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On Certain Results for Univalent Functions

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Abstract

Using the concept of differential subordination, we extend some results of Frasin and Darus [1] on univalent functions. Mathematica 7.0 is used to plot the extended regions of the complex plane.

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1 Introduction

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function f is said to be univalent in a domain $\mathbb{D}$ in the extended complex plane if and only if it is analytic in $\mathbb{D}$ except for at most one simple pole and $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2 (z_1; z_2 \in \mathbb{D})$. In this case, the equation $f(z) = w$ has at most one root in $\mathbb{D}$ for any complex number w. Such functions map $\mathbb{D}$ conformally onto a domain in the w-plane. Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the unit disk $\mathbb{E}$ and are normalized by the conditions $f(0) = f'(0) - 1 = 0$.

For two functions f and g analytic in the open unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function

w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{E}$ such that $f(z) = g(w(z))$, $z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\alpha)$ if it satisfies the inequality

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha, \ 0 \le \alpha < 1, \ z \in \mathbb{E}.$$

Nunokawa [2] proved the following result.

Theorem 1.1 *Let $f \in \mathcal{A}$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \ z \in \mathbb{E},$$

then f is univalent in $\mathbb{E}$.

Frasin and Darus [1] proved the following result.

Theorem 1.2 *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{1-\alpha}{2-\alpha}, \ 0 \le \alpha < 1, \ z \in \mathbb{E},$$

then $f \in \mathcal{B}(\alpha)$.

As a consequence of above two results, Frasin and Darus [1] also concluded the following result.

Theorem 1.3 *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right| < \frac{1}{2}, \ z \in \mathbb{E},$$

then f is univalent in $\mathbb{E}$.

The main objective of this paper is to extend the region of variability of the operator $\dfrac{(zf(z))''}{f'(z)} - \dfrac{2zf'(z)}{f(z)}$ in above stated results of Frasin and Darus [1] to get the same conclusion. Mathematica 7.0 is used to show the extended regions of the complex plane, pictorially. We shall use the following lemma to prove our main result.

Lemma 1.1 *([3], p.132, Theorem 3.4 h). Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re \frac{zh'(z)}{Q(z)} > 0,\ z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

2 Main Result

Theorem 2.1 *Let $f \in \mathcal{A}$, $\dfrac{z^2 f'(z)}{f^2(z)}$, satisfy the differential subordination*

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{az}{1+az},\ \ z \in \mathbb{E},$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec 1 + az,\ \ 0 < a \leq 1.$$

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = 0 \ \ \text{and} \ \ \phi(w) = \frac{1}{w}.$$

Obviously, the functions θ and ϕ are analytic in $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Also define the functions Q and h as under:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \frac{zq'(z)}{q(z)}.$$

Further, select the functions $p(z) = \dfrac{z^2 f'(z)}{f^2(z)}$, $f \in \mathcal{A}$ and $q(z) = 1 + az,\ 0 < a \leq 1$. Therefore,

$$Q(z) = h(z) = \frac{az}{1 + az},$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{zh'(z)}{Q(z)} = \frac{1}{1 + az}.$$

It can easily be verified that

$$\Re\frac{zQ'(z)}{Q(z)} = \Re\frac{zh'(z)}{Q(z)} > 0,\ z \in \mathbb{E},\ 0 < a \leq 1.$$

Hence, in view of Lemma 1.1, we obtain

$$\frac{z^2 f'(z)}{f^2(z)} \prec 1 + az,\ z \in \mathbb{E},\ 0 < a \leq 1.$$

Remark 2.1 *We, here, make a comparison of the above theorem with the result of Frasin [1] stated in Theorem 1.2. We notice that our result extends the region of variability of the differential operator* $\dfrac{(zf(z))''}{f'(z)} - \dfrac{2zf'(z)}{f(z)}$ *to get the required implication. We compare the results by considering the following particular cases. For* $a = \dfrac{3}{4}$ *and* f *same as in above theorem, from Theorem 2.1, we obtain:*

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{3z}{4 + 3z} = F(z) \tag{2}$$

$$\Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \frac{3}{4}.$$

For $\alpha = \dfrac{1}{4}$ *and all* z *in* $\mathbb{E}$*, Theorem 1.2 gives:*

$$\left|\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{3}{7} \tag{3}$$

$$\Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \frac{3}{4}$$

To show our claim pictorially, we plot $F(\mathbb{E})$ *(given in (2)) and the disk of radius* $\dfrac{3}{7}$ *with center at origin. According to the result in (3), the operator* $\dfrac{(zf(z))''}{f'(z)} -$

$\dfrac{2zf'(z)}{f(z)}$ *takes values in the disk of radius* $\dfrac{3}{7}$ *with center at origin shown by dark shaded region in Figure 2.1 to give the conclusion* $\left|\dfrac{z^2f'(z)}{f^2(z)} - 1\right| < \dfrac{3}{4}$, *whereas in view of the result in (2), the same operator can take values in the total shaded region (light and dark) to conclude the same result. Thus, the light shaded portion is the claimed extension.*

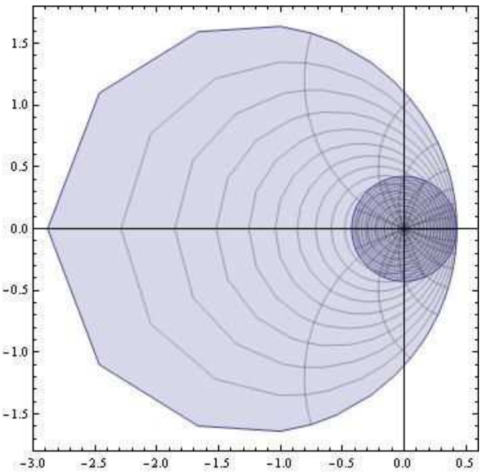

Figure 2.1

Remark 2.2 *For $a = 1$ and f same as in Theorem 2.1, we obtain:*

$$\Re\left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right) < \frac{1}{2}, \ z \in \mathbb{E},$$

implies

$$\left|\frac{z^2f'(z)}{f^2(z)} - 1\right| < 1,$$

and hence f is univalent in $\mathbb{E}$. We notice that the above result extends the region of variability of the differential operator $\dfrac{(zf(z))''}{f'(z)} - \dfrac{2zf'(z)}{f(z)}$ *over the result of Frasin and Darus [1] stated in Theorem 1.3 to get the required conclusion. By Theorem 1.3, the operator* $\dfrac{(zf(z))''}{f'(z)} - \dfrac{2zf'(z)}{f(z)}$ *takes values in the disk of radius* $\dfrac{1}{2}$ *with center at origin shown by dark shaded region in Figure 2.2 to imply the univalence of $f \in \mathcal{A}$ whereas according to the above result, the same operator can take values in the total shaded region (light and dark) to conclude the univalence of $f \in \mathcal{A}$. Therefore, the light shaded portion shows the extension.*

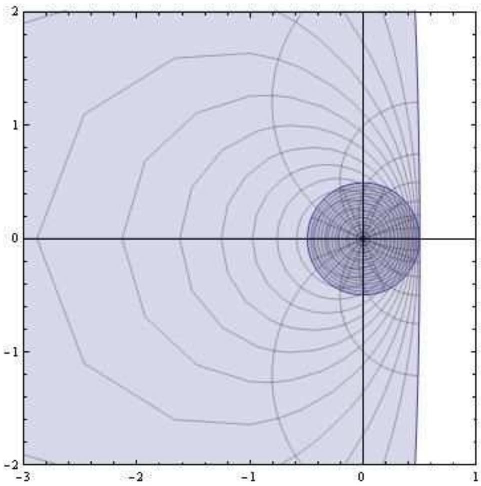

Figure 2.2

3 Open Problem

Using the technique of differential subordination, the regions of variability of differential operators involving with other classes of analytic univalent functions can also be extended.

References

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