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On Sakaguchi-Type Harmonic Univalent Functions

Elif Yaşar and Sibel Yalçın

Department of Mathematics
Faculty of Arts and Science
Uludag University, Bursa, Turkey.
email: elifyasar@uludag.edu.tr and syalcin@uludag.edu.tr

Abstract

We consider the Sakaguchi functions which are starlike with respect to symmetrical points in the open unit disk U and extend it to class $SH(\alpha, t)$ concerning with Sakaguchi-type complex-valued harmonic univalent functions in U. A sufficient coefficient condition and a necessary and sufficient convolution characterization for such harmonic functions are determined. In addition, by using the coefficient inequality, a subclass of $SH(\alpha, t)$ and a new class $FH(\alpha, t)$ are defined. Distortion bounds and other properties are investigated for the class $FH(\alpha, t)$.

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1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense- preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$ in D. See [3].

Denote by SH the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0)-$

$1 = 0$. Then for $f = h + \overline{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

In 1984 Clunie and Sheil-Small [3] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avci and Zlotkiewicz [1], Silverman [11], Silverman and Silvia [12], Jahangiri [7] and Duren [4] studied the harmonic univalent functions.

Note that SH reduces to S, the class of analytic univalent functions, if the co-analytic part of $f = h + \overline{g}$ is identically zero.

The class of uniformly starlike functions, UST, introduced by Goodman [5] is following:

$$UST = \left\{ h(z) \in S : Re \frac{(z-\xi)h'(z)}{h(z) - h(\xi)} > 0 \right\}, (z, \xi) \in U \times U$$

As for UST, Rϕnning [9] showed the following important result.

Remark 1.1 *$h(z) \in UST$ if and only if for every $z \in U$, $|t| = 1$*

$$Re \left\{ \frac{(1-t)zh'(z)}{h(z) - h(tz)} \right\} > 0.$$

Let $SH(\alpha, t)$ denote the subclass of SH consisting of functions f of the form (1) that satisfy the condition

$$Re \left\{ \frac{(1-t)(zh'(z) - \overline{zg'(z)})}{(h(z) - h(tz)) + \left(\overline{g(z)} - \overline{g(tz)}\right)} \right\} \geq \alpha, \tag{2}$$

where $0 \leq \alpha < 1$ and $|t| \leq 1$, $t \neq 1$.

The class $S(0, -1)$, which consist of analytic univalent functions in U that are starlike with respect to symmetrical points, was introduced by Sakaguchi [10]. Ahuja and Jahangiri extended the class $S(0, -1)$ to $SH(\alpha)$ which include the harmonic functions. Recently, Owa et al. [8] defined and studied the class $S(\alpha, t)$.

The aim of this study is to extend the class $S(\alpha, t)$ in [8] to the class $SH(\alpha, t)$. We introduce a sufficient coefficient condition to be in the class $SH(\alpha, t)$ and determine a convolution characterization.Then, by using the coefficient inequality, subclass of $SH(\alpha, t)$ i.e. $SH^0(\alpha, t)$ and a new class $FH(\alpha, t)$ are defined. Distortion bounds and covering result are given for the class $FH(\alpha, t)$. Also it is shown that, $FH(\alpha, t) \subset FH(\beta, t)$ for $0 \leq \beta \leq \alpha < 1$.

2 Main Results

Denote by $SH^*(\alpha)$, the class of functions $f = h + \overline{g}$ of the form (1) which are harmonic starlike or order α $(0 \leq \alpha < 1)$, satisfying the condition $\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) \geq \alpha$, for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$.

First, we need the following lemma which was obtained by Jahangiri [6, 7].

Lemma 2.1 *Let $f = h + \overline{g}$ be of the form (1) and suppose that the coefficients of h and g satisfy the condition*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\left|a_n\right| + \frac{n+\alpha}{1-\alpha}\left|b_n\right|\right) \leq 2, \; a_1 = 1, \; 0 \leq \alpha < 1. \tag{3}$$

Then f is sense-preserving, harmonic univalent, and $f \in SH^*(\alpha)$.

Theorem 2.2 *For h and g as in (1), if the harmonic function $f = h + \overline{g}$ satisfy*

$$\sum_{n=1}^{\infty}\left[\left(\left|n-u_n\right| + (1-\alpha)\left|u_n\right|\right)\left|a_n\right| + \left(\left|n+u_n\right| + (1-\alpha)\left|u_n\right|\right)\left|b_n\right|\right] \leq 2(1-\alpha), \tag{4}$$

then $f \in SH(\alpha, t)$, where $u_n = 1 + t + t^2 + ... + t^{n-1}$, $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha < 1$.

Proof Since

$$\begin{aligned}
\sum_{n=1}^{\infty} n(|a_n| + |b_n|) &\leq \sum_{n=1}^{\infty}\left[\left(\left|n-u_n\right| + \left|u_n\right|\right)\left|a_n\right| + \left(\left|n+u_n\right| + \left|u_n\right|\right)\left|b_n\right|\right] \\
&\leq \sum_{n=1}^{\infty}\left[\left(\frac{\left|n-u_n\right| + (1-\alpha)\left|u_n\right|}{1-\alpha}\right)\left|a_n\right|\right. \\
&\qquad \left. + \left(\frac{\left|n+u_n\right| + (1-\alpha)\left|u_n\right|}{1-\alpha}\right)\left|b_n\right|\right] \leq 2,
\end{aligned}$$

by Lemma 2.1, we conclude that f is sense-preserving, harmonic univalent and $f \in SH^*(0)$. To prove $f \in SH(\alpha, t)$, we show that if $f(z)$ satisfy (4), then

$$\left|\frac{(1-t)(zh'(z) - \overline{zg'(z)})}{(h(z) - h(tz)) + \left(\overline{g(z)} - \overline{g(tz)}\right)} - 1\right| < 1 - \alpha.$$

Evidently, since

$$\frac{(1-t)(zh'(z) - \overline{zg'(z)})}{(h(z) - h(tz)) + \left(\overline{g(z)} - \overline{g(tz)}\right)} - 1$$

$$= \frac{(1-t)\left[z+\sum_{n=2}^{\infty} na_n z^n - \sum_{n=1}^{\infty} n\overline{b_n z^n}\right]}{(1-t)\left[z+\sum_{n=2}^{\infty} u_n a_n z^n + \sum_{n=1}^{\infty} u_n\overline{b_n z^n}\right]} - 1$$

$$= \frac{\sum_{n=2}^{\infty}(n-u_n)a_n z^n - \sum_{n=1}^{\infty}(n+u_n)\overline{b_n z^n}}{z+\sum_{n=2}^{\infty} u_n a_n z^n + \sum_{n=1}^{\infty} u_n\overline{b_n z^n}}$$

$$= \frac{\sum_{n=2}^{\infty}(n-u_n)a_n r^{n-1}e^{i(n-1)\theta}}{1+\sum_{n=2}^{\infty} u_n a_n r^{n-1}e^{i(n-1)\theta} + \sum_{n=1}^{\infty} u_n\overline{b_n} r^{n-1}e^{-i(n+1)\theta}}$$
$$- \frac{\sum_{n=1}^{\infty}(n+u_n)\overline{b_n} r^{n-1}e^{-i(n+1)\theta}}{1+\sum_{n=2}^{\infty} u_n a_n r^{n-1}e^{i(n-1)\theta} + \sum_{n=1}^{\infty} u_n\overline{b_n} r^{n-1}e^{-i(n+1)\theta}},$$

we see that

$$\left|\frac{(1-t)(zh'(z)-\overline{zg'(z)})}{(h(z)-h(tz))+\left(\overline{g(z)}-\overline{g(tz)}\right)} - 1\right| \le \frac{\sum_{n=2}^{\infty}|n-u_n|\,|a_n| + \sum_{n=1}^{\infty}|n+u_n|\,|b_n|}{1-\sum_{n=2}^{\infty}|u_n|\,|a_n| - \sum_{n=1}^{\infty}|u_n|\,|b_n|}.$$

Therefore if $f(z)$ satisfy (4), then we have

$$\left|\frac{(1-t)(zh'(z)-\overline{zg'(z)})}{(h(z)-h(tz))+\left(\overline{g(z)}-\overline{g(tz)}\right)} - 1\right| \le 1-\alpha.$$

This completes the proof of Theorem 2.2.

We also define the convolution or Hadamard product of two power series $f(z)=\sum_{n=1}^{\infty} a_n z^n$ and $F(z)=\sum_{n=1}^{\infty} A_n z^n$ by

$$(f*F)(z) = f(z)*F(z) = \sum_{n=1}^{\infty} a_n A_n z^n.$$

Theorem 2.3 *Let α be a constant such that $0\le\alpha<1$. Then a harmonic function $f=h+\overline{g}$ is in $SH(\alpha,t)$ if and only if*

$$h(z)*\frac{(1-\alpha)2z+(\xi(1-t)+2\alpha-1-t)z^2}{(1-z)^2(1-tz)} - \overline{g(z)}*\frac{(\xi+\alpha)2\overline{z}+(-\xi(1+t)-2\alpha+1-t)\overline{z}^2}{(1-\overline{z})^2(1-t\overline{z})} \ne 0,$$

where $|\xi|=1$, $\xi\ne-1$ and $0<|z|<1$.

Proof For $0\le\alpha<1$, a harmonic function $f=h+\overline{g}$ is in $SH(\alpha,t)$ if and only if the condition (2) holds. From (2) we obtain

$$Re\left\{\frac{(1-t)(zh'(z)-\overline{zg'(z)})-\alpha((h(z)-h(tz))+\left(\overline{g(z)}-\overline{g(tz)}\right))}{(1-\alpha)((h(z)-h(tz))+\left(\overline{g(z)}-\overline{g(tz)}\right))}\right\}\ge 0.$$

Or equivalently,

$$\frac{(1-t)(zh'(z)-\overline{zg'(z)})-\alpha((h(z)-h(tz))+\left(\overline{g(z)}-\overline{g(tz)}\right))}{(1-\alpha)[(h(z)-h(tz))+\left(\overline{g(z)}-\overline{g(tz)}\right)]}\neq\frac{\xi-1}{\xi+1},\quad (5)$$

where $|\xi|=1$, $\xi\neq-1$ and $0<|z|<1$.

Simplifying (5) we obtain the equivalent condition

$$(1+\xi)(1-t)(zh'(z)-\overline{zg'(z)})+(1-2\alpha-\xi)(h(z)-h(tz))+(1-2\alpha-\xi)(\overline{g(z)}-g(tz))\neq 0. \quad (6)$$

Upon noting that

$$\begin{aligned} zh'(z) &= h(z)*\frac{z}{(1-z)^2},\ zg'(z)=g(z)*\frac{z}{(1-z)^2},\\ h(z)-h(tz) &= (1-t)h(z)*\frac{z}{(1-z)(1-tz)}, \end{aligned}$$

and

$$g(z)-g(tz)=(1-t)g(z)*\frac{z}{(1-z)(1-tz)},$$

the condition (6) yields the necessary and sufficient condition required by Theorem 2.3.

We now define

$$SH^0(\alpha,t)=\{f(z)\in SH(\alpha,t): f(z)\ satisfy\ (4)\}.$$

Example 2.4 *Let us consider a function $f(z)$ given by*

$$f(z)=z+(1-\alpha)\left(\frac{\delta_2}{2(3-\alpha)}\overline{z}+\frac{\delta_3}{4(2-\alpha)}z^2\right),\ |\delta_2|=|\delta_3|=1.$$

Then for any t ($|t|\leq 1$, $t\neq 1$), $f(z)\in SH^0(\alpha,t)\subset SH(\alpha,t)$.

We further denote a new class by $FH(\alpha,t)$, such that $f=h+\overline{g}\in SH(\alpha,t)$ and the coefficients of h and g satisfy the condition

$$\sum_{n=1}^{\infty}\left[\left(|n-u_n|+(1-\alpha)|u_n|\right)|a_n|+\left(|n+u_n|-(1-\alpha)|u_n|\right)|b_n|\right]\leq 2(1-\alpha), \quad (7)$$

$u_n=1+t+t^2+...+t^{n-1}$ for some α ($0\leq\alpha<1$).

Theorem 2.5 $SH^0(\alpha,t)\subset FH(\alpha,t)$.

Proof Let $f(z) \in SH^0(\alpha, t)$. If we consider this inequality,

$$\begin{aligned}
&\frac{(|n-u_n| + (1-\alpha)|u_n|)|a_n| + (|n+u_n| - (1-\alpha)|u_n|)|b_n|}{1-\alpha} \\
\leq\ &\frac{(|n-u_n| + (1-\alpha)|u_n|)|a_n| + (|n+u_n| + (1-\alpha)|u_n|)|b_n|}{1-\alpha} \\
\leq\ &2,
\end{aligned}$$

for all $n \geq 2$, then we have that $f(z) \in FH(\alpha, t)$.

Corollary 2.6 *$FH(\alpha, -1)$ coincide with $SH(\alpha)$ which was studied in [2].*

Theorem 2.7 *If $f(z) \in FH(\alpha, t)$, then*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\alpha}{|1-t| + (1-\alpha)|1+t|} - \frac{1+\alpha}{|1-t| + (1-\alpha)|1+t|} |b_1| \right) r^2,$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\alpha}{|1-t| + (1-\alpha)|1+t|} - \frac{1+\alpha}{|1-t| + (1-\alpha)|1+t|} |b_1| \right) r^2,$$

where $|z| = r < 1$.

Proof We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f(z) \in FH(\alpha, t)$. Taking the absolute value of $f(z)$ we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2 \\
&= (1+|b_1|)r + \frac{1-\alpha}{|1-t| + (1-\alpha)|1+t|} \sum_{n=2}^{\infty} \left(\frac{|1-t| + (1-\alpha)|1+t|}{1-\alpha} |a_n| \right. \\
&\quad \left. + \frac{|1-t| + (1-\alpha)|1+t|}{1-\alpha} |b_n| \right) r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{|1-t| + (1-\alpha)|1+t|} \sum_{n=2}^{\infty} \left(\frac{|n-u_n| + (1-\alpha)|u_n|}{1-\alpha} |a_n| \right. \\
&\quad \left. + \frac{|n+u_n| - (1-\alpha)|u_n|}{1-\alpha} |b_n| \right) r^2,
\end{aligned}$$

Using (7), we obtain

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\alpha}{|1-t| + (1-\alpha)|1+t|} - \frac{1+\alpha}{|1-t| + (1-\alpha)|1+t|} |b_1| \right) r^2.$$

Corollary 2.8 *If $f(z) \in FH(\alpha, t)$, then*

$$\left\{ w : |w| < \left(1 - \frac{1-\alpha}{|1-t| + (1-\alpha)|1+t|}\right) - \left(1 - \frac{1+\alpha}{|1-t| + (1-\alpha)|1+t|}\right)|b_1| \right\} \subset f(U).$$

Corollary 2.9 *For $0 \le \beta \le \alpha < 1$, $FH(\alpha, t) \subset FH(\beta, t)$.*

Proof Let $0 \le \beta \le \alpha < 1$ and $f(z) \in FH(\alpha, t)$,

$$\begin{aligned} & \frac{(|n-u_n| + (1-\beta)|u_n|)|a_n| + (|n+u_n| - (1-\beta)|u_n|)|b_n|}{(1-\beta)} \\ < & \frac{(|n-u_n| + (1-\alpha)|u_n|)|a_n| + (|n+u_n| - (1-\alpha)|u_n|)|b_n|}{(1-\alpha)} \\ < & \ 2, \end{aligned}$$

for all $n \ge 1$, then $f(z) \in FH(\beta, t)$.

3 Open Problem

In this paper, we obtained only a sufficient coefficient condition for the class $SH(\alpha, t)$. A necassary and sufficient coefficient condition is still an open problem for the class $SH(\alpha, t)$.

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