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Majorization Properties for Subclass of Analytic p-Valent Functions Defined by Linear Operator

R. M. El-Ashwah

Department of Mathematics
Faculty of Science (Damietta Branch),
Mansoura University, New Damietta 34517, Egypt.
email: r_elashwah@yahoo.com

Abstract

The object of the present paper is to investigate the majorization properties of certain subclass of analytic and p-valent functions defined by linear operator.

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1 Introduction

Let f and g be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We say that f is majorized by g in U (see [9]) and write

$$f(z) << g(z) \quad (z \in U), \tag{1.1}$$

if there exists a function φ, analytic in U such that

$$|\varphi(z)| < 1 \quad and \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{1.2}$$

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

For $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows:

$$f \prec g \ \ or \ \ f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in$

U). Further, if the function $g(z)$ is univalent in U, then we have the following equivalent (see [10, p. 4])

$$f(z) \prec g(z) \ \Leftrightarrow f(0) = g(0) \text{ and } f(U) \prec g(U).$$

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.3}$$

which are analytic and p-valent in U.

Liu [8] defined the linear operator $\Im_{p,b}^{s} : A(p) \to A(p)$ as follows:

$$\begin{aligned} \Im_{p,b}^{s}(z) &= z^p + \sum_{n=1}^{\infty} \left(\frac{b+1}{n+b+1} \right)^{s} z^{n+p} \\ (b &\in \mathbb{C} \backslash \mathbb{Z}^{-} = \{-1, -2, ...\}; s \in \mathbb{C}; p \in \mathbb{N}; z \in U). \end{aligned} \tag{1.4}$$

Now, we define the operator $\aleph_{p,b}^{s}(z)$ as follows:

$$\Im_{p,b}^{s}(z) * \aleph_{p,b}^{s}(z) = \frac{z^p}{1-z} \quad (b \in \mathbb{C} \backslash \mathbb{Z}^{-}; s \in \mathbb{C}; p \in \mathbb{N}; z \in U), \tag{1.5}$$

then for $f(z)$ given by (1.3) and (1.5), we have

$$\aleph_{p,b}^{s} f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{n+b+1}{b+1} \right)^{s} a_{n+p} z^{n+p} \quad (b \in \mathbb{C} \backslash \mathbb{Z}^{-}; s \in \mathbb{C}; p \in \mathbb{N}; z \in U). \tag{1.6}$$

We can easily verify from (1.6) that

$$z \left(\aleph_{p,b}^{s} f(z) \right)' = (b+1) \aleph_{p,b}^{s+1} f(z) - (b+1-p) \aleph_{p,b}^{s} f(z) \tag{1.7}$$

We note that

(i) $\aleph_{p,b}^{0} f(z) = f(z)$;

(ii) $\aleph_{p,p-1}^{1} f(z) = z^p + \sum\limits_{n=1}^{\infty} \left(\frac{n+p}{p} \right) a_{n+p} z^{n+p} = \dfrac{z f'(z)}{p}$.

Now, by making use of the operator $\aleph_{p,b}^{s} f(z)$, we define a new subclass of functions $f \in A(p)$ as follows.

Definition 1. *Let* $-1 \leq B < A \leq 1, p \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C}^*$ *and* $f \in A(p)$*. Then* $f \in S_{p,b,s}^{j}(\gamma; A, B)$, *the class of p-valent functions of complex order* γ in U, *if and only if*

$$\left\{ p + \frac{1}{\gamma} \left(\frac{z \left(\aleph_{p,b}^{s} f(z) \right)^{(j+1)}}{\left(\aleph_{p,b}^{s} f(z) \right)^{(j)}} - p + j \right) \right\} \prec p \frac{1+Az}{1+Bz}, \tag{1.8}$$

Clearly, we have the following relationships:
(i) $S^{j}_{p,b,s}(\gamma;1,-1)=S^{j}_{p,b,s}(\gamma)$;
(ii) $S^{j}_{p,b,0}(\gamma;1,-1)=S^{j}_{p}(\gamma)$;
(ii) $S^{0}_{1,b,0}(\gamma;1,-1)=S(\gamma)\ (\gamma\in\mathbb{C}^*)$ (see [11]);
(iii) $S^{0}_{1,b,0}(1-\alpha;1,-1)=S^*(\alpha)\ \ (0\le\alpha<1)$ (see [13]).
Also, we note that:
(i) For $j=s=0$, $S^{j}_{p,b,s}(\gamma)$ reduces to the class $S_p(\gamma)\,(\gamma\in\mathbb{C}^*)$ of $p-$valently starlike functions of order γ $(\gamma\in\mathbb{C}^*)$ in U (see Deniz et al. [4, with $\alpha=0$]), where

$$S_p(\gamma)=\left\{f(z)\in A(p):\operatorname{Re}\left(p+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-p\right)\right)>0,p\in\mathbb{N},\gamma\in\mathbb{C}^*\right\};$$

(ii) For $j=0,s=1$ and $b=p-1(p\in\mathbb{N})$, $S^{j}_{p,b,s}(\gamma)$ we get the class $K_p(\gamma)\,(\gamma\in\mathbb{C}^*)$ of $p-$valently convex functions of order γ $(\gamma\in\mathbb{C}^*)$ in U (see Aouf [3]), where,

$$K_p(\gamma)=\left\{f(z)\in A(p):\operatorname{Re}\left(p+\frac{1}{\gamma}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right)>0,p\in\mathbb{N},\gamma\in\mathbb{C}^*\right\}.$$

We shall need the following lemma.
Lemma 1 [1]. Let $\gamma\in\mathbb{C}^*$ and $f\in K^{j}_{p}(\gamma)$. Then $f\in S^{j}_{p}(\frac{1}{2}\gamma)$, that is,

$$K^{j}_{p}(\gamma)\subset S^{j}_{p}(\frac{1}{2}\gamma)\quad(\gamma\in\mathbb{C}^*). \tag{1.9}$$

An majorization problem for the class $S(\gamma)(\gamma\in\mathbb{C}^*)$ has been investigated by Altintas et al. [2]. Also, majorization problem for the class $S^*=S^*(0)$ has been investigated by MacGregor [9]. Recently Goswami and Wang [5], Goyal and Goswami [6] and Goyal et al. [7] generalized these results for classes of multivalent function defined by fractional derivatives operator and Saitoh operator, respectively. In this paper we investigate majorization problem for the class $S^{j}_{p,b,s}(\gamma;A,B)$ and other related subclasses.

2 Main Results

Unless otherwise mentioned we shall assume throughout the paper that $-1\le B<A\le 1,\gamma\in\mathbb{C}^*,b\in\mathbb{C}\backslash\mathbb{Z}^-,s\in\mathbb{C},p\in\mathbb{N}$ and $j\in\mathbb{N}_0$.
Theorem 1. *Let the function* $f\in A(p)$ and suppose that $g\in S^{j}_{p,b,s}\left(\gamma;A,B\right)$. If $\left(\aleph^{s}_{p,b}f(z)\right)^{(j)}$ is majorized by $\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}$ in U, then

$$\left|\left(\aleph^{s+1}_{p,b}f(z)\right)^{(j)}\right|\le\left|\left(\aleph^{s+1}_{p,b}g(z)\right)^{(j)}\right|\qquad(|z|<r_0), \tag{2.1}$$

where $r_0 = r_0(p, \gamma, b, A, B)$ is the smallest positive root of the equation

$$|\gamma p(A-B)+(b+1)B|\, r^3 - (2\,|B| + |b+1|)r^2 - [2 + |\gamma p(A-B)+(b+1)B|]\, r + |b+1| = 0. \tag{2.2}$$

Proof. Since $g \in S^{j}_{p,b,s}(\gamma; A, B)$, we find from (1.8) that

$$p + \frac{1}{\gamma}\left(\frac{z\left(\aleph^{s}_{p,b}g(z)\right)^{(j+1)}}{\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}} - p + j\right) = p\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$. From (2.3), we have

$$\frac{z\left(\aleph^{s}_{p,b}g(z)\right)^{(j+1)}}{\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}} = \frac{(p-j) + [\gamma p(A-B) + (p-j)B]\, w(z)}{1+Bw(z)}. \tag{2.4}$$

Also from (1.7), we have

$$z\left(\aleph^{s}_{p,b}g(z)\right)^{(j+1)} = (b+1)\left(\aleph^{s+1}_{p,b}g(z)\right)^{(j)} - (b+1+j-p)\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\left|\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}\right| \le \frac{|b+1|\,(1+|B|\,|z|)}{|b+1| - |\gamma p(A-B) + (b+1)\,B|\,|z|}\left|\left(\aleph^{s+1}_{p,b}g(z)\right)^{(j)}\right|. \tag{2.6}$$

Next, since $\left(\aleph^{s}_{p,b}f(z)\right)^{(j)}$ is majorized by $\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}$ in U, from (1.2), we have

$$\left(\aleph^{s}_{p,b}f(z)\right)^{(j)} = \varphi(z)\left(\aleph^{s}_{p,b}g(z)\right)^{(j)}. \tag{2.7}$$

Differentiating (2.7) with respect to z and multiplying by z, we have

$$z\left(\aleph^{s}_{p,b}f(z)\right)^{(j+1)} = z\varphi'(z)\left(\aleph^{s}_{p,b}g(z)\right)^{(j)} + z\varphi(z)\left(\aleph^{s}_{p,b}g(z)\right)^{(j+1)}, \tag{2.8}$$

using (2.5) in (2.8), we have

$$\left(\aleph^{s+1}_{p,b}f(z)\right)^{(j)} = \frac{z\varphi'(z)}{(b+1)}\left(\aleph^{s}_{p,b}g(z)\right)^{(j)} + \varphi(z)\left(\aleph^{s+1}_{p,b}g(z)\right)^{(j)}. \tag{2.9}$$

Thus, by noting that $\varphi \in P$ satisfies the inequality (see [12]),

$$\left|\varphi'(z)\right| \le \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z \in U), \tag{2.10}$$

and making use of (2.6) and (2.10) in (2.9), we have

$$\left|\left(\aleph_{p,b}^{s+1} f(z)\right)^{(j)}\right| \leq$$

$$\left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{(1+|B|\,|z|)\,|z|}{|b+1| - |\gamma p(A-B)+(b+1)B\,|z||}\right)\left|\left(\aleph_{p,b}^{s+1} g(z)\right)^{(j)}\right|, \tag{2.11}$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

leads us to the inequality

$$\left|\left(\aleph_{p,b}^{s+1} f(z)\right)^{(j)}\right| \leq$$

$$\frac{\Psi(\rho)}{(1-r^2)\left[|b+1| - |\gamma p(A-B)+(b+1)B|\,r\right]}\left|\left(\aleph_{p,b}^{s+1} g(z)\right)^{(j)}\right|,$$

where

$$\begin{aligned}\Psi(\rho) &= -r\,(1+|B|\,r)\,\rho^2 + (1-r^2)\left[|b+1| - |\gamma p(A-B)+(b+1)B|\,r\right]\rho \\ &\quad + r\,(1+|B|\,r)\,,\end{aligned} \tag{2.12}$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p,\gamma,b,A,B)$, where $r_0(p,\gamma,b,A,B)$ is the smallest positive root of (2.2), then the function $\Phi(\rho)$ defined by

$$\begin{aligned}\Phi(\rho) &= -\sigma\,(1+|B|\,\sigma)\,\rho^2 + (1-\sigma^2)\left[|b+1| - |\gamma p(A-B)+(b+1)B|\,\sigma\right]\rho \\ &\quad + \sigma\,(1+|B|\,\sigma)\end{aligned} \tag{2.13}$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\begin{aligned}\Phi(\rho) &\leq \Phi(1) = (1-\sigma^2)\left[|b+1| - |\gamma p(A-B)+(b+1)B|\,\sigma\right] \\ &\quad (0 \leq \rho \leq 1;\ 0 \leq \sigma \leq r_0(p,\gamma,b,A,B))\,.\end{aligned} \tag{2.14}$$

Hence upon setting $\rho = 1$ in (2.13), we conclude that (2.1) holds true for $|z| \leq r_0 = r_0(p,\gamma,b,A,B)$, where $r_0(p,\gamma,b,A,B)$ is the smallest positive root of (2.2). This completes the proof of Theorem 1.

Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function* $f \in A(p)$ and suppose that $g \in S^{j}_{p,b,s}(\gamma)$. If $\left(\aleph^{s}_{p,b} f(z)\right)^{(j)}$ is majorized by $\left(\aleph^{s}_{p,b} g(z)\right)^{(j)}$ in U, then

$$\left|\left(\aleph^{s+1}_{p,b} f(z)\right)^{(j)}\right| \leq \left|\left(\aleph^{s+1}_{p,b} g(z)\right)^{(j)}\right| \qquad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma, b)$ is given by

$$r_0 = r_0(p, \gamma, b) = \frac{k - \sqrt{k^2 - 4\,|2\gamma p - (b+1)|\,|b+1|}}{2\,|2\gamma p - (b+1)|},$$

where $(k = 2 + |b+1| + |2\gamma p - (b+1)|)$.

Putting $s = 0$ and $b = p - 1 (p \in \mathbb{N})$ in Corollary 1, we obtain the following corollary.

Corollary 2 [1, Theorem 1]. *Let the function* $f \in A(p)$ and suppose that $g \in S^{j}_{p}(\gamma)$. If $f^{(j)}(z)$ is majorized by $g^{(j)}(z)$ in U, then

$$\left|f^{(j+1)}(z)\right| \leq \left|g^{(j+1)}(z)\right| \qquad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma, j)$ is given by

$$r_0 = r_0(p, \gamma, j) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma p - p + j|}}{2\,|2\gamma p - p + j|},$$

where $(k = 2 + p - j + |2\gamma p - p + j|)$.

Putting $j = 0$ in Corollary 2, we obtain the following corollary.

Corollary 3. *Let the function* $f \in A(p)$ and suppose that $g \in S_p(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma)$ is given by

$$r_0 = r_0(p, \gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma p - p|}}{2\,|2\gamma p - p|},$$

where $(k = 2 + p + |2\gamma p - p|)$.

Putting $j = 0, s = 1$ and $b = p - 1 (p \in \mathbb{N})$, in Corollary 1, with the aid of Lemma 1 (with $j = 0$), we obtain the following corollary.

Corollary 4. *Let the function* $f \in A(p)$ and suppose that $g \in K_p(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma)$ is given by

$$r_0 = r_0(p, \gamma) = \frac{k - \sqrt{k^2 - 4p\,|\gamma - p|}}{2\,|\gamma - p|},$$

where ($k = 2 + p + |\gamma p - p|$).

Putting $A = 1, B = -1, p = 1$ and $s = b = j = 0$, in Theorem 1, we obtain the following corollary.

Corollary 5 [2, Theorem 1]. *Let the function* $f \in A$ and suppose that $g \in S(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0),$$

where $r_0 = r_0(\gamma)$ is given by

$$r_0 = r_0(\gamma) = \frac{k - \sqrt{k^2 - 4|2\gamma - 1|}}{2|2\gamma - 1|},$$

where ($k = 3 + |2\gamma - 1|$).

Letting $\gamma \to 1$ in Corollary 5, we obtain the following corollary.

Corollary 6 [9]. *Let the function* $f \in A$ and suppose that $g \in S^*$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0),$$

where r_0 is given by

$$r_0 = 2 - \sqrt{3}.$$

3 Open Problem

The class $S^j_{p,b,s}(\gamma; A, B)$ can be redefined by making use of a differential multiplier transformations to get new class. So, new results similar or parallel to what obtained in this paper can be derived for the new class.

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