Int. J. Open Problems Complex Analysis, Vol. 4, No. 3, November 2012
ISSN 2074-2827;
www.i-csrs.org

Sandwich Theorems for Higher-Order Derivatives of p-Valent Functions Involving a Generalized Differential Operator

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Abstract

The purpose of this paper is to obtain some applications of first order differential subordination, superordination and sandwich results for higher-order derivatives of $p-$valent functions involving a generalized differential operator. Some of our results improve and generalize previously known results.

Keywords: *Analytic function, Hadamard product, differential subordination, superordination, sandwich theorems, linear operator.*
2000 Mathematical Subject Classification: 30C45.

1 Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} ... \ (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, ...\}).$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1.1}$$

which are $p-$valent in U. We write $\mathcal{A}(1) = \mathcal{A}$.

If $f,\ g \in H(U)$, we say that f is subordinate to g or g is superordinate to f, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[9], [17] and [18]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h be univalent function in U. If β is analytic function in U and satisfies the first order differential subordination:

$$\phi\left(\beta(z), z\beta'(z); z\right) \prec h(z), \tag{1.2}$$

then β is a solution of the differential subordination (1.2). The univalent function q is called a dominant of the solutions of the differential subordination (1.2) if $\beta(z) \prec q(z)$ for all β satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If β and ϕ are univalent functions in U and if satisfies first order differential superordination:

$$h(z) \prec \phi\left(\beta(z), z\beta'(z); z\right), \tag{1.3}$$

then β is a solution of the differential superordination (1.3). An analytic function q is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec \beta(z)$ for all β satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants of (1.3) is called the best subordinant.

Using the results of Miller and Mocanu [18], Bulboaca [8] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [9]. Ali et al. [1], have used the results of Bulboaca [8] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$

to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [24] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [22] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions $f \in \mathcal{A}(p)$ given by (1.1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{1.4}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.5}$$

Upon differentiating both sides of (1.5) $j-$times with respect to z, we have

$$(f * g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k; j) a_k b_k z^{k-j}, \tag{1.6}$$

where

$$\delta(p; j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.7}$$

For functions $f, g \in \mathcal{A}(p)$, we define the linear operator $D_{\lambda,p}^n (f * g)^{(j)} :$ $\mathcal{A}(p) \rightarrow \mathcal{A}(p)$ by:

$$D_{\lambda,p}^0 (f * g)^{(j)}(z) = (f * g)^{(j)}(z),$$

$$\begin{aligned} D_{\lambda,p}^1 (f * g)^{(j)}(z) &= D_{\lambda,p} (f * g)^{(j)}(z) \\ &= (1-\lambda)(f * g)^{(j)}(z) + \frac{\lambda}{p-j} z \left((f * g)^{(j)} \right)' (z) \\ &= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left(\frac{p - j + \lambda(k-p)}{p-j} \right) \delta(k; j) a_k b_k z^{k-j}, \end{aligned}$$

$$\begin{aligned} D^2_{\lambda,p}\left(f*g\right)^{(j)}(z) &= D\left(D^1_p\left(f*g\right)^{(j)}(z)\right) \\ &= \delta(p;j)\,z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+\lambda(k-p)}{p-j}\right)^2\delta(k;j)\,a_kb_kz^{k-j}, \end{aligned}$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}\left(f*g\right)^{(j)}(z) &= D(D^{n-1}_p\left(f*g\right)^{(j)}(z)) \\ &= \delta(p;j)\,z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+\lambda(k-p)}{p-j}\right)^n\delta(k;j)\,a_kb_kz^{k-j} \\ & \quad (\lambda\geq 0;p>j;p\in\mathbb{N};j,n\in\mathbb{N}_0;z\in U)\,. \end{aligned} \tag{1.8}$$

From (1.8), we can easily deduce that

$$\begin{aligned} \frac{\lambda z}{p-j}\left(D^n_{\lambda,p}\left(f*g\right)^{(j)}(z)\right)' &= D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}(z)\ -(1-\lambda)\,D^n_{\lambda,p}\left(f*g\right)^{(j)}(z) \\ & \quad (\lambda>0;p>j;p\in\mathbb{N};n,j\in\mathbb{N}_0;z\in U)\,. \end{aligned} \tag{1.9}$$

We observe that the linear operator $D^n_{\lambda,p}\left(f*g\right)^{(j)}(z)$ reduces to several interesting many other linear operators considered earlier for different choices of j,n,λ and the function g:

(i) For $j=0$, $D^n_{\lambda,p}\left(f*g\right)^{(j)}(z)=D^n_{\lambda,p}\left(f*g\right)(z)$, where the operator $D^n_{\lambda,p}\left(f*g\right)$ $(\lambda\geq 0,\ p\in\mathbb{N},n\in\mathbb{N}_0)$ was introduced and studied by Selvaraj et al. [21] (see also [7]), and $D^n_{\lambda,1}\left(f*g\right)(z)=D^n_{\lambda}\left(f*g\right)(z)$, where the operator $D^n_{\lambda}\left(f*g\right)$ was introduced by Aouf and Mostafa [6];

(ii) For

$$g(z)=\frac{z^p}{1-z}\ \ (p\in\mathbb{N};z\in U\ ) \tag{1.10}$$

we have $D^n_{\lambda,p}\left(f*g\right)^{(j)}(z)=D^n_{\lambda,p}f^{(j)}(z)$, $D^n_{\lambda,p}f^{(0)}(z)=D^n_{\lambda,p}f(z)$, where the operator $D^n_{\lambda,p}$ is the $p-$valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [12], $D^n_{1,p}f^{(j)}(z)=D^n_pf^{(j)}(z)$, where the operator $D^n_pf^{(j)}$ $(p>j,p\in\mathbb{N},n,j\in\mathbb{N}_0)$ was introduced and studied by Aouf [3,4], and $D^n_{1,p}f^{(0)}(z)=D^n_pf(z)$, where the operator D^n_p is the $p-$valent Sălăgean operator which was introduced and studied by Kamali and Orhan [13] (see also [5]);

(iii) For

$$g(z)=z^p+\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}}\frac{z^k}{(1)_{k-p}}\qquad (z\in U), \tag{1.11}$$

(for complex parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $(\beta_j\notin\mathbb{Z}_0^-=\{0,-1,-2,...\}\,,$ $j=1,...,s)$; $q\leq s+1;p\in\mathbb{N};\ q,\ s\in\mathbb{N}_0)$ where $(\nu)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ, by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1, & (k=0), \\ \nu(\nu+1)(\nu+2)...(\nu+k-1), & (k\in\mathbb{N}). \end{cases}$$

we have $D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z) = D_{\lambda,p}^{n}\left(H_{p,q,s}(\alpha_1)f\right)^{(j)}(z)$, and $D_{\lambda,p}^{0}\left(f*g\right)^{(0)}(z) = H_{p,q,s}(\alpha_1)f(z)$, where the operator $H_{p,q,s}(\alpha_1) = H_{p,q,s}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)$ is the Dziok-Srivastava operator which was introduced and studied by Dziok and Srivastava [11] and which contains in turn many interesting operators;

(iv) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+l+\alpha\left(k-p\right)}{p+l}\right)^m z^k \tag{1.12}$$
$$(\alpha\geq 0;\ l\geq 0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0; z\in U\ ),$$

we have $D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z) = D_{\lambda,p}^{n}\left(I_p(m,\alpha,l)f\right)^{(j)}(z)$, and $D_{\lambda,p}^{0}\left(f*g\right)^{(0)}(z) = I_p(m,\alpha,l)f(z)$, where the operator $I_p(m,\alpha,l)$ was introduced and studied by Cătas [10] which contains in turn many interesting operators such as, $I_p(m,1,l) = I_p(m,l)$, where $I_p(m,l)$ was investigated by Kumar et al. [14];

(v) For

$$g(z) = z^p + \frac{\Gamma\left(p+\alpha+\beta\right)}{\Gamma\left(p+\beta\right)}\sum_{k=p+1}^{\infty}\frac{\Gamma\left(k+\beta\right)}{\Gamma\left(k+\alpha+\beta\right)}z^k \tag{1.13}$$
$$(\ \alpha\geq 0;\ p\in\mathbb{N};\ \beta>-1; z\in U\ )$$

we have $D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z) = D_{\lambda,p}^{n}\left(Q_{\beta,p}^{\alpha}f\right)^{(j)}(z)$, and $D_{\lambda,p}^{0}\left(f*g\right)^{(0)}(z) = Q_{\beta,p}^{\alpha}f(z)$, where the operator $Q_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [15];

(vi) For $j=0$ and g of the form (1.11) with $p=1$, we have $D_{\lambda,1}^{n}\left(f*g\right)(z) = D_{\lambda}^{n}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)(z)$, where the operator $D_{\lambda}^{n}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)$ was introduced and studied by Selvaraj and Karthikeyan [20];

(vii) For $j=0$, $p=1$ and

$$g(z) = z + \sum_{k=2}^{\infty}\left[\frac{\Gamma\left(k+1\right)\Gamma\left(2-m\right)}{\Gamma\left(k+1-m\right)}\right]^n z^k$$
$$(n\in\mathbb{N}_0; 0\leq\ m<1; z\in U\ )$$

we have $D_{\lambda,1}^{n}\left(f*g\right)(z) = D_{\lambda}^{n,m}f(z)$, where the operator $D_{\lambda}^{n,m}$ was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator $D_{\lambda,p}^{n}\left(f*g\right)^{(j)}$.

2 Definitions and preliminarie

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 2.1 [18]. Denote by Q, the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\rightarrow\zeta}f\left(z\right)=\infty\right\},$$

and are such that $f^{'}\left(\zeta\right)\neq0$ for $\zeta\in\partial U\backslash E\left(f\right)$.

Lemma 2.1 [18]. Let q be univalent in U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi\left(w\right)\neq0$ when $w\in q(U)$. Set

$$\psi\left(z\right)=zq^{'}\left(z\right)\varphi\left(q\left(z\right)\right)\quad\text{and}\quad h\left(z\right)=\theta\left(q\left(z\right)\right)+\psi\left(z\right). \tag{2.1}$$

Suppose that

(i) $\psi\left(z\right)$ is starlike univalent in U,

(ii) $\Re\left\{\frac{zh^{'}(z)}{\psi(z)}\right\}>0$ for $z\in U$.

If β is analytic with $\beta(0)=q(0)$, $\beta(U)\subset D$ and

$$\theta\left(\beta\left(z\right)\right)+z\beta^{'}\left(z\right)\varphi\left(\beta\left(z\right)\right)\prec\theta\left(q\left(z\right)\right)+zq^{'}\left(z\right)\varphi\left(q\left(z\right)\right), \tag{2.2}$$

then $\beta\left(z\right)\prec q\left(z\right)$ and q is the best dominant.

Lemma 2.2 [8]. Let q be convex univalent in U and θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(i) $\Re\left\{\frac{\theta^{'}(q(z))}{\phi(q(z))}\right\}>0$ for $z\in U$,

(ii) $\Psi\left(z\right)=zq^{'}\left(z\right)\phi\left(q\left(z\right)\right)$ is starlike univalent in U.

If $\beta(z)\in H[q(0),1]\cap Q$, with $\beta(U)\subseteq D$, and $\theta\left(\beta\left(z\right)\right)+z\beta^{'}\left(z\right)\phi\left(\beta\left(z\right)\right)$ is univalent in U and

$$\theta\left(q\left(z\right)\right)+zq^{'}\left(z\right)\phi\left(q\left(z\right)\right)\prec\theta\left(\beta\left(z\right)\right)+zp^{'}\left(z\right)\phi\left(\beta\left(z\right)\right), \tag{2.3}$$

then $q\left(z\right)\prec\beta\left(z\right)$ and q is the best subordinant.

3 Subordination resuts

Unless otherwise mentioned, we assume throughout this paper that α, β, $\gamma_i\in\mathbb{C}(i=1,2)$, such that $\alpha+\beta\neq0$, γ_3, $\mu\in\mathbb{C}^*\left(\mathbb{C}\backslash\left\{0\right\}\right)$, $\lambda>0$, $\delta\left(p;j\right)$ is given by (1.7), $p>j$; $p\in\mathbb{N}$, $n,j\in\mathbb{N}_0$ and the powers are understood as the principle values.

Theorem 3.1. Let q be convex univalent in U with $q(0)=1$ and assume that

$$\Re\left\{\frac{\gamma_2}{\gamma_3}q(z)+1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0 \quad (z\in U). \tag{3.1}$$

If $f,\ \Phi,\ \Psi\in\mathcal{A}(p)$ satisfy the following subordination condition:

$$\begin{aligned}&\gamma_1+\gamma_2\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\\&+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^{n+1}_{\lambda,p}(f*\Psi)^{(j)}(z)}{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}-1\right\}\\ \prec\;&\gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)},\end{aligned}$$

then

$$\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\prec q(z)$$

and q is the best dominant.

Proof. Define a function ϱ by

$$\varrho(z)=\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu} \quad (z\in U). \tag{3.2}$$

Then the function ϱ is analytic in U and $\varrho(0)=1$. Therefore, differentiating (3.2) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we have

$$\begin{aligned}&\gamma_1+\gamma_2\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\\&+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^{n+1}_{\lambda,p}(f*\Psi)^{(j)}(z)}{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}-1\right\}\\ =\;&\gamma_1+\gamma_2\varrho(z)+\gamma_3\frac{z\varrho'(z)}{\varrho(z)},\end{aligned}$$

that is,

$$\gamma_1+\gamma_2\varrho(z)+\gamma_3\frac{z\varrho'(z)}{\varrho(z)}\prec\gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w)=\gamma_1+\gamma_2 w \text{ and } \varphi(w)=\frac{\gamma_3}{w},$$

it can be easily observed that θ is analytic function in $\mathbb{C}$, φ is analytic function in $\mathbb{C}^*$ and $\varphi(w) \neq 0$. Also we see that

$$\psi(z) = zq'(z)\varphi(q(z)) = \gamma_3 \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + \psi(z) = \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)},$$

it is clear that ψ is starlike univalent in U and

$$\Re\left\{\frac{zh'(z)}{\psi(z)}\right\} = \Re\left\{\frac{\gamma_2}{\gamma_3} q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0$$

Therefore, Theorem 3.1 now follows by applying Lemma 2.1. ■

Putting $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, it easy to check that the assumption (3.1) holds whenever $-1 \le B < A \le 1$, hence we obtain the following corollary.

Corollary 3.1. Let $-1 \le B < A \le 1$ and assume that

$$\Re\left\{\frac{\gamma_2}{\gamma_3}\frac{1+Az}{1+Bz} + \frac{1-Bz}{1+Bz} - \frac{(A-B)z}{(1+Az)(1+Bz)}\right\} > 0 \quad (z \in U), \tag{3.3}$$

holds. If f, Φ, $\Psi \in \mathcal{A}(p)$ satisfy the following subordination condition:

$$\begin{aligned} &\gamma_1 + \gamma_2 \left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z) + \beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} \\ &+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)^{(j)}(z) + \beta D^{n+1}_{\lambda,p}(f*\Psi)^{(j)}(z)}{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z) + \beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)} - 1\right\} \\ \prec\ &\gamma_1 + \gamma_2\frac{1+Az}{1+Bz} + \gamma_3\frac{(A-B)z}{(1+Az)(1+Bz)}, \end{aligned}$$

then

$$\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z) + \beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $j = 0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. Let q be convex univalent in U with $q(0) = 1$ and assume that (3.1) holds. If f, Φ, $\Psi \in \mathcal{A}(p)$ satisfy the following subordination condition:

$$\gamma_1+\gamma_2\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)(z)+\beta D^n_{\lambda,p}(f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu}$$
$$+\gamma_3\mu\frac{p}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)(z)+\beta D^{n+1}_{\lambda,p}(f*\Psi)(z)}{\alpha D^n_{\lambda,p}(f*\Phi)(z)+\beta D^n_{\lambda,p}(f*\Psi)(z)}-1\right\}$$
$$\prec\ \gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)(z)+\beta D^n_{\lambda,p}(f*\Psi)(z)}{(\alpha+\beta)z^{p-j}}\right)^{\mu}\prec q(z)$$

and q is the best dominant.

Taking $p=\lambda=1$ and $n=0$ in Corollary 3.2, we obtain the following corollary which improves the result of Magesh et al. [16, Theorem 3.1].

Corollary 3.3. Let q be convex univalent in U with $q(0)=1$ and assume that (3.1) holds. If f, Φ, $\Psi\in\mathcal{A}$ satisfy the following subordination condition:

$$\gamma_1+\gamma_2\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}$$
$$+\gamma_3\mu\left\{\frac{\alpha z(f*\Phi)'(z)+\beta z(f*\Psi)'(z)}{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}-1\right\}$$
$$\prec\ \gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}\prec q(z)$$

and q is the best dominant.

Taking $\Phi(z)=\frac{z}{1-z}$, and $\Psi(z)=\frac{z}{(1-z)^2}$ in Corollary 3.3, we obtain the following corollary which improves the result of Magesh et al. [16, Corollary 3.2].

Corollary 3.4. Let q be convex univalent in U with $q(0)=1$ and assume that (3.1) holds. If $f\in\mathcal{A}$ satisfy the following subordination condition:

$$\gamma_1+\gamma_2\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)z}\right)^{\mu}+\gamma_3\mu\left\{\frac{\alpha zf'(z)+\beta zf'(z)+\beta z^2f''(z)}{\alpha f(z)+\beta zf'(z)}-1\right\}$$
$$\prec\ \gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)\,z}\right)^{\mu}\prec q\,(z)$$

and the function q is the best dominant.

Taking $\alpha=\beta=1$ in Corollary 3.4, we obtain the following corollary which improves the result of Magesh et al. [16, Corollary 3.3]

Corollary 3.5. Let q be convex univalent in U with $q(0)=1$ and (3.1) holds true. If $f\in\mathcal{A}$ satisfy the following subordination condition:

$$\begin{aligned}&\gamma_1+\gamma_2\left(\frac{f(z)+zf'(z)}{2z}\right)^{\mu}+\gamma_3\mu\left\{\frac{z^2f''(z)+2zf'(z)}{f(z)+zf'(z)}-1\right\}\\ \prec\;&\gamma_1+\gamma_2 q\,(z)+\gamma_3\frac{zq'(z)}{q\,(z)},\end{aligned}$$

then

$$\left(\frac{f(z)+zf'(z)}{2z}\right)^{\mu}\prec q\,(z)$$

and the function q is the best dominant.

Remark 3.1. **(i)** Taking $\alpha=1$ and $\beta=0$ in Corollary 3.4, we obtain the result obtained by Magesh et al. [16, Corollary 3.4];

(ii) Taking $\alpha=0$ and $\beta=1$ in Corollary 3.4, we obtain the result obtained by Magesh et al. [16, Corollary 3.5].

Taking $\gamma_1=\alpha=1$, $\gamma_2=\beta=0$, $q\,(z)=\frac{1}{(1-z)^{2ab}}$ $(a,b\in\mathbb{C}^*)$, $\mu=a$ and $\gamma_3=\frac{1}{ab}$ in Corollary 3.4, we obtain the following corollary obtained by Obradovič et al. [19, Theorem 1].

Corollary 3.6. Let q be convex univalent in U with $q(0)=1$ and (3.1) holds true. If $f\in\mathcal{A}$ satisfy the following subordination condition:

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a}\prec\frac{1}{(1-z)^{2ab}}$$

and the function $\frac{1}{(1-z)^{2ab}}$ is the best dominant.

Taking $\gamma_1=\beta=1$, $\gamma_2=\alpha=0$, $q\,(z)=\frac{1}{(1-z)^{2b}}$ $(b\in\mathbb{C}^*)$, $\mu=1$ and $\gamma_3=\frac{1}{b}$ in Corollary 3.4, we obtain the following corollary obtained by Srivastava and Lashin [23, Theorem 3].

Corollary 3.7. Let q be convex univalent in U with $q(0)=1$, and (3.1) holds true. If $f\in\mathcal{A}$ satisfy the following subordination condition:

$$1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\prec\frac{1+z}{1-z},$$

then

$$f'(z)\prec\frac{1}{(1-z)^{2b}}$$

and the function $\frac{1}{(1-z)^{2b}}$ is the best dominant.

4 Superordination results

Now, by appealing to Lemma 2.2 it can be easily prove the following theorem.

Theorem 4.1. Let q be convex univalent in U with $q(0)=1$ and assume that $\Re\left(\frac{\gamma_2}{\gamma_3}q(z)\right)>0$. If f, Φ, $\Psi\in\mathcal{A}(p)$ such that $\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu}\in H[q(0),1]\cap Q$,

$$\gamma_1+\gamma_2\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu}$$
$$+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D_{\lambda,p}^{n+1}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n+1}(f*\Psi)^{(j)}(z)}{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}-1\right\}\quad(4.1)$$

is univalent in U and the following superordination condition

$$\begin{aligned}&\gamma_1+\gamma_2q(z)+\gamma_3\frac{zq'(z)}{q(z)}\\ \prec\;&\gamma_1+\gamma_2\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu}\\ &+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D_{\lambda,p}^{n+1}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n+1}(f*\Psi)^{(j)}(z)}{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}-1\right\}\end{aligned}$$

holds, then

$$q(z)\prec\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu}$$

and q is the best subordinant.

Proof. Define a function ϱ by

$$\varrho(z)=\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\qquad(z\in U). \tag{4.2}$$

Then the function ϱ is analytic in U and $\varrho(0)=1$. Therefore, differentiating (4.2) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we have

$$\begin{aligned}&\gamma_1+\gamma_2\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\\&+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D_{\lambda,p}^{n+1}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n+1}(f*\Psi)^{(j)}(z)}{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}-1\right\}\\=\;&\gamma_1+\gamma_2\varrho(z)+\gamma_3\frac{z\varrho'(z)}{\varrho(z)},\end{aligned}$$

that is,

$$\gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)}\prec\gamma_1+\gamma_2\varrho(z)+\gamma_3\frac{z\varrho'(z)}{\varrho(z)}.$$

By setting

$$\theta(w)=\gamma_1+\gamma_2 w \text{ and } \varphi(w)=\frac{\gamma_3}{w},$$

it can be easily observed that θ is analytic function in $\mathbb{C}$. Also, φ is analytic function in $\mathbb{C}^*$ and $\varphi(w)\neq 0$. Also we see that

$$\psi(z)=zq'(z)\,\varphi(q(z))=\gamma_3\frac{zq'(z)}{q(z)}$$

and

$$\Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\}=\Re\left\{\frac{\gamma_2}{\gamma_3}q(z)\right\}>0 \text{ for } z\in U,$$

it is clear that ψ is starlike univalent in U.

Therefore, Theorem 4.1 now follows by applying Lemma 2.2. ■

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ in Theorem 4.1, we have the following corollary.

Corollary 4.1. Let $\Re\left(\frac{\gamma_2}{\gamma_3}\frac{1+Az}{1+Bz}\right)>0$. If $f,\ \Phi,\ \Psi\in\mathcal{A}(p)$ such that $\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu}\in H[q(0),1]\cap Q$,

$$\begin{aligned}&\gamma_1+\gamma_2\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\\&+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D_{\lambda,p}^{n+1}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n+1}(f*\Psi)^{(j)}(z)}{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}-1\right\}\end{aligned}$$

is univalent in U, and the following superordination condition

$$\begin{aligned}&\gamma_1+\gamma_2\frac{1+Az}{1+Bz}+\gamma_3\frac{(A-B)\,z}{(1+Bz)^2}\\ \prec\ &\gamma_1+\gamma_2\left(\frac{\alpha D^n_{\lambda,p}\,(f*\Phi)^{(j)}\,(z)+\beta D^n_{\lambda,p}\,(f*\Psi)^{(j)}\,(z)}{(\alpha+\beta)\,\delta\,(p;j)\,z^{p-j}}\right)^{\mu}\\ &+\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}\,(f*\Phi)^{(j)}\,(z)+\beta D^{n+1}_{\lambda,p}\,(f*\Psi)^{(j)}\,(z)}{\alpha D^n_{\lambda,p}\,(f*\Phi)^{(j)}\,(z)+\beta D^n_{\lambda,p}\,(f*\Psi)^{(j)}\,(z)}-1\right\}\end{aligned}$$

holds, then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{\alpha D^n_{\lambda,p}\,(f*\Phi)^{(j)}\,(z)+\beta D^n_{\lambda,p}\,(f*\Psi)^{(j)}\,(z)}{(\alpha+\beta)\,\delta\,(p;j)\,z^{p-j}}\right)^{\mu}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Taking $p=\lambda=1$ and $j=n=0$ in Theorem 4.1, we obtain the following corollary which improves the result obtained by Magesh et al. [16, Theorem 3.15] .

Corollary 4.2. Let q be convex univalent in U with $q(0)=1$ and assume that $\Re\left(\frac{\gamma_2}{\gamma_3}q\,(z)\right)>0$. If $f,\ \Phi,\ \Psi\in\mathcal{A}$ such that $\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}\in H\,[q\,(0)\,,1]\cap Q$,

$$\begin{aligned}&\gamma_1+\gamma_2\left(\frac{\alpha\,(f*\Phi)\,(z)+\beta\,(f*\Psi)\,(z)}{(\alpha+\beta)\,z}\right)^{\mu}\\ &+\gamma_3\mu\left\{\frac{\alpha z\,(f*\Phi)'\,(z)+\beta z\,(f*\Psi)'\,(z)}{\alpha\,(f*\Phi)\,(z)+\beta\,(f*\Psi)\,(z)}-1\right\}\end{aligned}$$

is univalent in U and the following superordination condition

$$\begin{aligned}&\gamma_1+\gamma_2 q\,(z)+\gamma_3\frac{zq'\,(z)}{q\,(z)}\\ \prec\ &\gamma_1+\gamma_2\left(\frac{\alpha\,(f*\Phi)\,(z)+\beta\,(f*\Psi)\,(z)}{(\alpha+\beta)\,z}\right)^{\mu}\\ &+\gamma_3\mu\left\{\frac{\alpha z\,(f*\Phi)'\,(z)+\beta z\,(f*\Psi)'\,(z)}{\alpha\,(f*\Phi)\,(z)+\beta\,(f*\Psi)\,(z)}-1\right\}\end{aligned}$$

holds, then

$$q\,(z)\prec\left(\frac{\alpha\,(f*\Phi)\,(z)+\beta\,(f*\Psi)\,(z)}{(\alpha+\beta)\,z}\right)^{\mu}$$

and q is the best subordinant.

Taking $\Phi(z)=\frac{z}{1-z}$, and $\Psi(z)=\frac{z}{(1-z)^2}$ in Corollary 4.2, we obtain the following corollary which improves the result obtained by Magesh et al. [16, Corollary 3.16].

Corollary 4.3. Let q be convex univalent in U with $q(0)=1$ and assume that $\Re\left(\frac{\gamma_2}{\gamma_3}q(z)\right)>0$. If $f\in\mathcal{A}$ such that $\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)z}\right)^{\mu}\in H[q(0),1]\cap Q$,

$$\gamma_1+\gamma_2\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)z}\right)^{\mu}+\gamma_3\mu\left\{\frac{\alpha zf'(z)+\beta zf'(z)+\beta z^2f''(z)}{\alpha f(z)+\beta zf'(z)}-1\right\}$$

is univalent in U and

$$\begin{aligned}\gamma_1+\gamma_2q(z)+\gamma\frac{zq'(z)}{q(z)} \prec\ & \gamma_1+\gamma_2\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)z}\right)^{\mu}\\ &+\gamma_3\mu\left\{\frac{\alpha zf'(z)+\beta zf'(z)+\beta z^2f''(z)}{\alpha f(z)+\beta zf'(z)}-1\right\}\end{aligned}$$

then

$$q(z)\prec\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)z}\right)^{\mu}$$

and q is the best subordinant.

Taking $\alpha=\beta=1$ in Corollary 4.3, we obtain the following corollary which improves the result obtained by Magesh et al. [16, Corollary 3.17].

Corollary 4.4. Let q be convex univalent in U with $q(0)=1$ and assume that $\Re\left(\frac{\gamma_2}{\gamma_3}q(z)\right)>0$. If $f\in\mathcal{A}$ such that $\left(\frac{f(z)+zf'(z)}{2z}\right)^{\mu}\in H[q(0),1]\cap Q$,

$$\gamma_1+\gamma_2\left(\frac{f(z)+zf'(z)}{2z}\right)^{\mu}+\gamma_3\mu\left\{\frac{z^2f''(z)+2zf'(z)}{f(z)+zf'(z)}-1\right\}$$

is univalent in U and

$$\gamma_1+\gamma_2q(z)+\gamma\frac{zq'(z)}{q(z)}\prec\gamma_1+\gamma_2\left(\frac{f(z)+zf'(z)}{2z}\right)^{\mu}+\gamma_3\mu\left\{\frac{z^2f''(z)+2zf'(z)}{f(z)+zf'(z)}-1\right\},$$

then

$$q(z)\prec\left(\frac{f(z)+zf'(z)}{2z}\right)^{\mu}$$

and q is the best subordinant.

5 Sandwich resuts

Combining Theorem 3.1 and Theorem 4.1, we get the following sandwich theorem for the linear operator $D_{\lambda,p}^{n}(f*g)^{(j)}$.

Theorem 5.1. Let $q_1(z)$ be convex univalent in U with $q_1(0) = 1$, $\Re\left(\frac{\gamma_2}{\gamma_3} q_1(z)\right) > 0$, $q_2(z)$ be univalent in U with $q_2(0) = 1$ and satisfies (3.1). If f, Φ, $\Psi \in \mathcal{A}(p)$ such that $\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} \in H[q(0),1] \cap Q$,

$$\gamma_1 + \gamma_2 \left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} + \gamma_3 \mu \frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^{n+1}_{\lambda,p}(f*\Psi)^{(j)}(z)}{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)} - 1\right\}$$

is univalent in U and

$$\begin{aligned} & \gamma_1 + \gamma_2 q_1(z) + \gamma_3 \frac{z q_1'(z)}{q_1(z)} \\ \prec\ & \gamma_1 + \gamma_2 \left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} \\ & + \gamma_3 \mu \frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^{n+1}_{\lambda,p}(f*\Psi)^{(j)}(z)}{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)} - 1\right\} \\ \prec\ & \gamma_1 + \gamma_2 q_2(z) + \gamma_3 \frac{z q_2'(z)}{q_2(z)} \end{aligned}$$

holds, then

$$q_1(z) \prec \left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Taking $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1, 2, -1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$ in Theorem 5.1, we obtain the following corollary.

Corollary 5.1. Let $\Re\left(\frac{\gamma_2}{\gamma_3}\frac{1+A_1 z}{1+B_1 z}\right) > 0$ and $q_2(z)$ satisfies (3.3). If f, Φ, $\Psi \in \mathcal{A}(p)$ such that $\left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} \in H[q(0),1] \cap Q$,

$$\gamma_1 + \gamma_2 \left(\frac{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\delta(p;j)z^{p-j}}\right)^{\mu} + \gamma_3 \mu \frac{(p-j)}{\lambda}\left\{\frac{\alpha D^{n+1}_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^{n+1}_{\lambda,p}(f*\Psi)^{(j)}(z)}{\alpha D^n_{\lambda,p}(f*\Phi)^{(j)}(z)+\beta D^n_{\lambda,p}(f*\Psi)^{(j)}(z)} - 1\right\}$$

is univalent in U and

$$\begin{aligned} & \gamma_1+\gamma_2\frac{1+A_1z}{1+B_1z}+\gamma_3\frac{(A_1-B_1)\,z}{(1+B_1z)^2} \\ \prec\ & \gamma_1+\gamma_2\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu} \\ & +\gamma_3\mu\frac{(p-j)}{\lambda}\left\{\frac{\alpha D_{\lambda,p}^{n+1}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n+1}(f*\Psi)^{(j)}(z)}{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}-1\right\} \\ \prec\ & \gamma_1+\gamma_2\frac{1+A_2z}{1+B_2z}+\gamma_3\frac{(A_2-B_2)\,z}{(1+B_2z)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\left(\frac{\alpha D_{\lambda,p}^{n}(f*\Phi)^{(j)}(z)+\beta D_{\lambda,p}^{n}(f*\Psi)^{(j)}(z)}{(\alpha+\beta)\,\delta(p;j)\,z^{p-j}}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are, respectively, the best subordinant and the best dominant.

Taking $p=\lambda=1$ and $j=n=0$ in Theorem 5.1, we obtain the following corollary which improves the result obtained by Magesh et al. [16, Theorem 4.1] .

Corollary 5.2. Let $q_1(z)$ be convex univalent in U with $q_1(0)=1$, $\Re\left(\frac{\gamma_2}{\gamma_3}q_1(z)\right)>0$, $q_2(z)$ be univalent in U with $q_2(0)=1$, and satisfies (3.1). If $f,\ \Phi,\ \Psi\in\mathcal{A}$ such that $\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}\in H[q(0),1]\cap Q$,

$$\begin{aligned} & \gamma_1+\gamma_2\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)\,z}\right)^{\mu} \\ & +\gamma_3\mu\left\{\frac{\alpha z(f*\Phi)'(z)+\beta z(f*\Psi)'(z)}{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}-1\right\} \end{aligned}$$

is univalent in U and

$$\begin{aligned} & \gamma_1+\gamma_2q_1(z)+\gamma_3\frac{zq_1'(z)}{q_1(z)} \\ \prec\ & \gamma_1+\gamma_2\left(\frac{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)\,z}\right)^{\mu} \\ & +\gamma_3\mu\left\{\frac{\alpha z(f*\Phi)'(z)+\beta z(f*\Psi)'(z)}{\alpha(f*\Phi)(z)+\beta(f*\Psi)(z)}-1\right\} \\ \prec\ & \gamma_1+\gamma_2q_2(z)+\gamma_3\frac{zq_2'(z)}{q_2(z)} \end{aligned}$$

holds, then

$$q_1(z) \prec \left(\frac{\alpha (f * \Phi)(z) + \beta (f * \Psi)(z)}{(\alpha + \beta) z} \right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

6 Open Problem

Considere the function

$$\left(\frac{(\alpha + \beta) \delta (p; j) z^{p-j}}{\alpha D^n_{\lambda,p} (f * \Phi)^{(j)} (z) + \beta D^n_{\lambda,p} (f * \Psi)^{(j)} (z)} \right)^{\mu}$$

$(\alpha, \beta \in \mathbb{C}(i = 1, 2),$ such that $\alpha+\beta \neq 0; \mu \in \mathbb{C}^*; \lambda > 0; p > j; p \in \mathbb{N}; n, j \in \mathbb{N}_0).$

So, derive the subordination, superordination and sandwich results.

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