Hadamard products of a function $F \in K_H$ (K_H^0) with some other harmonic functions f defined by some coefficient conditions and prove that such Hadamard products belong to some well known subclasses of univalent harmonic functions.

2 Preliminaries

We shall need the following results to prove our main theorems.

Lemma 2.1 *If $F \in K_H$ is given by (1), then for $n \in \mathbb{N}$*

$$|A_n| \leq \frac{n-1}{2}|B_1| + \frac{n+1}{2} \quad \text{and} \quad |B_n| \leq \frac{n-1}{2} + \frac{n+1}{2}|B_1|.$$

In particular, for $n = 2, 3, 4...$

$$|A_n| < n \quad \text{and} \quad |B_n| < n.$$

Lemma 2.2 *Let $F = H + \overline{G}$ be locally univalent in E and let $H + \epsilon G$ be convex for some ϵ $(|\epsilon| \leq 1)$. Then F is univalent close-to-convex in E.*

Lemma 2.3 *If $f = h+\overline{g}$, of the form (2), satisfies $\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq 1$, then $f \in K_H$ (or K_H^0 if $b_1 = 0$).*

Lemma 2.4 *Let $f = h+\overline{g}$, of the form (2), satisfy $\sum_{n=2}^{\infty} n|a_n|+\sum_{n=1}^{\infty} n|b_n| \leq 1$. Then $f \in S_H^{*}$(or S_H^{*0} if $b_1 = 0$).*

Lemma 2.5 *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in E,then f maps onto a convex domain if $\sum_{n=2}^{\infty} n^2|a_n| \leq 1$.*

Lemma 2.1 and 2.2 are due to Clunie and Shiel-Small [1] whereas Lemma 2.3 and 2.4 are due to Silverman [8] and Lemma 2.5 is by Silverman [7].

3 Main Results

To begin with, in the following theorem, we identify a class of harmonic functions f such that $f * F \in K_H^0$ for all $F \in K_H^0$

Theorem 3.1 *If a harmonic function f, where*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b}_n \overline{z}^n \tag{3}$$

satisfies

$$\sum_{n=2}^{\infty} n^3(|a_n| + |b_n|) \leq 1, \tag{4}$$

then $F * f \in K_H^0$ for all $F \in K_H^0$.

Proof. Let F given by

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B}_n \overline{z}^n \tag{5}$$

be any member of the class K_H^0. Then

$$\begin{aligned} F * f &= H * h + \overline{G * g} \\ &= z + \textstyle\sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=2}^{\infty} \overline{b_n B_n} \overline{z}^n \end{aligned}$$

Since

$$\begin{aligned} \textstyle\sum_{n=2}^{\infty} n^2(|a_n A_n| + |b_n B_n|) &= \textstyle\sum_{n=2}^{\infty} n^2(|a_n||A_n| + |b_n||B_n|) \\ &< \textstyle\sum_{n=2}^{\infty} n^2(\frac{n+1}{2}|a_n| + \frac{n-1}{2}|b_n|) \quad \text{(using Lemma 2.1, with } B_1 = 0) \\ &< \textstyle\sum_{n=2}^{\infty} n^2(n|a_n| + n|b_n|) \quad (\text{since } \frac{n+1}{2} < n, \frac{n-1}{2} < n) \\ &= \textstyle\sum_{n=2}^{\infty} n^3(|a_n| + |b_n|) \\ &\leq 1, \end{aligned}$$

in view of given condition (4). The result, now, follows by Lemma 2.3 (with $b_1 = 0$).

In the example below, we show that there do exist harmonic functions which satisfy the criteria in above theorem.

Example 3.2 *Let $f = z + \frac{1}{2^3}\overline{z}^2$ be a harmonic polynomial. Clearly, coefficients of f satisfy the condition (4) of Theorem 3.1. Now, let $F = H + \overline{G}$ be the right half-plane mapping such that, $H+G = \frac{z}{1-z}$ and dilatation $w(z) = -z$, which maps the unit disk E onto $R = \{w : Re(w) > -1/2\}$. A simple calculation gives*

$$H(z) = z + \sum_{n=2}^{\infty} \frac{1+n}{2} z^n \quad \text{and} \quad G(z) = \sum_{n=2}^{\infty} \frac{1-n}{2} z^n.$$

Then, obviously

$$F * f = z - \frac{1}{2^4}\overline{z^2},$$

satisfies the coefficient condition in Lemma 2.3 (with $b_1 = 0$). So, $F * f \in K_H^0$.

Next example (see Dorff [2]) shows that if coefficients of f do not satisfy condition (4) then $F * f \notin K_H^0$ for some $F \in K_H^0$. In fact, convolution may not even be univalent in E.

Example 3.3 *Let $f = h + \overline{g}$ be the right half-plane mapping as given in Example 3.2. So,*

$$h(z) = z + \sum_{n=2}^{\infty} \frac{1+n}{2} z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} \frac{1-n}{2} z^n.$$

Now

$$\begin{aligned} \sum_{n=2}^{\infty} n^3(|a_n| + |b_n|) &= \sum_{n=2}^{\infty} n^3\left(\left|\tfrac{1+n}{2}\right| + \left|\tfrac{1-n}{2}\right|\right) \\ &= \sum_{n=2}^{\infty} n^3\left(\tfrac{1+n}{2} + \tfrac{n-1}{2}\right) \\ &= \sum_{n=2}^{\infty} n^4 \\ &\not< 1. \end{aligned}$$

Therefore, coefficients of f do not satisfy condition (4). Let $F = H + \overline{G} \in K_H^0$ be a harmonic mapping that maps the unit disk E onto a $6-gon$, where

$$H(z) = z + \sum_{n=1}^{\infty} \frac{1}{6n+1} z^{6n+1} \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} \frac{-1}{6n-1} z^{6n-1}.$$

Then

$$\left|\frac{(G(z)*g(z))'}{(H(z)*h(z))'}\right| = \left|\frac{z^4(2+z^6)}{1+2z^6}\right| \not< 1.$$

It shows that the function $F*f$ is not sense-preserving. Hence, $F*f \notin K_H^0$.

Theorem 3.4 *Let f be a harmonic function of the form (3) whose coefficients satisfy*

$$\sum_{n=2}^{\infty} n^2(|a_n|+|b_n|) \leq 1.$$

Then, $F*f \in S_H^{*0}$ for all $F \in K_H^0$.

Proof. Proceeding as in proof of Theorem 3.1, we get

$$\begin{aligned}\textstyle\sum_{n=2}^{\infty} n(|a_nA_n|+|b_nB_n|) &= \textstyle\sum_{n=2}^{\infty} n(|a_n||A_n|+|b_n||B_n|)\\ &< \textstyle\sum_{n=2}^{\infty} n(n|a_n|+n|b_n|) \quad \text{(using Lemma 2.1)}\\ &= \textstyle\sum_{n=2}^{\infty} n^2(|a_n|+|b_n|)\\ &\leq 1.\end{aligned}$$

Proof, now, follows by Lemma 2.4.

The result in Theorem 3.1 can be extended to the class K_H by using the coefficients bounds as given in Lemma 2.1.

Theorem 3.5 *If f, given by (2), is any harmonic function such that*

$$\sum_{n=2}^{\infty} n^3(|a_n|+|b_n|) \leq 1-|b_1|. \tag{6}$$

Then, $F*f \in K_H$ for any $F \in K_H$.

Proof. Let F given by (1) be any function in K_H.Then

$$\begin{aligned}F*f &= H*h+\overline{G*g}\\ &= z+\textstyle\sum_{n=2}^{\infty} a_nA_nz^n+\sum_{n=1}^{\infty}\overline{b_nB_n}\overline{z}^n.\end{aligned}$$

Since

$$\begin{aligned}\textstyle\sum_{n=2}^{\infty} n^2|a_nA_n|+\sum_{n=1}^{\infty} n^2|b_nB_n| &= \textstyle\sum_{n=2}^{\infty} n^2|a_n||A_n|+\sum_{n=1}^{\infty} n^2|b_n||B_n|\\ &= \textstyle\sum_{n=2}^{\infty} n^2|a_n||A_n|+\sum_{n=2}^{\infty} n^2|b_n||B_n|+|b_1B_1|\\ &< \textstyle\sum_{n=2}^{\infty} n^2(n|a_n|+n|b_n|)+|b_1B_1| \quad \text{(using Lemma 2.1)}\\ &< \textstyle\sum_{n=2}^{\infty} n^3(|a_n|+|b_n|)+|b_1|\\ &\leq 1,\end{aligned}$$

in view of given condition (6). Here, since F is sense-preserving, therefore $|B_1| < 1$. Hence, by Lemma 2.3, $F * f \in K_H$

Similarly, an application of Lemma 2.4 immediately gives

Theorem 3.6 *Let $F \in K_H$ and let f be a harmonic function given as in (2) which satisfies*

$$\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1 - |b_1|.$$

Then, $F * f \in S_H^*$.

Remark 3.7 *We denote by $S_H^{*0}(\alpha)$ and $K_H^0(\alpha)$, the subclasses of S_H^0 consisting of starlike and convex functions of order α, $(0 \leq \alpha < 1)$ respectively. Jahangiri [4] proved that $f \in S_H^{*0}(\alpha)$ if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n| \leq 1$$

and $f \in K_H^0(\alpha)$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1.$$

We, now, state (without proof) the following two theorems pertaining to the classes $S_H^{*0}(\alpha)$ and $K_H^0(\alpha)$.

Theorem 3.8 *Let f be a harmonic function given as in (3) which satisfies*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1.$$

Then, $F * f \in S_H^{*0}(\alpha)$, for every $F \in K_H^0$.

Theorem 3.9 *Let f be harmonic function given as in (3) such that*

$$\sum_{n=2}^{\infty} \frac{n^2(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=2}^{\infty} \frac{n^2(n+\alpha)}{1-\alpha}|b_n| \leq 1.$$

Then, $F * f \in K_H^0(\alpha)$ for all $F \in K_H^0$.

We close this paper by presenting a class of harmonic functions whose convolutions with functions in K_H are close-to-convex harmonic.

Theorem 3.10 *Let f be a harmonic function given by (2) for which*

$$\sum_{n=2}^{\infty} n^3|a_n| \leq 1.$$

Assume that F, given by (1), belongs to K_H. If $F * f$ is locally univalent, then $F * f$ is close to convex harmonic.

Proof: We write the convolution of F and f as

$$\begin{aligned} F * f &= H * h + \overline{G * g} \\ &= H_1 + \overline{G_1} \quad \text{(say)}. \end{aligned}$$

We will show that H_1 is convex. The result will, then, follow by Lemma 2.2 (with $\epsilon = 0$). Now

$$\begin{aligned} \textstyle\sum_{n=2}^{\infty} n^2|a_n A_n| &= \textstyle\sum_{n=2}^{\infty} n^2|a_n||A_n| \\ &< \textstyle\sum_{n=2}^{\infty} n^2|na_n| \quad \text{(using Lemma 2.1)} \\ &= \textstyle\sum_{n=2}^{\infty} n^3|a_n| \\ &\leq 1 \quad \text{(given)}. \end{aligned}$$

Therefore, by Lemma 2.5, H_1 is convex. Hence the result.

4 Open Problem

In this paper, we investigated convolution properties of univalent harmonic convex functions only. Study of convolution properties of functions from classes S_H^* and C_H is still an open problem.

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