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Meromorphic Functions That Share One Small Function CM or IM with Their First Derivative

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Abstract

In this paper we obtain a unicity theorem of a meromorphic function and its first derivative that share one small function CM or IM. So we generalize some results given in [1], [2] and [3].

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1 Introduction

A meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, $m(r, f)$ etc (see [4], [5]). By $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r with finite linear measure. Then the meromorphic function β is called a small function of f if $T(r, \beta) = S(r, f)$. We say that two non-constant meromorphic functions f and g share a small function β IM (ignoring multiplicities), if f and g have the same β-points. If f and g have the same β-points with the same multiplicities, we say that f and g share the small function β CM (counting multiplicities). Let k be a positive integer, and let b be a small function of f or ∞, we denote by $N_{k)}(r, \frac{1}{f-b})$ the counting function of b-points of f with multiplicity $\leq k$ and by $N_{(k}(r, \frac{1}{f-b})$ the counting function of b-points of f with multiplicity $> k$. In like manner we define $\bar{N}_{k)}(r, \frac{1}{f-b})$ and $\bar{N}_{(k}(r, \frac{1}{f-b})$ where in counting the b-points of f we ignore the multiplicities.

In [2] G. G. Gundersen proved the following theorem:

Theorem 1.1 *Let f be a non-constant meromorphic function. If f and f' share two distinct values $0, a \neq \infty$ CM, then $f \equiv f'$*

In 2009, A. H. H. Al-Khaladi [1] proved the following theorems which are improvement and extension of Theorem 1.1:

Theorem 1.2 *Let f be a non-constant meromorphic function. If f and f' share the value $a \neq 0, \infty$ CM and if $\bar{N}(r, \frac{1}{f}) = S(r, f)$, then either $f \equiv f'$ or $f(z) = \frac{az+A}{1+ce^{-z}}$, where A and $c \neq 0$ are constants,*

Theorem 1.3 *Let f be a non-constant meromorphic function. If f and f' share the value $a \neq 0, \infty$ IM and if $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r, f)$, then either $f \equiv f'$ or*

$$f(z) = \frac{2a}{1 + ce^{-2z}}, \tag{1}$$

where c is a nonzero constant.

On the other hand, Q. C. Zhang [3] proved the following theorem:

Theorem 1.4 *Let f be a non-constant meromorphic function, a be a nonzero finite complex constant. If f and f' share 0 CM, and share a IM, then $f \equiv f'$ or f is given as (1).*

In this paper we will generalize the above results (Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4).

2 Main Results

Lemma 2.1 *Let f' be a non-constant meromorphic function, and let β be a small function of f' such that $\beta' \equiv \beta \not\equiv 0, \infty$. Then*

$$m\Big(r, \frac{1}{f' - \beta}\Big) \leq 2\bar{N}\Big(r, \frac{1}{f'}\Big) + 2\bar{N}_{(2}(r, f) + S(r, f')$$

Proof Set

$$W = \Big(\frac{F'}{F}\Big)^2 - 2\Big(\frac{F'}{F}\Big)' + 2\frac{F'}{F}, \tag{2}$$

where $F = \frac{f'}{\beta}$. Then from Nevanlinna's fundamental estimate of the logarithmic derivative we obtain

$$m(r,W) \leq 4m\Big(r,\frac{F'}{F}\Big) + S\Big(r,\frac{F'}{F}\Big) + O(1) = S(r,F) + S\Big(r,\frac{F'}{F}\Big).$$

Since

$$\begin{aligned} T\Big(r,\frac{F'}{F}\Big) &= N\Big(r,\frac{F'}{F}\Big) + m\Big(r,\frac{F'}{F}\Big) \leq \bar{N}(r,F) + \bar{N}\Big(r,\frac{1}{F}\Big) + S(r,F) \\ &\leq 2T(r,F) + S(r,F), \end{aligned}$$

this means that

$$m(r,W) = S(r,F) = S(r,f'). \tag{3}$$

Suppose that z_∞ is a simple pole of f. Then the Laurent expansion of f about z_∞ is

$$f(z) = a_{-1}(z - z_\infty)^{-1} + O(1)$$

where a_{-1} be the residue of f at z_∞. Hence

$$\frac{F'}{F} = -2(z - z_\infty)^{-1} - 1 + O(z - z_\infty).$$

Substitution of this into (2) gives

$$W(z_\infty) = O(1). \tag{4}$$

It follows from (2) that the poles of f with multiplicity $p \geq 2$ are poles of W with multiplicity 2 at most. We can also conclude from (2) that the zeros of f' with multiplicity $q \geq 1$ are poles of W with multiplicity 2. Thus, from (4) we get

$$N(r,W) \leq 2\bar{N}_{(2}(r,f) + 2\bar{N}\Big(r,\frac{1}{f'}\Big). \tag{5}$$

We distinguish the following the two cases:

Case 1. $W \not\equiv 0$. We write (2) in the form

$$\frac{1}{F-1} = \frac{1}{W}\Big(\frac{F'}{F-1} - \frac{F'}{F}\Big)\Big(\frac{3F'}{F} - \frac{2F''}{F'} + 2\Big).$$

Then it is clear that

$$\begin{aligned} m\Big(r,\frac{1}{F-1}\Big) &\leq m\Big(r,\frac{1}{W}\Big) + S(r,F) \leq T(r,W) + S(r,F) \\ &\leq m(r,W) + N(r,W) + S(r,f'). \end{aligned}$$

Combining this with (3) and (5), we have

$$m\Big(r,\frac{1}{F-1}\Big) \leq 2\bar{N}_{(2}(r,f) + 2\bar{N}\Big(r,\frac{1}{f'}\Big) + S(r,f').$$

That is,
$$m\Big(r, \frac{1}{f'-\beta}\Big) \le 2\bar{N}_{(2}(r,f) + 2\bar{N}\Big(r, \frac{1}{f'}\Big) + S(r,f').$$

Case 2. $W \equiv 0$. If $\frac{F'}{F} \equiv 0$, then $f' = c\beta$ and so $T(r,f') = S(r,f')$ a contradiction. Therefore $\frac{F'}{F} \not\equiv 0$ and (2) becomes

$$\frac{y'}{y+2} - \frac{y'}{y} = \frac{1}{2}, \tag{6}$$

where $y = \frac{F'}{F}$. Integrating (6) twice we obtain

$$f' = \beta A\Big(c - e^{-\frac{1}{2}z}\Big)^4,$$

where A and $c \neq 0$ are constants. So

$$T(r,f') = 4T\Big(r, e^{-\frac{1}{2}z}\Big) + S(r,f').$$

But
$$T(r,\beta) = 2T\Big(r, e^{-\frac{1}{2}z}\Big) + O(1).$$

Therefore
$$T(r,f') = 2T(r,\beta) + S(r,f') = S(r,f').$$

This is a contradiction. □

The following lemma belongs to [4].

Lemma 2.2 *Let f be a non-constant meromorphic function, and a_1, a_2, a_3 be distinct small functions of f. Then*

$$T(r,f) \le \sum_{j=1}^{3} \bar{N}\Big(\frac{1}{f-a_j}\Big) + S(r,f).$$

Theorem 2.3 *Let f be a non-constant meromorphic function, and let β be a small meromorphic function of f such that $\beta \not\equiv 0, \infty$. If f and f' share β CM and if $\bar{N}(r, \frac{1}{f}) = S(r,f)$, then either $f \equiv f'$ or*

$$f(z) = \frac{\int_0^z \beta(t)dt + A}{1 + ce^{-z}}, \tag{7}$$

where A and $c \neq 0$ are constants.

Proof Suppose that $f \not\equiv f'$ and let Ω be the function defined by

$$\begin{aligned}\Omega &= \frac{1}{f}\Big[\frac{(f'/\beta)'}{f'-\beta} - \frac{(f/\beta)'}{f-\beta}\Big] \\ &= \frac{1}{\beta^2}\Big[\frac{f'}{f}\Big(\frac{(f'/\beta)'}{(f'/\beta)-1} - \frac{(f'/\beta)'}{f'/\beta}\Big) - \Big(\frac{(f/\beta)'}{(f/\beta)-1} - \frac{(f/\beta)'}{f/\beta}\Big)\Big]. \qquad (8)\end{aligned}$$

Then from Nevanlinna's fundamental estimate of the logarithmic derivative we obtain

$$\begin{aligned}m(r,\Omega) &\le m\Big(r,\frac{1}{\beta^2}\Big) + m\Big(r,\frac{f'}{f}\Big) + m\Big(r,\frac{(f'/\beta)'}{(f'/\beta)-1}\Big) + \\ &\quad m\Big(r,\frac{(f'/\beta)'}{f'/\beta}\Big) + m\Big(r,\frac{(f/\beta)'}{(f/\beta)-1}\Big) + m\Big(r,\frac{(f/\beta)'}{f/\beta}\Big) + O(1) \\ &= S(r,f) + S(r,f').\end{aligned}$$

Since

$$T(r,f') \le 2T(r,f) + S(r,f),$$

this means that

$$m(r,\Omega) = S(r,f). \qquad (9)$$

It follows from (8) that if z_∞ is a pole of f with multiplicity $p \ge 1$ and $\beta(z_\infty) \ne 0, \infty$, then

$$\Omega(z) = O\Big((z-z_\infty)^{p-1}\Big). \qquad (10)$$

Since f and f' share β CM, we find from (8) that Ω is holomorphic at the zeros of $f-\beta$ and $f'-\beta$. Thus the pole of Ω can only occur at zeros of f. However the zeros of f with multiplicity $q \ge 2$ are pole of Ω with multiplicity 2. Thus, from $\bar{N}(r,\frac{1}{f}) = S(r,f)$ we get

$$\begin{aligned}N(r,\Omega) &\le \bar{N}\Big(r,\frac{1}{f}\Big) + \bar{N}_{(2}\Big(r,\frac{1}{f}\Big) + S(r,f) \\ &\le 2\bar{N}\Big(r,\frac{1}{f}\Big) + S(r,f) = S(r,f).\end{aligned}$$

Together with (9) we have

$$T(r,\Omega) = m(r,\Omega) + N(r,\Omega) = S(r,f). \qquad (11)$$

If $\Omega \equiv 0$, then from integration of (8) we get $f-\beta = c(f'-\beta)$, where c is some nonzero constant. This implies that $\bar{N}(r,f) = S(r,f)$. If $c=1$, then $f \equiv f'$, a contradiction. Therefore $c \ne 1$ and so

$$\frac{1}{f} = \frac{c}{\beta(c-1)}\Big(\frac{f'}{f} - 1\Big).$$

Hence, we obtain

$$\begin{aligned} T(r,f) &\leq T\Big(r,\frac{f'}{f}\Big)+S(r,f)=N\Big(r,\frac{f'}{f}\Big)+m\Big(r,\frac{f'}{f}\Big)+S(r,f)\\ &\leq \bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}(r,f)+S(r,f)=S(r,f), \end{aligned}$$

which is impossible. Therefore, we obtain $\Omega\not\equiv 0$. Writing (8) as

$$f=\frac{1}{\beta\Omega}\Big[\frac{(f'/\beta)'}{(f'/\beta)-1}-\frac{(f/\beta)'}{(f/\beta)-1}\Big].$$

Consequently, from (11),

$$m(r,f)\leq m\Big(r,\frac{1}{\beta}\Big)+m\Big(r,\frac{1}{\Omega}\Big)+S(r,f)\leq m\Big(r,\frac{1}{\Omega}\Big)+S(r,f) \tag{12}$$

$$\leq T(r,\Omega)+S(r,f)=S(r,f). \tag{13}$$

Furthermore, from (10) and (11) we deduce that

$$\begin{aligned} N_{(2}(r,f)-\bar{N}_{(2}(r,f) &\leq N\Big(r,\frac{1}{\Omega}\Big)+S(r,f)\\ &\leq T(r,\Omega)+S(r,f)=S(r,f), \end{aligned}$$

so that,

$$N_{(2}(r,f)=S(r,f). \tag{14}$$

We set

$$\omega=\frac{f'-f}{f(f-\beta)}=\frac{1}{f-\beta}\Big(\frac{f'}{f}-1\Big). \tag{15}$$

Then

$$\begin{aligned} m(r,\omega) &\leq m\Big(r,\frac{1}{f-\beta}\Big)+m\Big(r,\frac{f'}{f}\Big)+O(1)\\ &= m\Big(r,\frac{1}{f-\beta}\Big)+S(r,f). \end{aligned} \tag{16}$$

Since f and f' share β CM, from (15) we deduce that ω is holomorphic at the zeros of $f-\beta$. Also it is clear that the poles of f being not the poles of ω. Thus,

$$N(r,\omega)\leq \bar{N}\Big(r,\frac{1}{f}\Big)+S(r,f)=S(r,f). \tag{17}$$

Further, if z_∞ is a simple pole of f and $\beta(z_\infty)\neq 0,\infty$, by a simple computation, we deduce from (8) and (15) that

$$\Omega(z_\infty)=\frac{-1}{\beta(z_\infty)a_{-1}} \quad \textit{and} \quad \omega(z_\infty)=\frac{-1}{a_{-1}}, \tag{18}$$

where a_{-1} be the residue of f at z_∞. In the following we shall treat two cases $\beta\Omega \equiv \omega$ and $\beta\Omega \not\equiv \omega$ separately.

Case 1. $\beta\Omega \equiv \omega$. From (8) and (15) we know that if

$$h = \frac{f' - \beta}{f - \beta} = \frac{f'/\beta - 1}{f/\beta - 1}, \tag{19}$$

$$\beta\Omega = \frac{1}{f}\Big(\frac{h'}{h}\Big) \quad \textit{and} \quad \omega = \frac{1}{f}(h-1).$$

Hence,

$$\frac{h'}{h-1} - \frac{h'}{h} = 1.$$

By integration, we get $h(z) = \frac{1}{1-ce^z}$, where c nonzero constant. Combining this with (19) yields

$$f' - \frac{1}{1-ce^z} f = \frac{-c\beta e^z}{1-ce^z},$$

which leads to

$$\frac{d}{dz}\Big[f(z)\Big(1 - \frac{1}{c}e^{-z}\Big)\Big] = \beta(z).$$

From this we arrive at (7).

Case 2. $\beta\Omega \not\equiv \omega$. Then from (18), (11), (16) and (17) we see that

$$\begin{aligned} N_{1)}(r,f) &\le N\Big(r, \frac{1}{\beta\Omega - \omega}\Big) \le T(r, \beta\Omega - \omega) + O(1) \\ &\le T(r,\Omega) + T(r,\omega) + S(r,f) \le m\Big(r, \frac{1}{f-\beta}\Big) + S(r,f). \end{aligned}$$

Combining this, (14) and (13), we obtain

$$\begin{aligned} T(r,f) &= m(r,f) + N(r,f) = N_{1)}(r,f) + S(r,f) \\ &\le m\Big(r, \frac{1}{f-\beta}\Big) + S(r,f). \end{aligned}$$

Hence, we find that

$$N\Big(r, \frac{1}{f-\beta}\Big) = S(r,f). \tag{20}$$

We define

$$\mu = \frac{(f/\beta)'}{f(f-\beta)} = \frac{1}{\beta^2}\Big[\frac{(f/\beta)'}{(f/\beta)-1} - \frac{(f/\beta)'}{f/\beta}\Big]. \tag{21}$$

Then it is clear that

$$m(r,\mu) = S(r,f). \tag{22}$$

If z_∞ is a simple pole of f and $\beta(z_\infty)\neq 0,\infty$, by a simple calculation on the local expansions we find that

$$\mu(z_\infty)=\frac{-1}{\beta(z_\infty)a_{-1}}. \tag{23}$$

Thus, it can be obtained from (22), (23), (14), (20) and $\bar{N}(r,\frac{1}{f})=S(r,f)$ that

$$\begin{aligned}T(r,\mu)&=m(r,\mu)+N(r,\mu)=N(r,\mu)+S(r,f)\\&\leq \bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)=S(r,f).\end{aligned} \tag{24}$$

Further, from (23) and (18) we have

$$\Omega(z_\infty)=\mu(z_\infty). \tag{25}$$

If $\Omega\equiv\mu$, we know from (8) and (21) that

$$2\frac{(f/\beta)'}{f/\beta-1}=\frac{(f'/\beta)'}{f'/\beta-1}.$$

By integration once, $(f-\beta)^2=c\beta(f'-\beta)$, where c is a nonzero constant. We rewrite this in the form

$$\frac{\beta'-\beta}{f-\beta}=\frac{f-\beta}{c\beta}-\frac{(f-\beta)'}{f-\beta}. \tag{26}$$

If $\beta'-\beta\not\equiv 0$, from this, (13) and (20) we conclude that

$$\begin{aligned}T(r,f)&=T\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)\\&=m\Big(r,\frac{1}{f-\beta}\Big)+N\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)\\&\leq m\Big(r,\frac{1}{\beta'-\beta}\Big)+m(r,f)+N\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)\\&=S(r,f)\end{aligned}$$

This is impossible. Therefore we have $\beta'-\beta\equiv 0$, and (26) becomes

$$\frac{(f'-\beta)'}{(f-\beta)^2}=\frac{\beta'}{c\beta^2}.$$

By integration, we get

$$\frac{-1}{f-\beta}=\frac{-1}{c\beta}+A,$$

where A is a constant. So $T(r,f) = S(r,f)$, a cotradiction. Thus $\Omega \not\equiv \mu$. It follows from this, (13), (14), (25), (11) and (24) that

$$\begin{aligned} T(r,f) &= N(r,f) + m(r,f) = N_{1)}(r,f) + N_{(2}(r,f) + m(r,f) \\ &= N_{1)}(r,f) + S(r,f) \leq N\Big(r, \frac{1}{\Omega - \mu}\Big) + S(r,f) \\ &\leq T(r,\Omega) + T(r,\mu) + S(r,f) = S(r,f). \end{aligned}$$

This is impossible. The proof of Theorem 2.3 is complete. □

Theorem 2.4 *Let f be a non-constant meromorphic function, and let β be a small meromorphic function of f such that $\beta \not\equiv 0, \infty$. If f and f' share β IM and if $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r,f)$, then either $f \equiv f'$ or β is a constant and f is given as (1) when $\beta = a$.*

Proof In the following, we assume that $f \not\equiv f'$. Suppose z_0 is a zero of $f - \beta$ with multiplicity $n \geq 1$ and $\beta(z_\infty) \neq 0, \infty$. Then the Taylor expansion of $f - \beta$ about z_0 is

$$f(z) - \beta = a_n(z - z_0)^n + \dots, \quad a_n \neq 0. \tag{27}$$

Since f and f' share β IM,

$$f'(z) - \beta = b_m(z - z_0)^m + \dots, \quad b_m \neq 0. \tag{28}$$

Differentiating (27) and then using (28), we obtain

$$\beta(z) - \beta'(z) = na_n(z - z_0)^{n-1} - b_m(z - z_0)^m + \dots. \tag{29}$$

We consider the following two cases.

Case I. $\beta - \beta' \not\equiv 0$. Then we get from (29) that

$$\begin{aligned} \bar{N}_{(2}\Big(r, \frac{1}{f - \beta}\Big) &\leq N\Big(r, \frac{1}{\beta' - \beta}\Big) + S(r,f) \leq T(r, \beta' - \beta) + S(r,f) \\ &\leq 3T(r,\beta) + S(r,f) = S(r,f). \end{aligned} \tag{30}$$

If z_0 is a simple zero of $f - \beta$ and $f' - \beta$, from (8) we see that Ω is holomorphic at z_0. It follows from this, (8), (10), f and f' share β IM, $\bar{N}(r, \frac{1}{f}) = S(r,f)$ and (30) that

$$\begin{aligned} N(r,\Omega) &\leq \bar{N}\Big(r, \frac{1}{f}\Big) + \bar{N}_{(2}\Big(r, \frac{1}{f}\Big) + \bar{N}_{(2}\Big(r, \frac{1}{f' - \beta}\Big) + S(r,f) \\ &\leq \bar{N}_{(2}\Big(r, \frac{1}{f' - \beta}\Big) + S(r,f). \end{aligned}$$

Combining with (9) we obtain

$$T(r,\Omega) \le \bar{N}_{(2}\Big(r, \frac{1}{f'-\beta}\Big) + S(r,f). \tag{31}$$

Also we know from (10), (12) and (31) that

$$\begin{aligned} N_{(2}(r,f) - \bar{N}_{(2}(r,f) &\le N\Big(r, \frac{1}{\Omega}\Big) + S(r,f) \\ &\le T(r,\Omega) - m\Big(r,\frac{1}{\Omega}\Big) + S(r,f) \\ &\le \bar{N}_{(2}\Big(r, \frac{1}{f'-\beta}\Big) - m(r,f) + S(r,f). \end{aligned} \tag{32}$$

We set

$$H = \frac{(f'/\beta)'(f-\beta)}{f'(f'-\beta)} = \frac{f-\beta}{\beta^2}\Big[\frac{(f'/\beta)'}{(f'/\beta)-1} - \frac{(f'/\beta)'}{f'/\beta}\Big]. \tag{33}$$

Then it is clear that

$$m(r,H) \le m(r,f) + S(r,f). \tag{34}$$

From (33) we deduce that if z_∞ is a pole of f with multiplicity $p \ge 1$ and $\beta(z_\infty) \ne 0, \infty$,

$$H(z_\infty) = \frac{1}{\beta(z_\infty)}\Big(\frac{p+1}{p}\Big). \tag{35}$$

Substituting (27) and (28) into (33) gives

$$H(z) = O\Big((z-z_\infty)^{n-1}\Big). \tag{36}$$

Thus the pole of H can only occur at zeros of f'. However, the zeros of f' with multiplicity $s \ge 1$ are poles of H with multiplicity 1. Therefore from this, (35), (36) and $\bar{N}(r, \frac{1}{f'}) = S(r,f)$ we get

$$N(r,H) \le \bar{N}\Big(r, \frac{1}{f'}\Big) + S(r,f) = S(r,f).$$

Together with (34) we have

$$T(r,H) \le m(r,f) + S(r,f). \tag{37}$$

If z_∞ is a simple pole of f, then by (35) there are two cases.

Case 1. $H \equiv \frac{2}{\beta}$. This and (33) imply that

$$\frac{1}{f-\beta} = \frac{1}{2\beta}\Big[\frac{(f'/\beta)'}{(f'/\beta)-1} - \frac{(f'/\beta)'}{f'/\beta}\Big]. \tag{38}$$

Obviously, by logarithmic derivative lemma $m(r, \frac{1}{f-\beta}) = S(r, f)$. Combining with (16) we get

$$m(r, \omega) = S(r, f). \tag{39}$$

From (38), (27) and (28) we know that if z_0 is a zero of $f - \beta$ with multiplicity $n \geq 1$ and $\beta(z_0) \neq 0, \infty$, then $n = 1$. In addition since f and f' share β IM, from (15) we see ω is holomorphic at z_0. Also it is easily verified that the pole of f being not the pole of ω. Thus, from $\bar{N}(r, \frac{1}{f}) = S(r, f)$ we obtain

$$N(r, \omega) \leq \bar{N}\Big(r, \frac{1}{f}\Big) + S(r, f) = S(r, f).$$

Together with (39) we have

$$T(r, \omega) = S(r, f). \tag{40}$$

If $\omega \equiv 0$, then $f \equiv f'$ a contradiction. In the following we assume $\omega \not\equiv 0$. Further, it can be obtained from (15), (40) and $\bar{N}(r, \frac{1}{f}) = S(r, f)$ that

$$\begin{aligned} T(r, f) &\leq T\Big(r, \frac{1}{\omega}\Big) + T\Big(r, \frac{f'}{f}\Big) + S(r, f) \\ &= N\Big(r, \frac{f'}{f}\Big) + m\Big(r, \frac{f'}{f}\Big) + S(r, f) \\ &\leq \bar{N}\Big(r, \frac{1}{f}\Big) + \bar{N}(r, f) + S(r, f) \\ &= \bar{N}(r, f) + S(r, f), \end{aligned}$$

which shows that

$$T(r, f) = N_{1)}(r, f) + S(r, f). \tag{41}$$

Using an argument similar to that in the proof of Theorem 2.3, we can conclude that $\beta\Omega \equiv \omega$ or $\beta\Omega \not\equiv \omega$. If $\beta\Omega \equiv \omega$, then (see Case 1 in the proof of Theorem 2.3)

$$f(z) = \frac{\int_0^z \beta(t)dt + A}{1 + ce^{-z}}.$$

Hence

$$f'(z) = \frac{\beta + \Big(\beta + A + \int_0^z \beta(t)dt\Big)ce^{-z}}{1 + ce^{-z}}.$$

Since $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r, f)$, we must have $\beta + A + \int_0^z \beta(z)dz \equiv 0$. Differentiating this we obtain $\beta' + \beta \equiv 0$. Integrating once, $\beta(z) = c_1 e^{-z}$, where c_1 is a nonzero constant. So β can not small function of f. Therefore, we get $\beta\Omega \not\equiv \omega$. It follows from this, (41), (18), (40) and (31) that

$$\begin{aligned} T(r,f) &= N_{1)}(r,f) + S(r,f) \leq N\Big(r, \frac{1}{\beta\Omega - \omega}\Big) + S(r,f) \\ &\leq T(r,\beta) + T(r,\Omega) + T(r,\omega) + S(r,f) \\ &\leq \bar{N}_{(2}\Big(r, \frac{1}{f'-\beta}\Big) + S(r,f). \end{aligned} \tag{42}$$

Since f and f' share β IM,

$$\bar{N}\Big(r, \frac{1}{f-\beta}\Big) = \bar{N}\Big(r, \frac{1}{f'-\beta}\Big).$$

By this and (42) we have

$$N_{1)}\Big(r, \frac{1}{f'-\beta}\Big) = S(r,f). \tag{43}$$

If we rewrite (38) and (15) in the form

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = 2\Big(\frac{f'-\beta}{f-\beta}\Big) \quad \textit{and} \quad f\omega + 1 = \frac{f'-\beta}{f-\beta}$$

respectively, and then elimination $\frac{f'-\beta}{f-\beta}$ between the last two equations we obtain

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = 2(f\omega + 1). \tag{44}$$

Let z_0 be a zero of $f' - \beta$ with multiplicity $m \geq 2$ and $\beta(z_0) \neq 0, \infty$. By (27), (28) and (44) we find that

$$\beta(z_0)\omega(z_0) + 1 = 0 \tag{45}$$

If $\beta\omega + 1 \not\equiv 0$, then from this, (42), (45) and (40) we see that

$$\begin{aligned} T(r,f) &\leq \bar{N}_{(2}\Big(r, \frac{1}{f'-\beta}\Big) + S(r,f) \leq N\Big(r, \frac{1}{\beta\omega + 1}\Big) + S(r,f) \\ &\leq T(r,\beta) + T(r,\omega) + S(r,f) = S(r,f) \end{aligned}$$

This is impossible. Therefore, $\beta\omega + 1 \equiv 0$. Thus, from this, (44) and (15) we get

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = -\frac{2}{\beta}(f - \beta)$$

and

$$-\frac{1}{\beta}(f - \beta) = \frac{f'}{f} - 1 \tag{46}$$

respectively. If we now eliminate $-\frac{1}{\beta}(f-\beta)$ between the last two equations leads to

$$\frac{f''}{f'}-\frac{\beta'}{\beta}=2\Big(\frac{f'}{f}-1\Big)$$

By integrating once,

$$f'(z)=c\beta(z)f^2e^{-2z}.$$

Substituting this into (46) gives

$$f(z)=\frac{2\beta(z)}{1+c\beta^2(z)e^{-2z}}.$$

Hence,

$$f'(z)=\frac{2\beta'-2\beta^2ce^{-2z}(\beta'-2\beta)}{(1+c\beta^2e^{-2z})^2}.$$

Since $\bar{N}(r,\frac{1}{f'})=S(r,f)$, therefore we must have $\beta'\equiv 0$ and so β is a constant. Thus (1) holds when $\beta=a$.

Case 2. $H\not\equiv\frac{2}{\beta}$. Then from (35) and (37) we have

$$\begin{aligned}N_{1)}(r,f)\le N\Big(r,\frac{1}{H-\frac{2}{\beta}}\Big)+S(r,f)&\le T(r,H)+S(r,f)\\ &\le m(r,f)+S(r,f).\end{aligned}\tag{47}$$

From Lemma 2.2 ($a_1=0$, $a_2=\beta$ and $a_3=\infty$), $\bar{N}(r,\frac{1}{f'})=S(r,f)$, (47), (37) and (32) we get

$$\begin{aligned}T(r,f')&\le \bar{N}\Big(r,\frac{1}{f'}\Big)+\bar{N}\Big(r,\frac{1}{f'-\beta}\Big)+\bar{N}(r,f)+S(r,f)\\ &\le \bar{N}\Big(r,\frac{1}{f'-\beta}\Big)+\bar{N}_{1)}(r,f)+\bar{N}_{(2}(r,f)+S(r,f)\\ &\le \bar{N}\Big(r,\frac{1}{f'-\beta}\Big)+m(r,f)+\bar{N}_{(2}(r,f)+S(r,f)\\ &\le \bar{N}\Big(r,\frac{1}{f'-\beta}\Big)+\bar{N}_{(2}\Big(r,\frac{1}{f'-\beta}\Big)+S(r,f)\end{aligned}$$

which results in

$$N_{(3}\Big(r,\frac{1}{f'-\beta}\Big)=S(r,f).\tag{48}$$

Writing (15) as

$$f=\beta+\frac{1}{\omega}\Big(\frac{f'}{f}-1\Big),$$

which implies

$$m(r,f)\le m(r,\beta)+m\Big(r,\frac{1}{\omega}\Big)+m\Big(r,\frac{f'}{f}\Big)+O(1)=m\Big(r,\frac{1}{\omega}\Big)+S(r,f).\tag{49}$$

Also we know from (15) that if z_∞ is a pole of f with multiplicity $p \geq 1$, then z_∞ is a zero of ω with multiplicity $p-1$. Thus

$$N(r,f) - \bar{N}(r,f) \leq N\Big(r,\frac{1}{\omega}\Big) + S(r,f). \tag{50}$$

Combining (49), (50) and (16) we have

$$\begin{aligned} m(r,f) + N(r,f) - \bar{N}(r,f) &\leq T(r,\omega) + S(r,f) \\ &= m(r,\omega) + N(r,\omega) + S(r,f) \\ &\leq N(r,\omega) + m\Big(r,\frac{1}{f-\beta}\Big) + S(r,f). \end{aligned} \tag{51}$$

From (15), (27), (28), (50) and $\bar{N}(r,\frac{1}{f}) = S(r,f)$, we conclude that

$$\begin{aligned} N(r,\omega) &\leq \bar{N}\Big(r,\frac{1}{f}\Big) + N_{(2}\Big(r,\frac{1}{f-\beta}\Big) + S(r,f) \\ &= N_{(2}\Big(r,\frac{1}{f-\beta}\Big) + S(r,f). \end{aligned}$$

Together with (51) we get

$$\begin{aligned} m(r,f) + N_{(2}(r,f) - \bar{N}_{(2}(r,f) \leq N_{(2}\Big(r,\frac{1}{f-\beta}\Big) \\ +m\Big(r,\frac{1}{f-\beta}\Big) + S(r,f). \end{aligned} \tag{52}$$

From (31), it is easily verified that $H \not\equiv 0$ and

$$m\Big(r,\frac{1}{f-\beta}\Big) \leq m\Big(r,\frac{1}{H}\Big) + S(r,f). \tag{53}$$

By (36),

$$N_{(2}\Big(r,\frac{1}{f-\beta}\Big) - \bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big) \leq N\Big(r,\frac{1}{H}\Big) + S(r,f).$$

Combining this, (51) and (37) we obtain

$$\begin{aligned} m\Big(r,\frac{1}{f-\beta}\Big) + N_{(2}\Big(r,\frac{1}{f-\beta}\Big) - \bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big) &\leq T(r,H) + S(r,f) \\ &\leq m(r,f) + S(r,f). \end{aligned}$$

Adding this, (52) and (30) we deduce that

$$N_{(2}(r,f) = S(r,f), \tag{54}$$

and

$$m(r,f)=N_{(2}\Big(r,\frac{1}{f-\beta}\Big)+m\Big(r,\frac{1}{f-\beta}\Big)+S(r,f). \tag{55}$$

By (55) we see that

$$m(r,f)+N_{1)}\Big(r,\frac{1}{f-\beta}\Big)=T(r,f)+S(r,f).$$

Hence from (54) and (30) we get

$$\begin{aligned} N_{1)}(r,f) &= N_{1)}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)=\bar{N}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f) \\ &= \bar{N}\Big(r,\frac{1}{f'-\beta}\Big)+S(r,f). \end{aligned}$$

From this, (47) and (32) we see

$$\begin{aligned} \bar{N}\Big(r,\frac{1}{f'-\beta}\Big) &= N_{1)}(r,f)+S(r,f)\le m(r,f)+S(r,f) \\ &\le \bar{N}_{(2}\Big(r,\frac{1}{f'-\beta}\Big)+S(r,f), \end{aligned}$$

which results in

$$N_{1)}\Big(r,\frac{1}{f'-\beta}\Big)=S(r,f) \tag{56}$$

Set

$$G=\frac{1}{f}\Big[\frac{(f'/\beta)'}{f'-\beta}-2\frac{(f/\beta)'}{f-\beta}\Big]. \tag{57}$$

Similarly as (8) we have

$$m(r,G)=S(r,f). \tag{58}$$

if z_0 is a zero of $f-\beta$ and $f'-\beta$ with multiplicity 1 and 2 respectively, and $\beta(z_0)\neq 0,\infty$, then G is holomorphic. Also, if z_∞ is a simple pole of f, then by (57) we see $G(z_\infty)=0$. We discuss the following two cases.

Case 1. $G\not\equiv 0$. Then from (58), (56), (48), (54), (30) and $\bar{N}(r,\frac{1}{f})=S(r,f)$, we deduce that

$$\begin{aligned} N_{1)}(r,f) &\le N\Big(r,\frac{1}{G}\Big)+S(r,f) \\ &\le -m\Big(r,\frac{1}{G}\Big)+N(r,G)+m(r,G)+S(r,f) \\ &\le -m\Big(r,\frac{1}{G}\Big)+N(r,G)+S(r,f) \\ &\le -m\Big(r,\frac{1}{G}\Big)+S(r,f) \end{aligned}$$

which shows that

$$N_{1)}(r,f)+m\Big(r,\frac{1}{G}\Big)=S(r,f). \tag{59}$$

Notaing that

$$f=\frac{1}{G}\Big[\frac{(f'/\beta)'}{f'-\beta}-2\frac{(f/\beta)'}{f-\beta}\Big],$$

by (57). This imply,

$$m(r,f)\le m\Big(r,\frac{1}{G}\Big)+S(r,f).$$

From this, (59) and (54), we can see that $T(r,f)=S(r,f)$ a contradiction.

Case 2. $G\equiv 0$. Integrating of (57) we have

$$\beta(f'-\beta)=c(f-\beta)^2, \tag{60}$$

where c is a nonzero constant. This yields

$$\begin{aligned}2m(r,f) &= m(r,f')+S(r,f)\\ &\le m\Big(r,\frac{f'}{f}\Big)+m(r,f)+S(r,f)\\ &= m(r,f)+S(r,f)\end{aligned}$$

which means $m(r,f)+S(r,f)$. Together with (47) and (54) gives the contradiction $T(r,f)=S(r,f)$.

Case II. $\beta-\beta'\equiv 0$. Then $N_{1)}(r,\frac{1}{f-\beta})\equiv 0$ and if z_0 is a zero of $f-\beta$ with multiplicity n, then z_0 is a zero of $f'-\beta$ with multiplicity $n-1$. Using an argument similar to that in the Case I, we can deduce from (8) and (33) that $\Omega\not\equiv 0$, $H\not\equiv 0$ and

$$\begin{aligned}N_{(2}(r,f)-\bar{N}_{(2}(r,f) &\le N\Big(r,\frac{1}{\Omega}\Big)+S(r,f)\\ &\le -m\Big(r,\frac{1}{\Omega}\Big)+N(r,\Omega)+S(r,f)\\ &\le -m\Big(r,\frac{1}{\Omega}\Big)+\bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)\end{aligned} \tag{61}$$

and

$$\begin{aligned}N_{(2}\Big(r,\frac{1}{f-\beta}\Big)-\bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big) &\le N\Big(r,\frac{1}{H}\Big)+S(r,f)\\ &\le -m\Big(r,\frac{1}{H}\Big)+m(r,H)+S(r,f)\\ &\le -m\Big(r,\frac{1}{H}\Big)+m(r,f)+S(r,f).\end{aligned} \tag{62}$$

Combining (61), (12), (62) and (53) we conclude

$$N_{(2}(r,f)+N_{(3}\Big(r,\frac{1}{f-\beta}\Big)+m\Big(r,\frac{1}{f-\beta}\Big)=S(r,f) \tag{63}$$

and

$$m(r,f)=\bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f). \tag{64}$$

Since $\bar{N}(r,\frac{1}{f'})=S(r,f)$, it follows from (63) and Lemma 2.1 that

$$m\Big(r,\frac{1}{f'-\beta}\Big)=S(r,f). \tag{65}$$

Set

$$\Delta=\frac{f-\beta}{f'-\beta}.$$

It is easy to see that $\Delta\not\equiv 0$, $N(r,\Delta)=S(r,f)$ and

$$\begin{aligned}
\bar{N}(r,f)+\bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big) &\leq N\Big(r,\frac{1}{\Delta}\Big)\\
&\leq T(r,\Delta)+O(1)\\
&\leq m(r,\Delta)+O(1)\\
&\leq m(r,f)+m\Big(r,\frac{1}{f'-\beta}\Big)+S(r,f).
\end{aligned}$$

Together with (64) and (65) we have $\bar{N}(r,f)=S(r,f)$. Finlly, from this, (63) and (64) we find that

$$\begin{aligned}
T(r,f) &= m(r,f)+S(r,f)=\bar{N}_{(2}\Big(r,\frac{1}{f-\beta}\Big)\\
&\leq \frac{1}{2}N_{(2}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)\leq\frac{1}{2}T(r,f)+S(r,f),
\end{aligned}$$

which gives the contradiction $T(r,f)=S(r,f)$. This completes the proof of Theorem 2.4. □

From Theorem 2.3 and Theorem 2.4 we immediately deduce the following corollary:

Corollary 2.5 *Let f be a non-constant meromorphic function, and let β be a small meromorphic function of f such that $\beta\not\equiv 0,\infty$. If f and f' share 0 and β CM, then $f\equiv f'$.*

Corollary 2.6 *Let f be a non-constant meromorphic function, and let β be a small meromorphic function of f such that $\beta\not\equiv 0,\infty$. If f and f' share 0 CM and β IM, then either $f\equiv f'$ or β is a constant and f is given as (1) when $\beta=a$.*

3 Open Problem

From Corollary 2.5 and Corollary 2.6 we establish the following:

Conjecture 3.1 *Let f be a non-constant meromorphic function, β and α two distinct small meromorphic functions of f with $\beta \not\equiv \infty$ and $\alpha \not\equiv \infty$. If f and f' share α and β CM, then $f \equiv f'$.*

Conjecture 3.2 *Let f be a non-constant meromorphic function, and let β be a small meromorphic function of f such that $\beta \not\equiv 0, \infty$. If f and f' share 0 and β IM, then either $f \equiv f'$ or β is a constant and f is given as (1) when $\beta = a$.*

Corollary 2.5 shows that Conjecture 3.1 is valid when $\alpha \equiv 0$ and Corollary 2.6 shows that Conjecture 3.2 is true if 0 IM replaced by 0 CM.

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