Int. J. Open Problems Complex Analysis, Vol. 4, No. 3, November 2012
ISSN 2074-2827;
www.i-csrs.org

Extreme points and support points of a class of analytic functions with fixed finitely many coefficients

Liangpeng Xiong and Xiaoli Liu

College of Engineering and Technical,
ChengDu University of Technology, 614007, Leshan, Sichuan, P.R. China.
email: xlpwxf@163.com and travel-lxl@163.com

Abstract

Let $T_n^*(C_i, A, B, \alpha)$ ***denote the family of analytic functions of the form*** $f(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2+(1-\alpha)j]} z^j - \sum_{n=k+1}^{\infty} a_n z^n \quad (a_n \geq 0)$***, which are univalent in the open unit disk*** U ***and satisfy the following subordination condition:***$f'(z) + \alpha z f''(z) \prec \frac{1+Az}{1+Bz}(0 < \alpha \leq 1, -1 \leq A < B \leq 1, C_j \geq 0, 0 \leq \sum_{j=2}^{k} C_j \leq 1)$. ***In this paper, we obtain the extreme points and support points of the class of functions*** $T_n^*(C_j, A, B, \alpha)$.

Keywords: *analytic functions; linear functional; topology of uniform convergence; extreme points.*

2000 Mathematical Subject Classification: 30C45, 30C50.

1 Introduction

we denote the space of functions analytic in the unit disk $U = \{z : |z| < 1\}$ by $\mathcal{A}$. The topology of $\mathcal{A}$ is defined to be the topology of uniform convergence on compact subsets of the unit disk U. Let K be a subset of the space $\mathcal{A}$, the extreme points of K can be expressed as follows: $x_o \in EK$ if and only if the condition x, y, $x_0 \in K$, $0 < t < 1$ and $tx + (1-t)y = x_0$ can make sure $x = y = x_0$, where EK denote the set of all extreme points of K. Furthermore, Suppose that $\mathcal{F}$ is a compact subset of $\mathcal{A}$. If there exists a continuous linear functional J on $\mathcal{A}$, which satisfies for a function $f_0 \in \mathcal{F}$ such that $ReJ(f)$ is non-constant on $\mathcal{F}$, and $ReJ(f_0) = \max\{ReJ(f) : f \in \mathcal{F}\}$, then f_0 is called a support point of $\mathcal{F}$. The set of all support points of $\mathcal{F}$ is denoted by $supp\mathcal{F}$. Let $T_n \subset \mathcal{A}$ be the subclass of univalent analytic which are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,\ z \in U. \tag{1}$$

We recall the definition of subordination between two functions, say f and F, analytic in U. This means that there is an analytic function ϕ such that $\phi(0) = 0$, $|\phi(z)| < 1$ and $f(z) = F(\phi(z))$ for $|z| < 1$. This relation shall be denoted by $f \prec F$.

Indeed, some authors have given the extreme points and support points of several subclass of T_n, see [2, 5, 6, 8, 10]. Here, we want to introduce and study another subclasses $T_n(A, B, \alpha)$, $T_n^*(C_j, A, B, \alpha)$ of T_n. A function $f(z) \in T_n$ is said to be in the class $T_n(A, B, \alpha)$ if and only if

$$f'(z) + \alpha z f''(z) \prec \frac{1 + Az}{1 + Bz} \quad (0 < \alpha \leq 1, -1 \leq A < B \leq 1). \tag{2}$$

Some related analytic function classes with $T_n(A, B, \alpha)$ were studied for different objects, see [1, 4, 9, 11]. Now we can fix the finitely many coefficients of functions in $T_n(A, B, \alpha)$ and obtain the following class $T_n^*(C_j, A, B, \alpha)$:

$$T_n^*(C_j, A, B, \alpha) = \Bigg\{ f(z) : f(z) = z - \sum_{j=2}^{k} \frac{(B - A)C_j}{(1 + B)[\alpha j^2 + (1 - \alpha)j]} z^j -$$

$$\sum_{n=k+1}^{\infty} a_k z^k \in T_n(A, B, \alpha),\ 0 \leq C_j \leq 1, 0 < \alpha \leq 1, a_n \geq 0, 0 \leq \sum_{j=2}^{k} C_j \leq 1 \Bigg\}.$$

In this paper, we obtain the extreme points and support points of the subclass $T_n^*(C_j, A, B, \alpha)$ of T_n. Two important Lemmas need to be given.

Lemma 1.1 *If the function $f(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n \in T_n$, then $f(z)$ is in the class $T_n(A, B, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} [\alpha n^2 + (1 - \alpha)n] a_n < \frac{B - A}{1 + B}. \tag{3}$$

Proof Firstly, suppose $f(z) \in T_n(A, B, \alpha)$, then we have

$$f'(z) + \alpha z f''(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, -1 \leq A < B \leq 1, w(0) = 0, |w(z)| = 1, \tag{4}$$

it is equivalent to

$$|w(z)| = \left|\frac{1 - f'(z) - \alpha z f''(z)}{B[f'(z) + \alpha z f''(z) - A}\right| = \left|\frac{\sum\limits_{n=2}^{\infty}[n + \alpha n(n-1)]a_n z^{n-1}}{B - A - B\sum\limits_{n=2}^{\infty}[n + \alpha n(n-1)]a_n z^{n-1}}\right| < 1. \tag{5}$$

We consider real values of z, then the denominator in (5) cannot vanish in U, and is positive for $z = 0$. Therefore, it is positive on the line segment (0, 1) . Putting $z = r(0 < r < 1)$ in (5), we infer that

$$\sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n r^{n-1} < B - A - B\sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n r^{n-1}$$

or

$$\sum_{n=2}^{\infty}[\alpha n^2 + (1-\alpha)n]a_n r^{n-1} < \frac{B - A}{1 + B}.$$

Letting $r \to 1^-$, it yields the assertion (3).

Conversely, suppose $f(z) \in T_n$ and satisfies (3). Then, in view of (5), it is sufficient to prove that

$$\left|\sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n z^{n-1}\right| - \left|B - A - B\sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n z^{n-1}\right| < 0$$

In fact,

$$\begin{aligned}
\left|\sum_{n=2}^{\infty}[n \right. &+ \left. \alpha n(n-1)]a_n z^{n-1}\right| - \left|B - A - B\sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n |z|^{n-1} \\
&\quad -\{(B - A) - B\sum_{n=2}^{\infty}[n + \alpha n(n-1)]a_n |z|^{n-1}\} \\
&\leq \sum_{n=2}^{\infty}(1 + B)[\alpha n^2 + (1-\alpha)n]a_n |z|^{n-1} - (B - A) < 0 \\
&\leq \sum_{n=2}^{\infty}(1 + B)[\alpha n^2 + (1-\alpha)n]a_n - (B - A) < 0,
\end{aligned}$$

which complete the proof of Lemma 1.1.

Lemma 1.2 *If the function* $f(z) = z - \sum\limits_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2+(1-\alpha)j]} z^j - \sum\limits_{n=k+1}^{\infty} a_n z^n \in T_n$*, then* $f(z)$ *is in the class* $T_n^*(C_j, A, B, \alpha)$ *if and only if*

$$\sum_{n=k+1}^{\infty} [\alpha n^2 + (1-\alpha)n] a_n < \frac{B-A}{1+B}(1 - \sum_{j=2}^{k} C_j), \tag{6}$$

where $a_n \geq 0, 0 < \alpha \leq 1, 0 \leq C_j \leq 1, 0 \leq \sum\limits_{j=2}^{k} C_j \leq 1.$

Proof Putting $a_j = \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} (j = 1, 2, ..., k)$ in Lemma 1.1, we have

$$\sum_{j=2}^{k} \frac{B-A}{1+B} C_j + \sum_{n=k+1}^{\infty} [\alpha n^2 + (1-\alpha)n] a_n < \frac{B-A}{1+B},$$

which complete the proof of Lemma 1.2.

2 The extreme points of $T_n^*(C_j, A, B, \alpha)$

Lemma 2.1 *(See [3, 7])Suppose* $\mathcal{A}$ *is a topological vector space. If* $\mathcal{F}$ *is a nonempty compact subset in* $\mathcal{A}$*, then* $\mathcal{F} \subset \overline{co}(E(\mathcal{F}))$*. In particular, if* $\mathcal{F}$ *is a nonempty compact convex set in* $\mathcal{A}$*, then* $\mathcal{F}$ *is the closed convex hull of the set of its extreme points.*

Lemma 2.2 *Let* $f_k(z) = z - \sum\limits_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j, (k \geq 2)$ *and*

$f_n(z) = z - \sum\limits_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} z^n, (n \geq k+1)$*, then* $f(z) \in T_n^*(A, B, \alpha)$ *if and only if* $f(z) = \sum\limits_{n=k}^{\infty} \lambda_n f_n(z)$*, where* $\lambda_n \geq 0$ *and* $\sum\limits_{n=k}^{\infty} \lambda_n = 1.$

Proof Firstly, if $f(z) = \sum\limits_{n=k}^{\infty} \lambda_n f_n(z)$, then we have

$$f(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \sum_{n=k+1}^{\infty} \frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} \lambda_n z^n. \tag{7}$$

So

$$\begin{aligned}\sum_{n=k+1}^{\infty}[\alpha n^2 \quad &+ \quad (1-\alpha)n]\frac{(B-A)(1-\sum\limits_{j=2}^{k}C_j)}{(1+B)[\alpha n^2+(1-\alpha)n]}\lambda_n \\ &= \frac{B-A}{1+B}(1-\sum_{j=2}^{k}C_j)\sum_{n=k+1}^{\infty}\lambda_n \\ &\leq \frac{B-A}{1+B}(1-\sum_{j=2}^{k}C_j),\end{aligned}$$

from Lemma 1.2, we can know $f(z) \in T_n^*(C_j, A, B, \alpha)$.

Conversely, suppose $f(z) \in T_n^*(C_j, A, B, \alpha)$, then from Lemma 1.2, it is easy to know that

$$a_n \leq \frac{(B-A)(1-\sum\limits_{j=2}^{k}C_j)}{(1+B)[\alpha n^2+(1-\alpha)n]} \quad (n \geq k+1). \tag{8}$$

Setting

$$\lambda_n = \frac{(1+B)[\alpha n^2+(1-\alpha)n]}{(B-A)(1-\sum\limits_{j=2}^{k}C_j)}a_n \quad (n \geq k+1)\,,\ \lambda_k = 1-\sum_{n=k+1}^{\infty}\lambda_n,$$

then $f(z) = \sum\limits_{n=k}^{\infty}\lambda_n f_n(z)$.

Theorem 2.3 *The extreme points of the class $T_n^*(C_j, A, B, \alpha)$ are given by*

$$ET_n^*(C_j, A, B, \alpha) = V = \left\{ f_k(z) = z - \sum_{j=2}^{k}\frac{(B-A)C_j}{(1+B)[\alpha j^2+(1-\alpha)j]}z^j,\right.$$

$$\left. f_n(z) = z - \sum_{j=2}^{k}\frac{(B-A)C_j}{(1+B)[\alpha j^2+(1-\alpha)j]}z^j - \frac{(B-A)(1-\sum\limits_{j=2}^{k}C_j)}{(1+B)[\alpha n^2+(1-\alpha)n]}z^n \ (n \geq k+1)\right\}$$

Proof Suppose $z - \sum\limits_{j=2}^{k}\frac{(B-A)C_j}{(1+B)[\alpha j^2+(1-\alpha)j]}z^j - \frac{(B-A)(1-\sum\limits_{j=2}^{k}C_j)}{(1+B)[\alpha n^2+(1-\alpha)n]}z^n = tg_1(z) + (1-t)g_2(z)$, where $0 < t < 1$, $g_i(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \sum_{n=k+1}^{\infty} a_{n,i} z^n$ and $g_i(z) \in T_n^*(C_j, A, B, \alpha)$ $(i = 1, 2)$, then we have

$$\frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} = t a_{n,1} + (1-t) a_{n,2}. \tag{9}$$

Since $g_i(z) \in T_n^*(C_j, A, B, \alpha)$, (3.2) gives $a_{n,i} \leq \frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]}$ $(i = 1, 2)$. This implies that $a_{n,1} = a_{n,2} = \frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]}$, so $g_1(z) = g_2(z)$, this gives us

$$f_n(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} z^n \in$$

$ET_n^*(C_j, A, B, \alpha)$. Taking the same process, we can obtain:

$$f_k(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j \in ET_n^*(C_j, A, B, \alpha). \tag{10}$$

So $V \subset ET_n^*(C_j, A, B, \alpha)$. From Lemma 2.2, we know $T_n^*(C_j, A, B, \alpha) = HV$. Since V is a compact set, using Lemma 2.1, it gives $ET_n^*(C_j, A, B, \alpha) = EHV \subset V$. So $ET_n^*(C_j, A, B, \alpha) = V$.

3 The support points of $T_n^*(C_j, A, B, \alpha)$

Lemma 3.1 *(See [3]) J is a complex-valued continuous linear functional on $\mathcal{A}$ if and only if there is a sequence $\{b_n\}$ of complex numbers satisfying $\overline{\lim\limits_{n\to\infty}} (|b_n|)^{\frac{1}{n}} < 1$ and such that $J(f) = \sum\limits_{n=0}^{\infty} b_n a_n$, where $f \in \mathcal{A}$ and $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \quad (|z| < 1)$.*

Lemma 3.2 *(See [2]) Suppose f_0 is a support point of $\mathcal{F}$, and let J be a corresponding continuous linear functional on $\mathcal{F}$. Defining G_j by $G_j = \{f \in \mathcal{F} : ReJ(f) = ReJ(f_0)\}$, then G_j is convex, $EG_j \subset E\mathcal{F}$ and $G_j = \{f \in \mathcal{F} : f = \sum\limits_{i=1}^{\infty} \delta_i f_i, \delta_i \geq 0, \sum\limits_{i=1}^{\infty} \delta_i = 1, f_i \in EG_j, i = 1, 2, ...\}$.*

Lemma 3.3 *(See [3]) Let $\mathcal{F}$ be a compact subset of $\mathcal{A}$ and let J be a complex-valued continuous linear functional on $\mathcal{A}$.Then* $\max\{ReJ(f) : f \in H\mathcal{F}\} = \max\{ReJ(f) : f \in \mathcal{F}\} = \max\{ReJ(f) : f \in E(H\mathcal{F})\}$.

Theorem 3.4 *The support points of the class $T_n^*(C_j, A, B, \alpha)$ are given by*

$$SuppT_n^*(C_j,A,B,\alpha) = \left\{ f(z) \in T_n(A,B,\alpha) : f(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \right.$$

$$\left. \sum_{n=k+1}^{\infty} \frac{(B-A)(1-\sum_{j=2}^{k} c_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} \zeta_n z^n, \zeta_n \geq 0, \sum_{n=k+1}^{\infty} \zeta_n \leq 1, \zeta_i = 0 \, for \, some \; i \geq k+1 \right\}.$$

Proof Firstly, let a function $f_0(z) \in T_n^*(C_j, A, B, \alpha)$, and put

$$f_0(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \sum_{n=k+1}^{\infty} \frac{(B-A)(1-\sum_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} \zeta_n z^n, \tag{11}$$

where $\sum_{n=k+1}^{\infty} \zeta_n \leq 1, \zeta_n \geq 0, \zeta_i = 0$ for some $i \geq k+1$. Now, taking

$$\begin{cases} b_n = 0, & n > 1, n \neq i, \\ b_n = 1, & n = 1, n = i. \end{cases}$$

Then we have $\overline{\lim_{n\to\infty}} (|b_n|)^{\frac{1}{n}} < 1$. There we define a functional J on T_n by

$$J(f) = \sum_{n=0}^{\infty} a_n b_n, \quad where \quad f(z) = z - \sum_{n=2}^{\infty} a_n \in T_n.$$

It is clearly that the J is a continuous linear functional on T_n by Lemma 3.1. Moreover, we note that $J(f_0) = 1$, whenever, there are two cases for any function

$$f(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^n - \sum_{n=k+1}^{\infty} a_n z^n \in T_n^*(C_j, A, B, \alpha), \tag{12}$$

Case 1: For $2 \leq i \leq k$, $J(f) = a_1 b_1 + a_i b_i = 1 - a_i = 1 - \frac{(B-A)C_i}{(1+B)[\alpha j^2 + (1-\alpha)j]} \leq 1$.
Case 2: For $i \geq k+1$,$J(f) = a_1 b_1 + a_i b_i = 1 - a_i \leq 1 (a_i \geq 0)$.
So we have $ReJ(f_0) = \max\{ReJ(f) : f \in T_n^*(C_j, A, B, \alpha)\}$ and $ReJ(f)$ are not constant on $T_n^*(C_j, A, B, \alpha)$, hence f_0 is a support point of $T_n^*(C_j, A, B, \alpha)$.

Conversely, suppose that f_0 is a support point of $T_n^*(C_j, A, B, \alpha)$, and J is a continuous linear functional on $T_n^*(C_j, A, B, \alpha)$ given by Lemma 3.1 with sequence $\{b_n\}$. Note that ReJ is also a continuous linear on $T_n^*(C_j, A, B, \alpha)$, consequently, by the Lemma 3.3, we have

$$ReJ(f_0) = \max\{ReJ(f) : f \in T_n^*(C_j, A, B, \alpha)\} = \max\{ReJ(f) : f \in$$

$ET_n^*(C_j, A, B, \alpha)\}$. Set$M_i = \{f_n : ReJ(f_0) = ReJ(f_n), f_n \in ET_n^*(C_j, A, B, \alpha)\}$, if $M_i = ET_n^*(C_j, A, B, \alpha)$, then $ReJ(f)$ must be constant on $T_n^*(C_j, A, B, \alpha)$, this contradicts that $f_0(z)$ is a support point of $T_n^*(C_j, A, B, \alpha)$. Therefore, there exists i such that $ReJ(f_i) < ReJ(f_0)$. So we can obtain the relation $EM_i \subset \{f_n : f_n \in ET_n^*(C_j, A, B, \alpha), n = 1, 2, ...,$ and $n \neq i\}$. Hence, following the Lemma 3.2 , we have

$$f_0(z) = \sum_{n=k}^{\infty} \zeta_n f_n(z)$$

where $\zeta_n \geq 0$, $\sum\limits_{n=k+1}^{\infty} \zeta_n \leq \sum\limits_{n=k}^{\infty} \zeta_n = 1$ and $f_n(z) \in ET_n^*(C_j, A, B, \alpha)$. It follows from this and Theorem 2.1 that

$$f_0(z) = z - \sum_{j=2}^{k} \frac{(B-A)C_j}{(1+B)[\alpha j^2 + (1-\alpha)j]} z^j - \sum_{n=k+1, n\neq i}^{\infty} \frac{(B-A)(1 - \sum\limits_{j=2}^{k} C_j)}{(1+B)[\alpha n^2 + (1-\alpha)n]} \zeta_n z^n$$

which complete the proof of Theorem 3.1.

4 Open Problem

The method here is employed from the work done by W.DEEB [2].It is interesting to see similar results for different classes such as

$$\mathcal{G} = \left\{ f(z) : f'(z) + \alpha z f''(z) \prec \frac{1+Az}{1+Bz}, f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\},$$

where $0 < \alpha \leq 1, -1 \leq A < B \leq 1, a_n \in \mathcal{R}$. We did try, but failed to get any results and it is left for the readers to tackle this problem.

Acknowledgements: The authors are thankful to the referee for valuable suggestions which improve the results obtained in this research. The present investigation was supported by the College of Engineering and Technical of ChengDu University of Technology under Grant (NO.C122010003).

References

[1] H.E.Darwish , M.K.Aouf, *Generalizations of modified-Hadamard products of p-valent functions with negative coefficients*, Math. Comput. Modelling, 49(1-2), 2009, 38-45.

[2] W. DEEB, *Extreme points and support points of families of univalent functions with real coefficients*, Math Rep Toyama Univ, 8, 1985, 103-111.

[3] D.J. Hallenbeck, *Linear problems and convexity techniques in geometric function theorem*, Boston: Pitman Advanced Publishing Program, 1984.

[4] Y.C. Kim, *Mapping properties of differential inequalities related to univalent functions*, Appl. Math. Comput, 187, 2007, 272-279.

[5] Z.G. Peng, *Extreme points and support points of a class of analysis functions*, Acta Mathematic Scientia, 20B(1), 2000, 131-136.

[6] Z.G. Peng, F. Su, *Extreme points and support points of a family of analytic functions*, Acta Mathematic Scientia, 25A(3), 2005, 345-348.

[7] Walter Rudin: *Functional Analysis(Second Edition)*, China Machine press, BeiJing, 2004.

[8] H. Silverman, *univalent functions with negative coefficients*, Pro Am Math Soc, 51(1), 1975, 109-116.

[9] H.M. Srivastava, N.Xu and D.G. Yang, *Inclusion relations and convolution properties of a certain class of analytic functions associated with the Ruscheweyh derivatives*, J. Math. Anal. Appl, 331, 2007, 686-700.

[10] L.P. Xiong, *Some general results and extreme points of p-valent functions with negative coefficients*, Demonstratio Mathematica, 44(2), 2011, 261-272.

[11] D.G. Yang, J.L.Liu, *On a class of analytic functions with missing coefficients*, Applied Mathematics and Computation, 215, 2010, 3473-3481.